Mon. Not. R. Astron. Soc. **000**, 000–000 (0000) Printed 18 March 2012 (MN LaTeX style file v2.2)

Critical properties of spherically symmetric black hole accretion in Schwarzschild geometry

Ipsita Mandal,[1⋆] Arnab K. Ray[2]† and Tapas K. Das[1]‡
[1] *Harish–Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211019, India*
[2] *Inter–University Centre for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune University Campus, Pune 411007, India*

arXiv:astro-ph/0702733v1 28 Feb 2007

18 March 2012

ABSTRACT
The stationary spherically symmetric accretion flow in the Schwarzschild metric has been set up as an autonomous first-order dynamical system, and it has been studied completely analytically. Of the three possible critical points in the flow, the one that is physically realistic behaves like the saddle point of the standard Bondi accretion problem. One of the two remaining critical points exhibits the strange mathematical behaviour of being either a saddle point or a centre-type point, depending on the values of the flow parameters. The third critical point is always unphysical and behaves like a centre-type point. The treatment has been extended to pseudo-Schwarzschild flows for comparison with the general relativistic analysis.

Key words: accretion, accretion discs – black hole physics – hydrodynamics

1 INTRODUCTION

To researchers in astrophysics and general relativity, physical models of spherical symmetry have an abiding appeal. One especial advantage with these models is that almost always they lend themselves to an exact mathematical analysis, and in the process they allow a very clear insight to be had into the underlying physical principles. For this reason in particular, spherically symmetric models frequently serve as a firm foundation for constructing theoretical models of physical systems involving more realistic and, therefore, unavoidably complicated features.

Studies in accretion are no exception to this practice. Ever since the seminal work published by Bondi (1952), that effectively launched the subject in the form in which it is recognised today, the problem of spherically symmetric flows has been revisited time and again from various angles (Parker 1958, 1966; Axford & Newman 1967; Balazs 1972; Michel 1972; Mészáros 1975; Blumenthal & Mathews 1976; Mészáros & Silk 1977; Begelman 1978; Cowie et al. 1978; Stellingwerf & Buff 1978; Garlick 1979; Brinkmann 1980; Moncrief 1980; Petterson et al. 1980; Vitello 1984; Bonazzola et al. 1987, 1992; Theuns & David 1992; Kazhdan & Murzina 1994; Markovic 1995; Tsuribe et al. 1995; Titarchuk et al. 1996; Zampieri et al. 1996; Titarchuk et al. 1997; Kovalenko & Eremin 1998; Das 1999; Malec 1999; Toropin et al. 1999; Das 2000; Das & Sarkar 2001; Ray & Bhattacharjee 2002; Ray 2003; Das 2004; Ray & Bhattacharjee 2005; Gaite 2006).

The original work of Bondi (1952) introduced formal fluid dynamical equations in the Newtonian construct of space and time to study the stationary accretion problem. From here the connection to the proper general relativistic framework was not too long in coming. In particular it was Michel (1972) who made an important early foray into the general relativistic domain, which was followed by a spate of other works, of which some (to mention a few), along with the paper of Michel (1972), addressed various aspects related to the critical behaviour of general relativistic flows in spherical symmetry (Begelman 1978; Brinkmann 1980; Malec 1999; Das & Sarkar 2001).

The work being presented here is also along the same lines. Its direct purpose is to construct a pedagogical theory to understand the nature of the critical points of stationary spherically symmetric flows in the Schwarzschild metric, after starting with the set of basic stationary equations which govern the flow. To the extent that critical solutions — specifically transonic solutions in regard to spherically symmetric flows — can only pass through some critical points (which must arguably be saddle points), this treatment will also have a bearing on a very important general issue in accretion studies — the manner in which a compressible astrophysical fluid passes (either continuously or discontinuously) from infinity to the event horizon of a black hole. In addressing this question the mathematical method that has been adopted is a dynamical systems analysis, which is always an effective tool for researchers in non-linear dynamics, and which has been tried successfully

⋆ ipsita@mri.ernet.in
† akr@iucaa.ernet.in
‡ tapas@mri.ernet.in

before in a related astrophysical system, namely multitransonic flows in an inviscid and thin pseudo-Schwarzschild accretion disc (Chaudhury et al. 2006). While it is to be naturally expected that the same treatment could be carried over directly to a rotational flow in the Schwarzschild metric, it has also been a worthwhile exercise to consider spherically symmetric flows first, as a suitably uncomplicated starting point into more intricate general relativistic problems. An immediate advantage in studying this relatively simple system has been that the mathematical treatment could be carried out with full analytic rigour, something, as it has been stressed right at the beginning, that has inspired the choice of the spherically symmetric model in the first place. It has been found here that under practical restrictions on the conditions for critical behaviour, it will be possible to gain a complete mathematical understanding (in the sense of producing final results which are absolutely non-numerical) of the nature of the critical points (not all of whom may be physically realistic), and the pattern of the solution topologies in the neighbourhood of those points. In no way do these results contradict any of the earlier findings (Michel 1972; Begelman 1978), and if anything, many surprisingly new features in the flow, hitherto unrecognised, have been revealed.

The dynamical systems treatment has also been carried out on pseudo-Schwarzschild spherical flows driven by some of the established pseudo-Newtonian potentials (Paczyński & Wiita 1980; Nowak & Wagoner 1991; Artemova et al. 1996) to check for the consistency of this approach with the fully general relativistic methods. All the results have been in complete qualitative compatibility with one another in the sense that there is only one physically meaningful critical point in the flow through which a solution could pass transonically, connecting infinity to the event horizon of the black hole. In one important detail, however, a quantitative difference has appeared. For accretion governed by cold ambient conditions, transonicity has been shown to be very much more pronounced in a properly relativistic flow, than in a pseudo-Schwarzschild flow.

The case for a dynamical systems approach in studying a proper general relativistic problem has been argued cogently in this work. The mathematical methods demonstrated here can very well be applied to the more involved accretion disc system, described both by the Schwarzschild metric and the Kerr metric. This particular study will be reported separately. Meanwhile, following the treatment on pseudo-Schwarzschild flows reported by Chaudhury et al. (2006), the present work on spherically symmetric flows may be considered to be the second in a series that pedagogically underlines the conspicuous advantages of applying dynamical systems methods in standard astrophysical fluid flow problems.

2 THE GENERAL RELATIVISTIC FLOW AND ITS FIXED POINTS

In this general relativistic treatment of a spherically symmetric, stationary, compressible fluid flow, the two relevant flow variables will be the radial inflow three-velocity, v, and the local proper mass density of the fluid, ρ. The radial coordinate of the flow is scaled by the Schwarzschild radius, $r_{\rm g} = 2GM_{\rm BH}/c^2$ (here $M_{\rm BH}$ is the mass of the black hole), with any characteristic velocity in the flow being scaled by c. Setting $G = c = 1$, the general relativistic analogue of Bernoulli's equation will be given as (Chakrabarti 1990, 1996; Das 2004)

$$\mathcal{E} = hv_t = \frac{p+\epsilon}{\rho}\sqrt{\frac{1-r^{-1}}{1-v^2}}, \tag{1}$$

with the pressure, p, connected to the density, ρ, through an equation of state, $p = k\rho^\gamma$, and the specific enthalpy, h, expressed as

$$h = \frac{p+\epsilon}{\rho}, \tag{2}$$

in which the energy density, ϵ (which includes the rest mass density and the internal energy), is to be set down as,

$$\epsilon = \rho + \frac{p}{\gamma-1}. \tag{3}$$

It is then possible to arrive at the relation

$$\mathcal{E} = \left(\frac{\gamma-1}{\gamma-1-c_{\rm s}^2}\right)\sqrt{\frac{1-r^{-1}}{1-v^2}}, \tag{4}$$

with the speed of sound, $c_{\rm s}$, defined under conditions of constant entropy, $\mathcal{S}$, as

$$c_{\rm s}^2 = \left.\frac{\partial p}{\partial \epsilon}\right|_{\mathcal{S}}. \tag{5}$$

Through the equation state, $p = k\rho^\gamma$, the speed of sound can be connected to the density, ρ, as

$$\rho = \left[\frac{c_{\rm s}^2}{\gamma k\left(1-nc_{\rm s}^2\right)}\right]^n, \tag{6}$$

with n being given by the usual definition of the polytropic index (Chandrasekhar 1939) as $n = (\gamma-1)^{-1}$.

The stationary continuity condition, on the other hand, will give another relation connecting the velocity and density fields as,

$$4\pi\rho v r^2\sqrt{\frac{1-r^{-1}}{1-v^2}} = \dot{m}, \tag{7}$$

with $\dot{m}$ being an integration constant, which is to be physically identified as the mass accretion rate.

It is now easy to see that equations (4), (6) and (7) will give a complete description of the flow system. Making use of equation (6) in equation (7), and then going back to equation (4), it will be possible to express the gradient of solutions in the r — v^2 plane as

$$\frac{\rm d}{{\rm d}r}(v^2) = \frac{v^2\left(1-v^2\right)\left[c_{\rm s}^2\left(4r-3\right)-1\right]}{r\left(r-1\right)\left(v^2-c_{\rm s}^2\right)}, \tag{8}$$

with $c_{\rm s}$ being used as a characteristic local scale of velocity in the fluid, against which the local bulk velocity of the flow, v, is to be measured.

From the foregoing expression it is evident that there will be non-trivial singularities under the conditions $r = 1$ and $v^2 = c_{\rm s}^2$, unless the numerator in equation (8) vanishes simultaneously. The condition $r = 1$, of course, corresponds to the behaviour of the flow on the actual event horizon of the black hole (where $v^2 = 1$), but of immediate interest is the condition $v^2 = c_{\rm s}^2$, which will give a critical condition for the flow at $r > 1$, if and only if the requirement of

$c_{\rm s}^2\,(4r-3)=1$, is simultaneously satisfied in the numerator. In that event the critical point conditions in the flow will be expressed as

$$v_{\rm c}^2=c_{\rm sc}^2=\frac{1}{4r_{\rm c}-3}, \tag{9}$$

with the subscript "c" labelling the critical point values.

The next logical step from here is to represent the critical point coordinates in terms of the parameters of the system. This can be done in two ways — either by substituting $v_{\rm c}$ and $c_{\rm sc}$ in equation (4) and expressing $r_{\rm c}$ as a function of $\mathcal{E}$ and γ, or by substituting the same critical point values in equation (7) and expressing $r_{\rm c}$ as a function of $\dot{m}$ and γ. Either approach is entirely equivalent to the other, and here for simplicity of algebraic manipulations, the former approach is being adopted. This will deliver a cubic equation in $r_{\rm c}$ as

$$r_{\rm c}^3+\mathcal{A}_2r_{\rm c}^2+\mathcal{A}_1r_{\rm c}+\mathcal{A}_0=0, \tag{10}$$

where

$$\mathcal{A}_0=\frac{27}{64\left(\mathcal{E}^2-1\right)},$$

$$\mathcal{A}_1=\frac{\left(2-3\gamma\right)^2\mathcal{E}^2-27\left(\gamma-1\right)^2}{16\left(\mathcal{E}^2-1\right)\left(\gamma-1\right)^2}$$

and

$$\mathcal{A}_2=\frac{2\left(2-3\gamma\right)\mathcal{E}^2+9\left(\gamma-1\right)}{4\left(\mathcal{E}^2-1\right)\left(\gamma-1\right)}.$$

The solutions of equation (10) can be found completely analytically by employing the Cardano-Tartaglia-del Ferro method for solving cubic equations. To that end it should be first convenient to define

$$\Sigma_1=\frac{3\mathcal{A}_1-\mathcal{A}_2^2}{9},$$

$$\Sigma_2=\frac{9\mathcal{A}_1\mathcal{A}_2-27\mathcal{A}_0-2\mathcal{A}_2^3}{54},$$

$$\Psi=\Sigma_1^3+\Sigma_2^2,$$

$$\xi_1=\left(\Sigma_2+\sqrt{\Psi}\right)^{1/3}$$

and

$$\xi_2=\left(\Sigma_2-\sqrt{\Psi}\right)^{1/3},$$

following which, the three roots of $r_{\rm c}$ can ultimately be set down as

$$r_{\rm c1}=-\frac{\mathcal{A}_2}{3}+\left(\xi_1+\xi_2\right), \tag{11}$$

$$r_{\rm c2}=-\frac{\mathcal{A}_2}{3}-\left(\xi_1+\xi_2\right)+{\rm i}\frac{\sqrt{3}}{2}\left(\xi_1-\xi_2\right) \tag{12}$$

and

$$r_{\rm c3}=-\frac{\mathcal{A}_2}{3}-\left(\xi_1+\xi_2\right)-{\rm i}\frac{\sqrt{3}}{2}\left(\xi_1-\xi_2\right), \tag{13}$$

respectively.

The sign of Ψ should be crucial in determining the nature of the roots. If $\Psi>0$, then only one root of $r_{\rm c}$ will be real, while for $\Psi<0$, there will be three real roots. Since it is obvious that Ψ will have a dependence on $\mathcal{E}$ and γ, it will be instructive to consider the appropriate ranges of values for these two parameters.

The parameter $\mathcal{E}$ is scaled in terms of the rest mass energy and it includes the rest mass energy itself. So it might be argued that a lower limit of $\mathcal{E}$ would be $\mathcal{E}=1$. On the other hand, although $\mathcal{E}$ can in principle assume any value greater than unity, values of $\mathcal{E}>2$ will imply extremely hot conditions at the outer boundary, with the thermal energy being much greater than the rest mass energy. Such a situation could not conceivably prevail in realistic astrophysical systems, and so the practically admissible range of $\mathcal{E}$ will be restricted to $1<\mathcal{E}<2$. The non-relativistic range of $\mathcal{E}$, on the other hand, will be $0<\mathcal{E}<1$, without any involvement of the rest mass energy. This range will be considered for the pseudo-Schwarzschild treatment in Section 4.

The parameter γ is likewise restricted by the range $1<\gamma<2$. The lower limit, i.e. $\gamma=1$, corresponds to optically thin, isothermal accretion, while values of $\gamma>2$ will involve magneto-hydrodynamics in the general relativistic theory and the anisotropic nature of pressure. For most realistic purposes, it should be noted, the range of γ actually varies from 4/3 to 5/3. Detailed discussions devoted to these issues are to be found in the literature (Begelman 1978; Das 2004).

And so it is that with the ranges of $1<\mathcal{E}<2$ and $1<\gamma<2$, it should be easy to show that Ψ would always be negative. This will consequently imply that the three roots of $r_{\rm c}$, as given by equations (11), (12) and (13), are always real, and they can be represented in terms of a new variable,

$$\Theta=\arccos\left(\frac{\Sigma_2}{\sqrt{-\Sigma_1^3}}\right),$$

as

$$r_{{\rm c}j}=-\frac{\mathcal{A}_2}{3}+2\sqrt{-\Sigma_1}\cos\left[\frac{\Theta+2\pi\left(j-1\right)}{3}\right], \tag{14}$$

with the label j taking the values $\{j=1,2,3\}$, respectively, for the three distinct roots. Of these roots, $r_{\rm c2}$ is always negative and, therefore, is not of much physical interest. The other two roots are always positive, and of these two, the one at $r_{\rm c1}$ is always to be found at distances greater than the event horizon of the black hole, i.e. $r_{\rm c1}>1$. A physically meaningful transonic inflow solution, connecting infinity to the event horizon, seems to prefer the critical point at $r_{\rm c1}$ to the one at $r_{\rm c3}$, even when $r_{\rm c3}>1$. Through the latter point, the flow exhibits non-physical properties like the matter inflow rate, as given by equations (6) and (7), being reduced to an imaginary quantity. It was also pointed out by Das (2004) that a flow associated with this point becomes superluminal much before reaching the event horizon of the black hole. Nevertheless, for all its apparent barrenness from a physical perspective, this critical point is not entirely devoid of some very interesting mathematical properties, when a dynamical systems approach is made to study the nature of the critical points of the flow. This issue will be taken up in Section 3.

3 PROPERTIES OF THE FIXED POINTS : AN AUTONOMOUS DYNAMICAL SYSTEM

So far the flow variables have been ascertained only at the critical points. Since the flow equations are in general nonlinear differential equations, short of carrying out a numerical integration, there is no completely rigorous analytical

prescription for solving these differential equations to determine the global nature of the flow variables. Nevertheless, some analytical headway could be made after all by taking advantage of the fact that equation (8), which gives a complete description of the r — v^2 phase portrait of the flow, is an autonomous first-order differential equation, and as such, could easily be recast into the mathematical form $\dot{x} = X(x, y)$ and $\dot{y} = Y(x, y)$, which is that of the very familiar coupled first-order dynamical system (Jordan & Smith 1999).

With the adoption of this line of attack, equation (8) may be decomposed in terms of a mathematical parameter, τ, to read as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}(v^2) &= v^2\left(1-v^2\right)\left[c_{\mathrm{s}}^2\left(4r-3\right)-1\right] \\ \frac{\mathrm{d}r}{\mathrm{d}\tau} &= r\left(r-1\right)\left(v^2-c_{\mathrm{s}}^2\right) \end{aligned} \tag{15}$$

in both of which, it must be noted that the parameter τ does not make an explicit appearance in the right hand side, something of an especial advantage that derives from working with autonomous systems. This kind of parametrization is quite common in fluid dynamics (Bohr et al. 1993), and in accretion studies especially, this approach has been made before (Ray & Bhattacharjee 2002; Afshordi & Paczyński 2003; Chaudhury et al. 2006). A further point that has to be noted is that the function c_{s}^2 in the right hand side of equations (15) can be expressed entirely in terms of v^2 and r, with the help of equations (6) and (7). This will exactly satisfy the criterion of a first-order dynamical system.

The critical points of the foregoing dynamical system, as equations (9) give them, have already been identified, and as equation (10) indicates, they have also been fixed in terms of the physical flow parameters. Beyond this stage, the next task would be to make a linearised approximation about the fixed point coordinates and extract a linear dynamical system out of equations (15). This will give a direct way to establish the nature of the critical points (or fixed points), which will ultimately pave the way for an investigation into the global behaviour of the solutions in the phase portrait of the flow. Indeed, not infrequently, if the flow system is simple enough, with only an understanding of the features of its critical points, complete qualitative predictions can be made about the global solutions. In this regard the classical spherically symmetric Bondi flow, with its single critical point, provides an object lesson (Ray & Bhattacharjee 2002).

Expanding about the fixed point values, a perturbation scheme of the kind $v^2 = v_{\mathrm{c}}^2 + \delta v^2$, $c_{\mathrm{s}}^2 = c_{\mathrm{sc}}^2 + \delta c_{\mathrm{s}}^2$ and $r = r_{\mathrm{c}} + \delta r$ is now to be applied on equations (15), and then a set of coupled autonomous linear equations is to be derived from them. While doing so, it will also be necessary to express δc_{s}^2 itself in terms of δr and δv^2, with the help of equations (6) and (7), as

$$\frac{\delta c_{\mathrm{s}}^2}{c_{\mathrm{sc}}^2} = -\frac{\gamma-1-c_{\mathrm{sc}}^2}{2}\left[\frac{1}{1-v_{\mathrm{c}}^2}\left(\frac{\delta v^2}{v_{\mathrm{c}}^2}\right)+\frac{4r_{\mathrm{c}}-3}{r_{\mathrm{c}}-1}\left(\frac{\delta r}{r_{\mathrm{c}}}\right)\right]. \tag{16}$$

The resulting coupled linearised equations will then read as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\delta v^2) = -\mathcal{B}\,\delta v^2 + \left[4c_{\mathrm{sc}}^2 - \frac{\mathcal{B}\left(1-c_{\mathrm{sc}}^2\right)}{r_{\mathrm{c}}\left(r_{\mathrm{c}}-1\right)}\right]\delta r$$

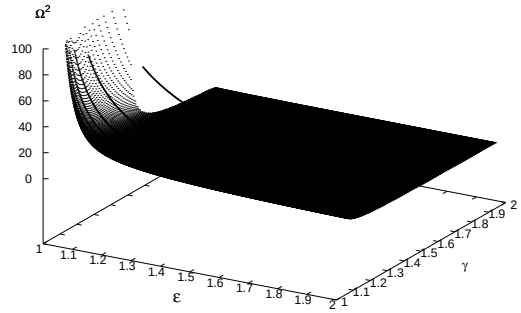

Figure 1. Variation of Ω^2 (associated with a physical saddle point) with respect to the parameters $\mathcal{E}$ and γ for the fully general relativistic spherically symmetric flow. All values of Ω^2 are positive for the chosen ranges of $\mathcal{E}$ and γ. For small values of these two parameters, the saddle-type feature is very robust.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\delta r) = r_{\mathrm{c}}\left(r_{\mathrm{c}}-1\right)\left(1+\frac{\mathcal{B}}{1-v_{\mathrm{c}}^2}\right)\delta v^2 + \mathcal{B}\,\delta r, \tag{17}$$

with $\mathcal{B} = (\gamma - 1 - c_{\mathrm{sc}}^2)/2$. Using solutions of the type $\delta v^2 \sim \exp(\Omega\tau)$ and $\delta r \sim \exp(\Omega\tau)$ in equations (17), the eigenvalues of the stability matrix associated with the critical points will be derived as

$$\Omega^2 = \frac{\mathcal{C}^2}{4} + \left(2r_{\mathrm{c}} - \mathcal{C}\right)\left[\frac{3\left(r_{\mathrm{c}}-1\right)}{4r_{\mathrm{c}}-3} - \frac{2-\gamma}{4}\right], \tag{18}$$

where

$$\mathcal{C} = \frac{4\left(\gamma-1\right)r_{\mathrm{c}} - \left(3\gamma-2\right)}{4r_{\mathrm{c}}-3}.$$

Once the position of a critical point, r_{c}, has become known, it is then quite easy to determine the nature of that critical point by using r_{c} in equation (18). Since r_{c} is a function of $\mathcal{E}$ and γ, it effectively implies that Ω^2 can, in principle, be regarded as a function of the flow parameters. From the form of Ω^2 in equation (18), a generic conclusion that can be immediately drawn is that the only possible critical points will be saddle points and centre-type points, and for the former, $\Omega^2 > 0$, while for the latter, $\Omega^2 < 0$. This is entirely to be expected because the physical system under study here is a conservative system, very much like, by analogy, the undamped simple harmonic oscillator, the fixed points of whose phase portrait also manifest identical properties (Jordan & Smith 1999).

Of the three critical points, as implied by equation (14), the one given by the unphysical negative root, r_{c2}, is always a centre-type point. This is something that by itself is also an equally unphysical trait as far as transonic accretion is concerned, where the whole objective is to have a solution that will connect infinity to the event horizon of the black hole, and in doing so will cross the sonic barrier with a finite gradient.

This requirement, on the other hand, is very eminently met at the critical point fixed at r_{c1}, which is always a saddle point and allows a physical transonic solution to pass throught itself without any hindrance whatsoever. The behaviour of this critical point has been graphically depicted

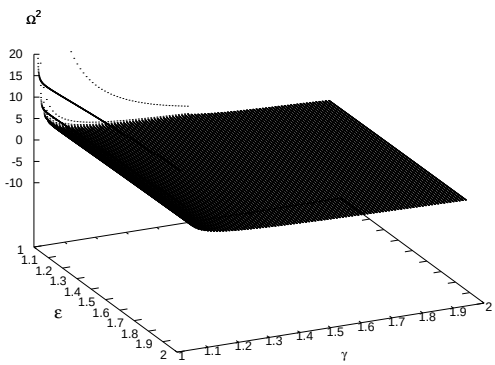

Figure 2. The general relativistic flow has a strange critical point which behaves sometimes like a saddle ($\Omega^2 > 0$) and at other times like a centre-type point ($\Omega^2 < 0$), depending on the values of the parameters $\mathcal{E}$ and γ. The dependence of Ω^2 does not exhibit widespread deviations, except for small values of $\mathcal{E}$ and γ.

in Fig. 1. An interesting fact that emerges from the plot is that for low values of $\mathcal{E}$ and γ, there is a strong growth behaviour for Ω^2. In fact, as $\mathcal{E}$ and γ approach unity, the value of Ω^2 increases by four orders of magnitude than what has been scaled along the vertical axis of Fig. 1[1], and this is exactly how it should be. Apart from the sign of Ω^2 — always positive and, therefore, indicative of a saddle point — its magnitude also conveys quantitative information about the strength of the saddle-type behaviour; the greater its magnitude, the more prominent in effect will be the transonic behaviour of the flow (which will pass through the saddle point). When $\mathcal{E}$ and γ approach unity, it will correspond more closely to a cold, isothermal distribution of matter, and this will make the flow easily submit to the strong gravitational influence of the black hole (a hotter and more pressure-dominated flow will be capable of building a much greater resistance against gravity). Transonicity can only occur when gravity triumphs over all other effects, and the general features of the flow indicated by Fig. 1 is an emphatic endorsement of physical transonicity in a general relativistic scenario.

On the other hand, a most intriguing and counter-intuitive behaviour is to be encountered at the critical point characterised by r_{c3}, which is placed between r_{c1} (always a saddle point) and r_{c2} (always a centre-type point). Depending on the values of the parameters $\mathcal{E}$ and γ, this point exhibits the properties of both a saddle point ($\Omega^2 > 0$) and a centre-type point ($\Omega^2 < 0$). As far as the former case is concerned, this is a very curious state of affairs indeed, because conventional wisdom about well-behaved fixed points will have it that no two adjacent fixed points can both be saddle points (Jordan & Smith 1999), quite contrary to what is being seen here — a saddle point (obviously realistic and physically meaningful) and a centre-type point (however physically unrealistic) flanking a point which, on some occasions at least, behaves like a saddle (and on other occasions like a centre). It is difficult to provide an analogy for this kind of behaviour from any other area in physics. One might conjecture that this waywardness could be intimately connected to the divergent behaviour (like superluminal motion) exhibited by solutions associated with this critical point (when it behaves like a saddle point). A quantitative graphical understanding of the nature of this critical point has been conveyed in Fig. 2. Once again it is to be seen that as $\mathcal{E}$ and γ both assume values closer to unity, strong evidence of a saddle-type behaviour results.

[1] This reduced scaling of the vertical axis will give a much better resolution of the overall behaviour of Ω^2 against the parameters on which it is dependent.

4 THE PSEUDO-SCHWARZSCHILD APPROACH : A COMPARATIVE STUDY

Frequently in studies of black hole accretion, it becomes convenient to dispense completely with the rigour of general relativity, and instead make use of an "effective" pseudo-Newtonian potential that will imitate general relativistic effects in the Newtonian construct of space and time. In that event the relevant stationary equations for the compressible spherically symmetric flow will look like

$$v\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} + \phi'(r) = 0 \tag{19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho v r^2\right) = 0 \tag{20}$$

respectively, with the former being the familiar Euler's equation and the latter the equation of continuity (Chakrabarti 1990; Frank et al. 2002). In equation (19), $\phi(r)$ is the generalised pseudo-Newtonian potential driving the flow (with the prime denoting its spatial derivative), and p is the pressure of the flowing gas, which is related to the density by the usual polytropic prescription. The local speed of sound, with which the bulk flow will have to be scaled, is defined by $c_{\rm s}^2 = \partial p/\partial\rho$, following which the connection between the ρ and $c_{\rm s}$ could be established as

$$\rho = \left(\frac{c_{\rm s}^2}{\gamma k}\right)^n, \tag{21}$$

whose form may, for academic interest, be compared with the general relativistic analogue given in equation (6).

Making use of equation (21) in equation (20), and then going back to equation (19), will lead to a relation for the gradient of solutions, which will read as

$$\frac{\mathrm{d}}{\mathrm{d}r}(v^2) = \frac{2v^2\left[2c_{\rm s}^2 - r\phi'(r)\right]}{r\left(v^2 - c_{\rm s}^2\right)}. \tag{22}$$

The critical points in the flow will be derived from the standard requirement that the flow solutions will have a finite gradient when they will cross the sonic horizon, which will mean that both the numerator and the denonimator will have to vanish simultaneously and non-trivially. This can only happen when

$$v_{\rm c}^2 = c_{\rm sc}^2 = \frac{r_{\rm c}\phi'(r_{\rm c})}{2}, \tag{23}$$

which gives the critical point conditions, with the subscript "c" labelling the critical point values, as usual.

It is not a difficult exercise to integrate equation (19) and then transform the variable ρ in it to $c_{\rm s}$, with the help

of equation (21). Once this has been done, the critical conditions, as given by equations (23), will have to be invoked, and all of these will deliver a relation for fixing the critical point coordinates in terms of the flow parameters, $\mathcal{E}$ (which is actually Bernoulli's constant) and γ. This relation will look like

$$\frac{1}{4}\left(\frac{\gamma+1}{\gamma-1}\right) r_{\rm c}\phi'(r_{\rm c}) + \phi(r_{\rm c}) = \mathcal{E}, \tag{24}$$

with the ranges of values of $\mathcal{E}$ and γ here, in what is essentially a non-relativistic approach, being accordingly chosen (Das & Sarkar 2001), as opposed to the relativistic values of $\mathcal{E}$ and γ adopted in Section 3.

The choice of the pseudo-Newtonian potential, $\phi(r)$, will obviously determine the number of roots of equation (24). Four such potentials have been considered here, and they have in general been labelled as $\phi \equiv \phi_i(r)$, with $\{i = 1, 2, 3, 4\}$. In an explicit form, each of these potentials will be given as

$$\begin{aligned}
\phi_1(r) &= -\frac{1}{2\,(r-1)} \\
\phi_2(r) &= -\frac{1}{2r}\left[1 - \frac{3}{2r} + 12\left(\frac{1}{2r}\right)^2\right] \\
\phi_3(r) &= -1 + \left(1 - \frac{1}{r}\right)^{1/2} \\
\phi_4(r) &= \frac{1}{2}\ln\left(1 - \frac{1}{r}\right)
\end{aligned} \tag{25}$$

in all of which, the length of the radial coordinate, r, has been scaled in units of the Schwarzschild radius, defined as $r_{\rm g} = 2GM_{\rm BH}/c^2$. Every potential mentioned above has been introduced in accretion literature at various stages to meet some specific physical requirement — ϕ_1 by Paczyński & Wiita (1980), ϕ_2 by Nowak & Wagoner (1991), and ϕ_3 and ϕ_4 by Artemova et al. (1996), respectively. With respect to spherically symmetric flows in particular, a comparative overview of the physical properties of these potentials has been given by Das & Sarkar (2001).

Considering each of the potentials separately in equation (24), it will be seen that two distinct roots will be obtained on using both ϕ_1 and ϕ_3, while from ϕ_2 three roots will be delivered. The fourth potential, ϕ_4, will lead to a transcendental equation, and any root, therefore, can only be extracted by numerical methods. Using the bisection algorithm, it can be shown that only one physical root is possible.

With each such physically feasible root, a critical point can evidently be associated. The way to have any appreciation of the behaviour of these critical points has already been outlined in Section 3. The first task would be to set up an autonomous dynamical system (in terms of a mathematical parameter, τ), which will be

$$\begin{aligned}
\frac{\rm d}{{\rm d}\tau}(v^2) &= 2v^2\left[2c_{\rm s}^2 - r\phi'(r)\right] \\
\frac{{\rm d}r}{{\rm d}\tau} &= r\left(v^2 - c_{\rm s}^2\right).
\end{aligned} \tag{26}$$

Subject to the perturbation scheme, $v^2 = v_{\rm c}^2 + \delta v^2$, $c_{\rm s}^2 = c_{\rm sc}^2 + \delta c_{\rm s}^2$ and $r = r_{\rm c} + \delta r$, equation (26) will lead to a set of coupled autonomous linear equations in the perturbed quantities δv^2 and δr, with $\delta c_{\rm s}^2$ having first been expressed in terms of δv^2 and δr from the continuity condition, as

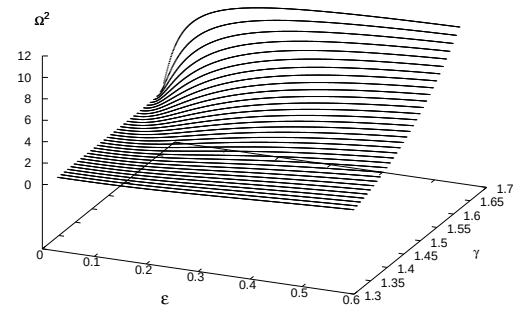

Figure 3. Dependence of Ω^2 (always positive) on the parameters $\mathcal{E}$ and γ for the pseudo-Newtonian potential, ϕ_1. The saddle-type behaviour is maximum for intermediate values of $\mathcal{E}$ and γ.

$$\frac{\delta c_{\rm s}^2}{c_{\rm sc}^2} = -\frac{1}{2n}\left[\frac{\delta v^2}{v_{\rm c}^2} + 4\frac{\delta r}{r_{\rm c}}\right]. \tag{27}$$

The coupled linear dynamical system will be

$$\begin{aligned}
\frac{\rm d}{{\rm d}\tau}(\delta v^2) &= -\frac{2c_{\rm sc}^2}{n}\,\delta v^2 - 2v_{\rm c}^2\mathcal{D}\,\delta r \\
\frac{\rm d}{{\rm d}\tau}(\delta r) &= r_{\rm c}\left(1 + \frac{1}{2n}\right)\delta v^2 + \frac{2c_{\rm sc}^2}{n}\,\delta r,
\end{aligned} \tag{28}$$

with

$$\mathcal{D} = \frac{4c_{\rm sc}^2}{nr_{\rm c}} + \phi'(r_{\rm c}) + r_{\rm c}\phi''(r_{\rm c}).$$

From here it is an easy passage to deriving the eigenvalues of the stability matrix associated with the critical points. With the use of solutions, $\delta v^2 \sim \exp(\Omega\tau)$ and $\delta r \sim \exp(\Omega\tau)$, in equations (28), these eigenvalues will be derived as

$$\Omega^2 = \frac{\left[r_{\rm c}\phi'(r_{\rm c})\right]^2}{2}\left[(3-5\gamma) - (\gamma+1)\,r_{\rm c}\frac{\phi''(r_{\rm c})}{\phi'(r_{\rm c})}\right], \tag{29}$$

from whose structure it can once again be claimed that the critical points can only be either saddle points or centre-type points. The dependence of Ω^2 on $\mathcal{E}$ and γ has been separately shown in Figs. 3 and 4, under the choice of ϕ_1 and ϕ_4, respectively. For ϕ_1 there are two critical points, of which only one is the physically relevant saddle point, while for ϕ_4 there is only one critical point, which has to be found numerically. It is always a saddle point. The two potentials, ϕ_1 and ϕ_4, have been chosen because they give a closer Newtonian approximation to fully general relativistic conditions, than the other two potentials, ϕ_2 and ϕ_3 (Das & Sarkar 2001). From both the plots in Figs. 3 and 4, it is quite evident that contrary to what it was for the fully general relativistic case, for pseudo-Schwarzschild flows, strongly transonic features (indicated by high positive values of Ω^2) occur at much greater values of $\mathcal{E}$ and γ.

5 CONCLUDING REMARKS

In trying to mathematically understand the nature of the three critical points in the general relativistic flow, it has been shown that there arises a situation, whereby, because

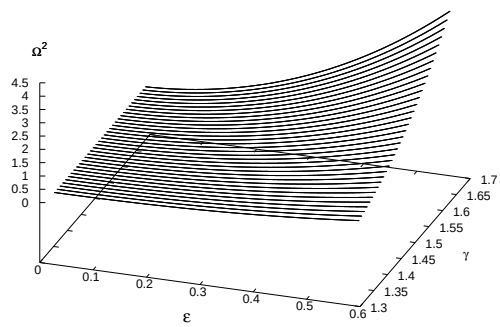

Figure 4. Dependence of Ω^2 (always positive) on the parameters $\mathcal{E}$ and γ for the pseudo-Newtonian potential, ϕ_4. The saddle-type behaviour has a monotonic growth with increasingly higher values of $\mathcal{E}$ and γ.

of the fluctuating nature of the middle critical point (sometimes a saddle point and sometimes a centre-type point), two contiguous critical points will be of the same kind. If these two particular points are both saddle-type then as an exercise of mathematical interest (if not of any direct physical relevance), it will be patently impossible to connect the two points by continuous solutions. However, going back to equation (8), and subjecting it to a closer examination, a possible way of bypassing this difficulty could be found. It has been discussed already that the only critical conditions selected from equation (8) will be the ones on length scales greater than that of the event horizon. On the other hand, if some of the unphysical criteria ($v^2 = 1$ at $r = 1$, or $v = 0$ at $r = 0$, etc.) for criticality in the flow were to be taken into account, then some new critical points (of mathematical interest only) could be found in the global phase portrait, even on unrealistic length scales. These possible critical points may then settle the difficulty which has arisen from the apparent existence of adjacent critical points of the same nature. Solutions could then be connected from one region to the other through these "hidden" critical points.

Another problem with saddle points is that solutions passing through them are notoriously sensitive to the fine tuning of the outer boundary condition of the flow. This is a standing problem with stationary flows and in consequence of this, it has been shown for the classical Bondi problem that transonicity could be achieved only if the evolution of the flow were to be followed through time (Ray & Bhattacharjee 2002). This is a relatively easy proposition in the Newtonian domain. When a flow is studied in the general relativistic regime, the time-dependent evolution will require much greater mathematical (both analytical and numerical) sophistication. Having made this point it should also be a fair expectation that transonicity would continue all the same to hold its primary position in spherically symmetric flows.

ACKNOWLEDGEMENTS

This research has made use of NASA's Astrophysics Data System. The authors express their indebtedness to J. K. Bhattacharjee, J. Ehlers, A. D. Gangal, T. Naskar and Y. Shtanov for some useful comments.

REFERENCES

Afshordi, N., Paczyński, B., 2003, ApJ, 592, 354
Artemova, I. V., Björnsson, G., Novikov, I. D., 1996, ApJ, 461, 565
Axford, W. I., Newman, R. C., 1967, ApJ, 147, 230
Balazs, N. L., 1972, MNRAS, 160, 79
Begelman, M. C., 1978, A&A, 70, 53
Blumenthal, G. R., Mathews, W. G., 1976, ApJ, 203, 714
Bohr, T., Dimon, P., Putkaradze, V., 1993, Journal of Fluid Mechanics, 254, 635
Bonazzola, S., Falgarone, E., Heyvaerts, J., Pérault, M., Puget, J. L., 1987, A&A, 172, 293
Bonazzola, S., Pérault, M., Puget, J. L., Heyvaerts, J., Falgarone, E., Panis, J. F., 1992, Journal of Fluid Mechanics, 245, 1
Bondi, H., 1952, MNRAS, 112, 195
Brinkmann, W., 1980, A&A, 85, 146
Chakrabarti, S. K., 1990, Theory of Transonic Astrophysical Flows, World Scientific, Singapore
Chakrabarti, S. K., 1996, Physics Reports, 266, 229
Chandrasekhar, S., 1939, An Introduction to the Study of Stellar Structure, The University of Chicago Press, Chicago
Chaudhury, S., Ray, A. K., Das, T. K., 2006, MNRAS, 373, 146
Cowie, L. L., Ostriker, J. P., Stark, A. A., 1978, ApJ, 226, 1041
Das, T. K., 1999, MNRAS, 308, 201
Das, T. K., 2000, MNRAS, 318, 294
Das, T. K., 2004, Classical and Quantum Gravity, 21, 5253
Das, T. K., Sarkar, A., 2001, A&A, 374, 1150
Frank, J., King, A., Raine, D., 2002, Accretion Power in Astrophysics, Cambridge University Press, Cambridge
Gaite, J., 2006, A&A, 449, 861
Garlick, A. R., 1979, A&A, 73, 171
Jordan, D. W., Smith, P., 1999, Nonlinear Ordinary Differential Equations, Oxford University Press, Oxford
Kazhdan, Y. M., Murzina, M., 1994, MNRAS, 270, 351
Kovalenko, I. G., Eremin, M. A., 1998, MNRAS, 298, 861
Malec, E., 1999, Phys. Rev. D, 60, 104043
Markovic, D., 1995, MNRAS, 277, 11
Mészáros, P., 1975, A&A, 44, 59
Mészáros, P., Silk, J., 1977, A&A, 55, 289
Michel, F. C., 1972, Astrophys. Space Sci., 15, 153
Moncrief, V., 1980, ApJ, 235, 1038
Nowak, A. M., Wagoner, R. V., 1991, ApJ, 378, 656
Paczyński, B., Wiita P. J., 1980, A&A, 88, 23
Parker, E. N., 1958, ApJ, 123, 664
Parker, E. N., 1958, ApJ, 143, 32
Petterson, J. A., Silk, J., Ostriker, J. P., 1980, MNRAS, 191, 571
Ray, A. K., 2003, MNRAS, 344, 1085
Ray, A. K., Bhattacharjee, J. K., 2002, Phys. Rev. E, 66, 066303

Ray, A. K., Bhattacharjee, J. K., 2005, ApJ, 627, 368
Stellingwerf, R. F., Buff, J., 1978, ApJ, 221, 661
Theuns, T., David, M., 1992, ApJ, 384, 587
Titarchuk, L., Mastichiadis, A., Kylafis, N. D., 1996, A&A, 120, 171
Titarchuk, L., Mastichiadis, A., Kylafis, N. D., 1997, ApJ, 487, 834
Toropin, Yu. M., Toropina, O. D., Savelyev, V. V., Romanova, M. M., Chechetkin, V. M., Lovelace, R. V. E., 1999, ApJ, 517, 906
Tsuribe, T., Umemura, M., Fukue, J., 1995, PASJ, 47, 73
Vitello, P., 1984, ApJ, 284, 394
Zampieri, L., Miller, J. C., Turolla, R., 1996, MNRAS, 281, 1183