Bianchi type I universe with viscous fluid: A qualitative analysis

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(Dated: June 5, 2015)

The nature of cosmological solutions for a homogeneous, anisotropic Universe given by a Bianchi type-I (BI) model in the presence of a Cosmological constant Λ is investigated by taking into account dissipative process due to viscosity. The system in question is thoroughly studied both analytically and numerically. It is shown the viscosity, as well as the Λ term exhibit essential influence on the character of the solutions. In particular a negative Λ gives rise to an ever-expanding Universe, whereas, a suitable choice of initial conditions plus a positive Λ can result in a singularity-free oscillatory mode of expansion. For some special cases it is possible to obtain oscillations in the exponential mode of expansion of the BI model even with a negative Λ, where oscillations arise by virtue of viscosity.

PACS numbers: 03.65.Pm, 04.20.Jb and 04.20.Ha
Keywords: Bianchi type I (BI) model, Cosmological constant, viscous fluid

arXiv:gr-qc/0410056v1 13 Oct 2004

I. INTRODUCTION

The investigation of relativistic cosmological models usually has the energy momentum tensor of matter generated by a perfect fluid. To consider more realistic models one must take into account the viscosity mechanisms, which have already attracted the attention of many researchers. Misner [1, 2] suggested that strong dissipative due to the neutrino viscosity may considerably reduce the anisotropy of the blackbody radiation. Viscosity mechanism in cosmology can explain the anomalously high entropy per baryon in the present universe [3, 4]. Bulk viscosity associated with the grand-unified-theory phase transition [5] may lead to an inflationary scenario [6, 7, 8].

A uniform cosmological model filled with fluid which possesses pressure and second (bulk) viscosity was developed by Murphy [9]. The solutions that he found exhibit an interesting feature that the big bang type singularity appears in the infinite past. Exact solutions of the isotropic homogeneous cosmology for open, closed and flat universe have been found by Santos et al [10], with the bulk viscosity being a power function of energy density.

The nature of cosmological solutions for homogeneous Bianchi type I (BI) model was investigated by Belinsky and Khalatnikov [11] by taking into account dissipative process due to viscosity. They showed that viscosity cannot remove the cosmological singularity but results in a qualitatively new behavior of the solutions near singularity. They found the remarkable property that during the time of the *big bang* matter is created by the gravitational field. BI solutions in case of stiff matter with a shear viscosity being the power function of of energy density were obtained by Banerjee [12], whereas BI models with bulk viscosity (η) that is a power function of energy density ε and when the universe is filled with stiff matter were studied by Huang [13]. The effect

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of bulk viscosity, with a time varying bulk viscous coefficient, on the evolution of isotropic FRW models was investigated in the context of open thermodynamics system was studied by Desikan [14]. This study was further developed by Krori and Mukherjee [15] for anisotropic Bianchi models. Cosmological solutions with nonlinear bulk viscosity were obtained in [16]. Models with both shear and bulk viscosity were investigated in [17, 18].

Though Murphy [9] claimed that the introduction of bulk viscosity can avoid the initial singularity at finite past, results obtained in [19] show that, it is, in general, not valid, since for some cases big bang singularity occurs in finite past.

We studied a self-consistent system of the nonlinear spinor and/or scalar fields in a BI spacetime in presence of a perfect fluid and a Λ term [20, 21] in order to clarify whether the presence of a singular point an inherent property of the relativistic cosmological models or is it only a consequence of specific simplifying assumptions underlying these models? Recently we have considered a system of nonlinear spinor field in a BI Universe filled with viscous fluid [22]. Since the viscous fluid itself presents a growing interest, we have studied the influence of viscous fluid and Λ term in the evolution of the BI Universe [23]. In that paper we consider only some special cases those allow exact solutions. In this paper along with those special cases we study some general cases, giving a qualitative analysis of the system of equations. We also perform some numerical calculations and compare the results obtained with those given in some pioneering papers in this field, e.g. [11]

II. DERIVATION OF BASIC EQUATIONS

Using the variational principle in this section we derive the fundamental equations for the gravitational field from the action (2.1) :

$$\mathscr{S}(g;\varepsilon) = \int \mathscr{L}\sqrt{-g}d\Omega \tag{2.1}$$

with

$$\mathscr{L} = \mathscr{L}_{\text{grav.}} + \mathscr{L}_{\text{vf}}. \tag{2.2}$$

The gravitational part of the Lagrangian (2.2) $\mathscr{L}_{\text{grav.}}$ is given by a Bianchi type-I metric, whereas the term $\mathscr{L}_{\text{vf}}$ describes a viscous fluid.

We also write the expressions for the metric functions explicitly in terms of the volume scale τ defined bellow (2.18). Defining Hubble constant (2.28) in analogy with a flat Friedmann-Robertson-Walker (FRW) Universe, we also derive the system of equations for τ, H and ε, with ε being the energy density of the viscous fluid, which plays the central role here.

A. The gravitational field

As a gravitational field we consider the Bianchi type I (BI) cosmological model. It is the simplest model of anisotropic universe that describes a homogeneous and spatially flat space-time and if filled with perfect fluid with the equation of state $p = \zeta\varepsilon, \quad \zeta < 1$, it eventually evolves into a FRW universe [24]. The isotropy of present-day universe makes BI model a prime candidate for studying the possible effects of an anisotropy in the early universe on modern-day data observations. In view of what has been mentioned above we choose the gravitational part of the Lagrangian (2.2) in the form

$$\mathscr{L}_{\text{g}} = \frac{R}{2\kappa}, \tag{2.3}$$

where R is the scalar curvature, $\kappa = 8\pi G$ being the Einstein's gravitational constant. The gravitational field in our case is given by a Bianchi type I (BI) metric

$$ds^2 = dt^2 - a^2 dx^2 - b^2 dy^2 - c^2 dz^2, \tag{2.4}$$

with a, b, c being the functions of time t only. Here the speed of light is taken to be unity.

B. Viscous fluid

The influence of the viscous fluid in the evolution of the Universe is performed by means of its energy momentum tensor, which acts as the source of the corresponding gravitational field. The reason for writing $\mathscr{L}_{\rm vf}$ in (2.2) is to underline that we are dealing with a self-consistent system. The energy momentum tensor of a viscous field has the form

$$T^{\nu}_{\mu\,(\rm m)} = (\varepsilon + p')u_\mu u^\nu - p'\delta^\nu_\mu + \eta g^{\nu\beta}[u_{\mu;\beta} + u_{\beta:\mu} - u_\mu u^\alpha u_{\beta;\alpha} - u_\beta u^\alpha u_{\mu;\alpha}], \tag{2.5}$$

where

$$p' = p - (\xi - \frac{2}{3}\eta)u^\mu_{;\mu}. \tag{2.6}$$

Here ε is the energy density, p - pressure, η and ξ are the coefficients of shear and bulk viscosity, respectively. Note that the bulk and shear viscosities, η and ξ, are both positively definite, i.e.,

$$\eta > 0, \quad \xi > 0. \tag{2.7}$$

They may be either constant or function of time or energy, such as :

$$\eta = |A|\varepsilon^\alpha, \quad \xi = |B|\varepsilon^\beta. \tag{2.8}$$

The pressure p is connected to the energy density by means of a equation of state. In this report we consider the one describing a perfect fluid :

$$p = \zeta\varepsilon, \quad \zeta \in (0,1]. \tag{2.9}$$

Note that here $\zeta \neq 0$, since for dust pressure, hence temperature is zero, that results in vanishing viscosity.

In a comoving system of reference such that $u^\mu = (1,0,0,0)$ we have

$$T^0_{0\,(\rm m)} = \varepsilon, \tag{2.10a}$$

$$T^1_{1\,(\rm m)} = -p' + 2\eta\frac{\dot a}{a}, \tag{2.10b}$$

$$T^2_{2\,(\rm m)} = -p' + 2\eta\frac{\dot b}{b}, \tag{2.10c}$$

$$T^3_{3\,(\rm m)} = -p' + 2\eta\frac{\dot c}{c}. \tag{2.10d}$$

Let us introduce the dynamical scalars such as the expansion and the shear scalar as usual

$$\theta = u^\mu_{;\mu}, \quad \sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \tag{2.11}$$

where

$$\sigma_{\mu\nu} = \frac{1}{2}\Big(u_{\mu;\alpha}P^{\alpha}_{\nu} + u_{\nu;\alpha}P^{\alpha}_{\mu}\Big) - \frac{1}{3}\theta P_{\mu\nu}. \tag{2.12}$$

Here P is the projection operator obeying

$$P^2 = P. \tag{2.13}$$

For the space-time with signature $(+,-,-,-)$ it has the form

$$P_{\mu\nu} = g_{\mu\nu} - u_{\mu}u_{\nu}, \quad P^{\mu}_{\nu} = \delta^{\mu}_{\nu} - u^{\mu}u_{\nu}. \tag{2.14}$$

For the BI metric the dynamical scalar has the form

$$\theta = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = \frac{\dot{\tau}}{\tau}, \tag{2.15}$$

and

$$2\sigma^2 = \frac{\dot{a}^2}{a^2} + \frac{\dot{b}^2}{b^2} + \frac{\dot{c}^2}{c^2} - \frac{1}{3}\theta^2. \tag{2.16}$$

C. Field equations and their solutions

Variation of (2.1) with respect to metric tensor $g_{\mu\nu}$ gives the Einstein's field equation. In account of the Λ-term for the BI space-time (2.4) this system of equations can be rewritten as

$$\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} = \kappa T^1_1 - \Lambda, \tag{2.17a}$$

$$\frac{\ddot{c}}{c} + \frac{\ddot{a}}{a} + \frac{\dot{c}}{c}\frac{\dot{a}}{a} = \kappa T^2_2 - \Lambda, \tag{2.17b}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} = \kappa T^3_3 - \Lambda, \tag{2.17c}$$

$$\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{b}}{b}\frac{\dot{c}}{c} + \frac{\dot{c}}{c}\frac{\dot{a}}{a} = \kappa T^0_0 - \Lambda, \tag{2.17d}$$

where over dot means differentiation with respect to t and T^{μ}_{ν} is the energy-momentum tensor of a viscous fluid given above (2.10).

1. *Expressions for the metric functions*

To write the metric functions explicitly, we define a new time dependent function $\tau(t)$

$$\tau = abc = \sqrt{-g}, \tag{2.18}$$

which is indeed the volume scale of the BI space-time.

Let us now solve the Einstein equations. In account of (2.10) from (2.17a), (2.17b), and (2.17c) one finds the following expressions for the metric functions explicitly [23]

$$a(t) = A_1\tau^{1/3}\exp\left[(B_1/3)\int\frac{e^{-2\kappa\int\eta dt}}{\tau}dt\right], \tag{2.19a}$$

$$b(t) = A_2\tau^{1/3}\exp\left[(B_2/3)\int\frac{e^{-2\kappa\int\eta dt}}{\tau}dt\right], \tag{2.19b}$$

$$c(t) = A_3\tau^{1/3}\exp\left[(B_3/3)\int\frac{e^{-2\kappa\int\eta dt}}{\tau}dt\right], \tag{2.19c}$$

where the constants A_i's and B_i's obey the following relations

$$\begin{aligned} A_1A_2A_3 &= 1, \\ B_1+B_2+B_3 &= 0. \end{aligned}$$

Thus, the metric functions are found explicitly in terms of τ and viscosity.

As one sees from (2.19a), (2.19b) and (2.19c), for $\tau = t^n$ with $n > 1$ the exponent tends to unity at large t, and the anisotropic model becomes isotropic one.

2. *Singularity analysis*

Let us now investigate the existence of singularity (singular point) of the gravitational case, which can be done by investigating the invariant characteristics of the space-time. In general relativity these invariants are composed from the curvature tensor and the metric one. In a 4D Riemann space-time there are 14 independent invariants. Instead of analyzing all 14 invariants, one can confine this study only in 3, namely the scalar curvature $I_1 = R$, $I_2 = R_{\mu\nu}R^{\mu\nu}$, and the Kretschmann scalar $I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ [25, 26].. At any regular space-time point, these three invariants I_1, I_2, I_3 should be finite. Let us rewrite these invariants in detail.

For the Bianchi I metric one finds the scalar curvature

$$I_1 = R = -2\Big(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}+\frac{\dot{c}}{c}\frac{\dot{a}}{a}\Big). \tag{2.20}$$

Since the Ricci tensor for the BI metric is diagonal, the invariant $I_2 = R_{\mu\nu}R^{\mu\nu} \equiv R^{\nu}_{\mu}R^{\mu}_{\nu}$ is a sum of squares of diagonal components of Ricci tensor, i.e.,

$$I_2 = \Big[\big(R^0_0\big)^2+\big(R^1_1\big)^2+\big(R^2_2\big)^2+\big(R^3_3\big)^2\Big]. \tag{2.21}$$

Analogically, for the Kretschmann scalar in this case we have $I_3 = R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu}$, a sum of squared components of all nontrivial $R^{\mu\nu}{}_{\mu\nu}$, which can be written as

$$\begin{aligned} I_3 &= 4\Big[\Big(R^{01}{}_{01}\Big)^2+\Big(R^{02}{}_{02}\Big)^2+\Big(R^{03}{}_{03}\Big)^2+\Big(R^{12}{}_{12}\Big)^2+\Big(R^{23}{}_{23}\Big)^2+\Big(R^{31}{}_{31}\Big)^2\Big] \\ &= 4\Big[\Big(\frac{\ddot{a}}{a}\Big)^2+\Big(\frac{\ddot{b}}{b}\Big)^2+\Big(\frac{\ddot{c}}{c}\Big)^2+\Big(\frac{\dot{a}}{a}\frac{\dot{b}}{b}\Big)^2+\Big(\frac{\dot{b}}{b}\frac{\dot{c}}{c}\Big)^2+\Big(\frac{\dot{c}}{c}\frac{\dot{a}}{a}\Big)^2\Big]. \end{aligned} \tag{2.22}$$

Let us now express the foregoing invariants in terms of τ. From Eqs. (2.19) we have

$$a_i = A_i\tau^{1/3}\exp\left((B_i/3)\int\frac{e^{-2\kappa\int\eta dt}}{\tau(t)}dt\right), \tag{2.23a}$$

$$\frac{\dot{a}_i}{a_i} = \frac{\dot{\tau}+B_ie^{-2\kappa\int\eta dt}}{3\tau} \quad (i=1,2,3,), \tag{2.23b}$$

$$\frac{\ddot{a}_i}{a_i} = \frac{3\tau\ddot{\tau}-2\dot{\tau}^2-\dot{\tau}B_ie^{-2\kappa\int\eta dt}-6\kappa\eta\tau B_ie^{-2\kappa\int\eta dt}+B_i^2e^{-4\kappa\int\eta dt}}{9\tau^2}, \tag{2.23c}$$

i.e., the metric functions a,b,c and their derivatives are in functional dependence with τ. From Eqs. (2.23) one can easily verify that [23]

$$I_1 \propto \frac{1}{\tau^2}, \quad I_2 \propto \frac{1}{\tau^4}, \quad I_3 \propto \frac{1}{\tau^4}.$$

Thus we see that at any space-time point, where $\tau = 0$ the invariants $I_1, I_2,$ and I_3 become infinity, hence the space-time becomes singular at this point.

D. Equations for determining τ

In the foregoing subsection we wrote the corresponding metric functions in terms of volume scale τ. In what follows, we write the equation for τ and study it in details.

Summation of Einstein equations (2.17a), (2.17b), (2.17c) and 3 times (2.17d) gives

$$\ddot{\tau} - \frac{3}{2}\kappa\xi\,\dot{\tau} = \frac{3}{2}\kappa\big(\varepsilon - p\big)\tau - 3\Lambda\tau. \tag{2.24}$$

For the right-hand-side of (2.24) to be a function of τ only, the solution to this equation is well-known [27].

The energy-momentum conservation law, i.e.,

$$T^{\nu}_{\mu;\nu} = T^{\nu}_{\mu,\nu} + \Gamma^{\nu}_{\rho\nu}T^{\rho}_{\mu} - \Gamma^{\rho}_{\mu\nu}T^{\nu}_{\rho} = 0, \tag{2.25}$$

in our case gives the following equation for ε :

$$\dot{\varepsilon} + \frac{\dot{\tau}}{\tau}\omega - (\xi + \frac{4}{3}\eta)\frac{\dot{\tau}^2}{\tau^2} + 4\eta(\kappa T^0_0 - \Lambda) = 0, \tag{2.26}$$

where

$$\omega = \varepsilon + p, \tag{2.27}$$

is the thermal function.

Defining a generalized Hubble constant H :

$$\frac{\dot{\tau}}{\tau} = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = 3H. \tag{2.28}$$

the Eqs. (2.24) and (2.26) in account of (2.10) can be rewritten as

$$\dot{H} = \frac{\kappa}{2}\big(3\xi H - \omega\big) - \big(3H^2 - \kappa\varepsilon + \Lambda\big), \tag{2.29a}$$

$$\dot{\varepsilon} = 3H\big(3\xi H - \omega\big) + 4\eta\big(3H^2 - \kappa\varepsilon + \Lambda\big). \tag{2.29b}$$

In terms of dynamical scalars θ and σ the system (2.29) takes a very simple form

$$\dot{\theta} = \frac{3\kappa}{2}\big(\xi\theta - \omega\big) - 3\sigma^2, \tag{2.30a}$$

$$\dot{\varepsilon} = \theta\big(\xi\theta - \omega\big) + 4\eta\sigma^2. \tag{2.30b}$$

Note that the Eqs. (2.30) coincide with the ones given in [12].

III. QUALITATIVE ANALYSIS AND SOME SPECIAL SOLUTIONS

In this subsection we simultaneously solve the system of equations for τ, H, and ε. It is convenient to rewrite the Eqs. (2.28) and (2.29) as a single system :

$$\dot{\tau} = 3H\tau, \tag{3.1a}$$
$$\dot{H} = \frac{\kappa}{2}\big(3\xi H - \omega\big) - \big(3H^2 - \kappa\varepsilon + \Lambda\big), \tag{3.1b}$$
$$\dot{\varepsilon} = 3H\big(3\xi H - \omega\big) + 4\eta\big(3H^2 - \kappa\varepsilon + \Lambda\big). \tag{3.1c}$$

In account of (2.27),(2.8) and (2.9) the Eqs. (3.1) now can be rewritten as

$$\dot{\tau} = 3H\tau, \tag{3.2a}$$
$$\dot{H} = \frac{\kappa}{2}\big(3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon\big) - \big(3H^2 - \kappa\varepsilon + \Lambda\big), \tag{3.2b}$$
$$\dot{\varepsilon} = 3H\big(3B\varepsilon^{\beta}H - (1+\zeta)\varepsilon\big) + 4A\varepsilon^{\alpha}\big(3H^2 - \kappa\varepsilon + \Lambda\big). \tag{3.2c}$$

The system (3.1) have been extensively studied in literature either partially [9, 12, 13] or in general [11]. In what follows, we consider the system (3.1) for some special choices of the parameters.

A. Qualitative analysis

Following Belinski and Khalatnikov [11] let us now study the characters of the solutions of the dynamical system (3.1) or (3.2). We first rewrite the system (3.1), namely (3.1b) and (3.1c) in the matrix form :

$$\begin{pmatrix} \dot{H} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} \kappa/2 & -1 \\ 3H & 4\eta \end{pmatrix} \begin{pmatrix} 3\xi H - \omega \\ 3H^2 - \kappa\varepsilon + \Lambda \end{pmatrix}. \tag{3.3}$$

Note that unlike the system studied by Belinski and Khalatnikov the system in consideration contains a Cosmological constant Λ.

1. *General properties of the system*

Easy to note that the solutions cannot intersect the axis $\varepsilon = 0$, since $\dot{\varepsilon}|_{\varepsilon=0} = 0$, as well as the parabola

$$3H^2 - \kappa\varepsilon + \Lambda = 0, \tag{3.4}$$

as far as (3.4) is itself the integral curve. Thus, starting from the point $(H,\varepsilon) = (+\infty, 0)$, the solutions cannot enter into the "prohibited region" inside the parabola (3.4). Whether they may achieve $H < 0$ depends on the value of Λ. Note that, unlike the system considered by Belinski *et. al* [11] the system in this report contains a nonzero Λ term.

2. *Critical points of the dynamical system*

a) By virtu of linear independence of the columns of the matrix of the Eq. (3.3) the critical points are the solutions of the equations

$$3\xi H - \omega = 0, \tag{3.5a}$$
$$3H^2 - \kappa\varepsilon + \Lambda = 0. \tag{3.5b}$$

i.e., they necessarily lie on the parabola (3.4). Solutions to the system (3.5) will be the roots of the equation

$$3\kappa B^2\varepsilon^{1+2\beta} - (1+\zeta)^2\varepsilon^2 \;-\; 3\Lambda B^2\varepsilon^{2\beta} = 0, \tag{3.6a}$$

$$H \;=\; \frac{1+\zeta}{3B}\varepsilon^{1-\beta}. \tag{3.6b}$$

The quantity of the positive roots of the Eq. (3.6) according to Cartesian law is equal to the number of changes of sign of coefficients of equations or less than that by an even number. So, for

$$\begin{array}{rlll} & \Lambda < 0 & \text{and} & 1/2 < \beta < 1 \;\; (Fig.2, Fig.3) \\ or & \Lambda > 0 & \text{and} & \beta < 1/2 \;\; (Fig.5, Fig.6) \end{array}$$

the number of roots is either 2 or zero. For the remaining cases

$$\begin{array}{lll} \Lambda < 0 & \text{and} & \beta > 1 \;\; (Fig.1), \\ \Lambda < 0 & \text{and} & \beta < 1/2 \;\; (Fig.4), \\ \Lambda > 0 & \text{and} & \beta > 1/2 \;\; (Fig.7) \end{array}$$

there exists only one root. The corresponding pictures of the phase curves are given in figures cited above. The critical points are denoted by small circles. Note that here we consider the case with $\eta = 0$, i.e., $A = 0$. In case if $\eta \neq 0$, with the increase of A the separatrix of the saddle tilts (inclines) to the left. Since the overall picture for $A \neq 0$ remains qualitatively unaltered, we only show the corresponding phase portrait for two cases, namely Fig. 8 corresponds to Fig. 1, Fig. 9 corresponds to Fig. 4. Note that for numerical calculations we set $\kappa = 1$, $\zeta = 0.333$ (if not mentioned otherwise). In the Figs. 1 - 7 η is taken to be zero. Note that in the Figs. E and T stand for ε and τ, respectively.

Since, the equation for ε only contains η, the energy density for nontrivial η undergoes essential changes, whereas H and τ remain virtually unchanged.

The types of critical points lying on the integral curve alternate: ... saddle, attracting knot, saddle .... So it is sufficient to consider the case with maximum number of roots. Taking into account the Eqs. (3.1c) and (3.4) let us now calculate

$$\begin{aligned} \lim_{\varepsilon\to+\infty}\frac{\dot\varepsilon}{3H\varepsilon} \;&=\; \lim_{\varepsilon\to+\infty}\frac{3B\varepsilon^{\beta}\sqrt{\kappa\varepsilon-\Lambda}-\varepsilon(1+\zeta)}{\varepsilon} \\ &=\; 3B\sqrt{\kappa\varepsilon^{(2\beta-1)}-\Lambda\varepsilon^{-2}}-(1+\zeta) = \begin{cases} -(1+\zeta) \;<\; 0, \;\; \beta < 1/2, \\ \quad +\infty \quad\;\; > \; 0, \;\; \beta > 1/2. \end{cases} \end{aligned} \tag{3.7}$$

So, the latest critical point for $\beta < 1/2$ is attracting knot and for $\beta > 1/2$ is saddle.

b) It is obvious that if $Lambda \leq 0$ the points of intersection of the boundary are the critical points

$$H \;=\; \pm\sqrt{-\Lambda/3}, \tag{3.8a}$$

$$\varepsilon \;=\; 0. \tag{3.8b}$$

c) For $H < 0$ there may exist critical points , if the columns of the matrix of (3.3) are linearly dependent. In that case the critical points are the roots of the equation

$$3\kappa(\zeta-1)\varepsilon + 6\kappa^2 AB\varepsilon^{\alpha+\beta} + 8\kappa^2 A^2\varepsilon^{2\alpha} + 6\Lambda = 0, \tag{3.9}$$

and

$$H = -\frac{2}{3}\kappa A \varepsilon^{\alpha}. \tag{3.10}$$

In case of $\eta = 0$ the roots of the characteristic equation

$$\left|\frac{D(\dot{H}, \dot{\varepsilon})}{D(H, \varepsilon)} - \mu\right| = 0, \tag{3.11}$$

are

$$\mu_{1,2} = \frac{3\kappa\xi \pm \sqrt{9\kappa^2\xi^2 - 48\Lambda(1+\zeta)}}{4}. \tag{3.12}$$

The critical point $(H, \varepsilon) = (0, 2\Lambda/[\kappa(1-\zeta)])$ is of type divergent focus if $\Lambda > 9\kappa^2\xi^2/[48(1+\zeta)]$ or divergent knot if $\Lambda < 9\kappa^2\xi^2/[48(1+\zeta)]$.

In the cases illustrated in Figs. 5 and 7, $H \to \infty$ and $\varepsilon \to \infty$ as $t \to \infty$, whereas, for the cases given in Fig. 6 one observes increasing oscillation bounded by the attracting parabola (3.4).

3. *Integral curves*

For $\Lambda \leq 0$ the solutions starting from the upper half-plane $H > 0$ cannot enter into the lower one. For $\Lambda > 0$ some of the solutions may enter into the lower half-plane through the segment $H = 0$ and $0 \leq \varepsilon \leq \Lambda$ and never returns back, since $\dot{H}|_{H=0} < 0$.

B. Numerical solutions

In this subsection solutions to the system of equations (3.1) has been obtained numerically. Evolution of the Hubble constant H, energy density ε and volume scale τ corresponding to the cases studied above with different B, β and Λ has been illustrated in the Figs. 12 - 32. As one sees, for a negative Λ the volume scale τ expands exponentially, whereas, for a positive Λ there exist solutions where τ initially expands and after reaching some maximum begins to contract and finally collapses into a point, thus giving rise to space-time singularity. Beside this, as one sees from Fig. 11, a suitable choice of initial conditions gives rise to a singularity-free oscillatory mode of expansion of the Universe.

C. exact solutions

In this subsection we consider some special cases allowing exact solutions.

1. *Case with bulk viscosity*

Let us first consider the case when the real fluid possesses the bulk viscosity only. The corresponding system of Eqs. can then be obtained by setting $\eta = 0$ in (3.1) or $A = 0$ in (3.2). In this case the Eqs. (3.1a) and (3.1b) remain unaltered, while (3.1c) takes the form

$$\dot{\varepsilon} = 3H\big(3\xi H - \omega\big). \tag{3.13}$$

In view of (3.13) the system (3.1) admits the following first integral

$$\tau^2\big(\kappa\varepsilon - 3H^2 - \Lambda\big) = C_1, \quad C_1 = \text{const}. \tag{3.14}$$

The relation (3.14) can be interpreted as follows. At the initial stage of evolution the volume scale τ tends to zero, while, the energy density ε tends to infinity. Since the Hubble constant and the Λ term are finite, the relation (3.14) is in correspondence with the current line of thinking. Let us see what happens as the Universe expands. It is well known that with the expansion of the Universe, i.e., with the increase of τ, the energy density ε decreases. Suppose at some stage of expansion $\tau \to \infty$, hence $\varepsilon \to 0$. Then from (3.14) follows that at the stage in question

$$3H^2 + \Lambda \to 0. \tag{3.15}$$

In case of $\Lambda = 0$, we find $H = 0$, i.e., in absence of a Λ term, once $\tau \to \infty$, the process of evolution is terminated. As one sees from (3.15), for the H to make any sense, the Λ term should be negative. In presence of a negative λ term the evolution process of the Universe never comes to a halt, it either expands further or begin to contract depending on the sign of $H = \pm\sqrt{-\Lambda/3}, \quad \Lambda < 0$.

Let us now consider the case when the bulk viscosity is inverse proportional to expansion, i.e.,

$$\xi\,\theta = C_2, \quad C_2 = \text{const}. \tag{3.16}$$

Now keeping into mind that $\theta = \dot{\tau}/\tau = 3H$, also the relations (3.1a), (2.27) and (2.9) the Eq. (3.13) can be written as

$$\frac{\dot{\varepsilon}}{C_2 - (1+\zeta)\varepsilon} = \frac{\dot{\tau}}{\tau}. \tag{3.17}$$

From the Eq. (3.17) one finds

$$\varepsilon = \frac{1}{1+\zeta}\left[C_2 + C_3\tau^{-(1+\zeta)}\right], \tag{3.18}$$

with C_3 being some arbitrary constant. Further, inserting ε from (3.18) into (2.24) one finds the expression for τ explicitly.

Taking into account the equation of state (2.9) in view of (3.16) and (3.18), the Eq. (2.24) admits the following solution in quadrature :

$$\int \frac{d\tau}{\sqrt{C_2^2 + C_0^0\tau^2 + C_1^1\tau^{1-\zeta}}} = t + t_0, \tag{3.19}$$

where C_2^2 and t_0 are some constants. Further we set $t_0 = 0$. Here, $C_0^0 = 3\kappa C_2/(1+\zeta) - 3\Lambda$ and $C_1^1 = 3\kappa C_3/(1+\zeta)$. As one sees, C_0^0 is negative for

$$\Lambda > \kappa C_2/(1+\zeta). \tag{3.20}$$

It means that for a positive Λ obeying (3.20) (we assume that the constant C_2 is a positive quantity) τ should be bound from above as well. It should be noted that for a suitable choice of C_2^2 and τ_0 (the initial value of τ), it is possible to obtain oscillatory mode of expansion with τ being always positive, i.e., a singularity free evolution of the Universe. The phase portrait of the (H, ε) plane and the evolution of the BI Universe corresponding to this portrait allowing oscillatory solutions are given in Figs. 10 and 11.

As a second example we consider the case, when $\zeta = 1$. From (3.19) one then finds

$$\tau(t) \;=\; \left(\exp(\sqrt{C_0^0}\,t) - C_2^2\exp(\sqrt{C_0^0}\,t)\right)/(2\sqrt{C_0^0}), \quad C_0^0 > 0, \tag{3.21a}$$

$$\tau(t) \;=\; (C_2^2/\sqrt{|C_0^0|})\sin(\sqrt{|C_0^0|}\,t). \quad C_0^0 < 0. \tag{3.21b}$$

Taking into account that $C_0^0 > 0$ for any non-positive Λ, from (3.21a) one sees that, in case of $\Lambda \leq 0$ the Universe may be infinitely large (there is no upper bound), which is in line with the conclusion made above. On the other hand, C_0^0 may be negative only for some positive value of Λ. Thus we see that a positive Λ can generate a oscillatory mode of expansion of a BI Universe. The oscillation takes place around the critical point $(H, \varepsilon) = (0, (2\Lambda - \kappa C_2)/[\kappa(1-\zeta)])$ having the type of cycle under the condition $\Lambda > \kappa C_2/(1+\zeta)$. It was shown in [20, 21] that in case of a perfect fluid a positive Λ always invokes oscillations in the model, whereas, in the present model with viscous fluid, it is the case only when Λ obeys (3.20). Unlike the case with radiation where BI admits a singularity-free oscillatory mode of evolution, here, in case of a stiff matter one finds the BI Universe first expands, reaches its maximum and then contracts into a point, thus giving rise to space-time singularity.

2. *Case with shear and bulk viscosity*

Let us now consider the general case with the shear viscosity η being proportional to the expansion, i.e.,

$$\eta \propto \theta = 3H. \tag{3.22}$$

We will consider the case when

$$\eta = -\frac{3}{2\kappa}H. \tag{3.23}$$

In this case from (3.1b) and (3.1c) one easily find

$$3H^2 = \kappa\varepsilon + C_4, \quad C_4 = \text{const}. \tag{3.24}$$

From (3.24) it follows that at the initial state of expansion, when ε is large, the Hubble constant is also large and with the expansion of the Universe H decreases as does ε. Inserting the relation (3.24) into the Eqs. (3.1b) one finds

$$\int \frac{dH}{\sqrt{AH^2 + BH + C}} = t, \tag{3.25}$$

where, $A = -1.5(1+\zeta)$, $B = 1.5\kappa\xi$, and $C = 0.5C_4(\zeta - 1) - \Lambda$. For ξ being a constant, (3.25) admits sinusoidal solution, i.e., H evolves oscillatory. Further, from (3.1a) one finds the expression for τ, which is exponential one accompanied by a sinusoidal mode [23].

IV. CONCLUSION

We investigated the cosmological solutions to the equations of General Relativity for the homogeneous anisotropic Bianchi type I model by taking into account dissipative processes due to viscosity and Cosmological constant (Λ term). A detailed analysis showed that the viscosity, as well as the Λ term exhibit essential influence on the character of the solutions. The classification of the solutions was pursued for the viscosity being some power law of energy density, namely, $\eta = A\varepsilon^{\alpha}$ and $\xi = B\varepsilon^{\beta}$. It was noticed that for $\Lambda < 0$ the Universe expands forever with a logarithmic velocity H, which, depending on the viscosity either becomes constant or increases infinitely. In the process behavior of the energy density ε is analogous to that of H except the case when $\varepsilon \to 0$. For $\Lambda > 0$, beside the variants mentioned above, there exists few other possibilities: contraction of the Universe into a point, thus giving rise to a space-time singularity; a regime of increasing oscillation corresponding to suitable initial conditions. It was also noticed that a special

case with $\Lambda > 0$, $\eta = 0$ and $\xi H =$ const. the model admits a singularity-free oscillatory mode of expansion.

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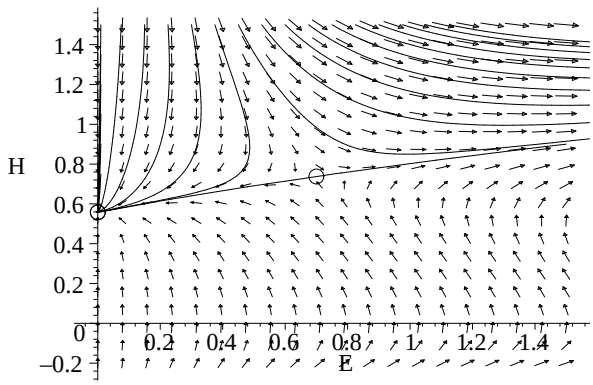

FIG. 1: Phase diagram on $H-\varepsilon$ plane for $\beta = 1.5,\ \Lambda = -.933,\ B = .720.$

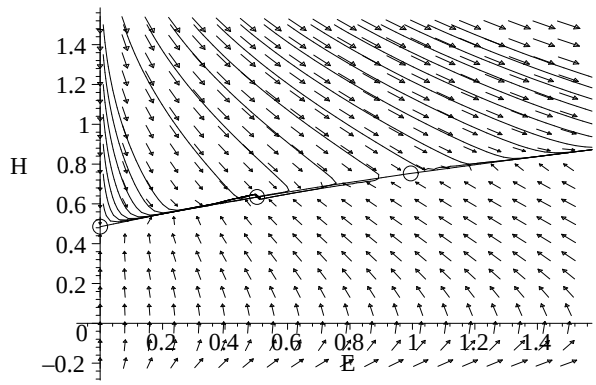

FIG. 2: Phase diagram on $H-\varepsilon$ plane for $\beta = .75,\ \Lambda = -.707,\ B = .589.$

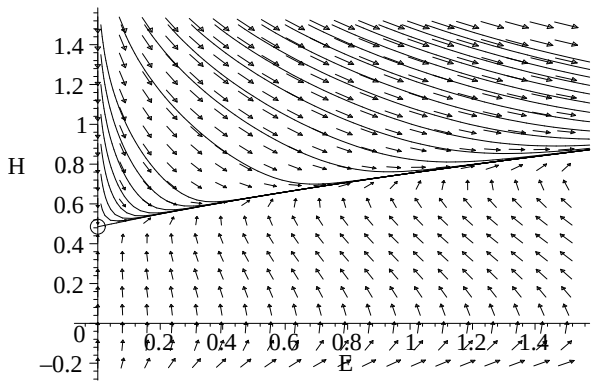

FIG. 3: Phase diagram on $H-\varepsilon$ plane for $\beta = .75,\ \Lambda = -.707,\ B = .667$

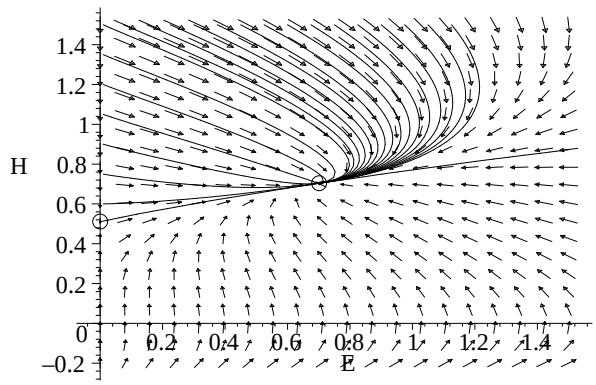

FIG. 4: Phase diagram on $H-\varepsilon$ plane for $\beta = .05,\ \Lambda = -.785,\ B = .451.$

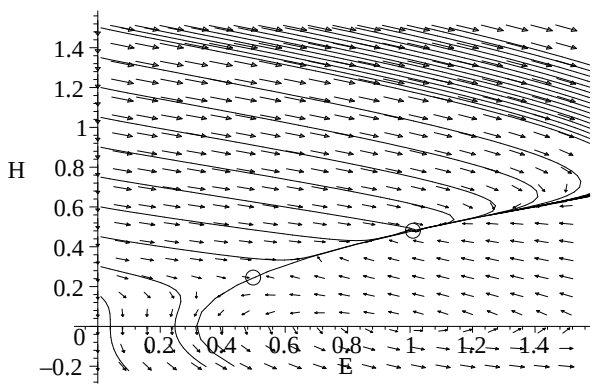

FIG. 5: Phase diagram on $H-\varepsilon$ plane for $\beta = .05,\ \Lambda = .317,\ B = 0.933$

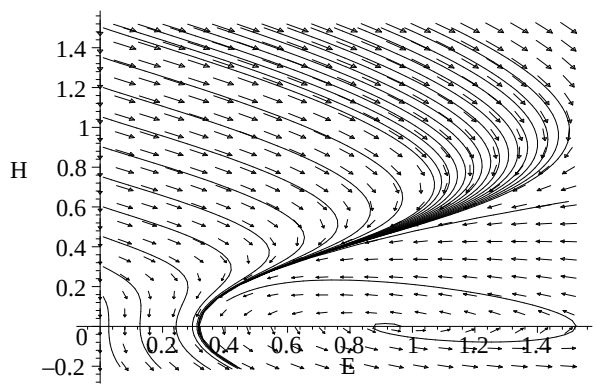

FIG. 6: Phase diagram on $H-\varepsilon$ plane for $\beta = .05,\ \Lambda = .317,\ B = .667.$

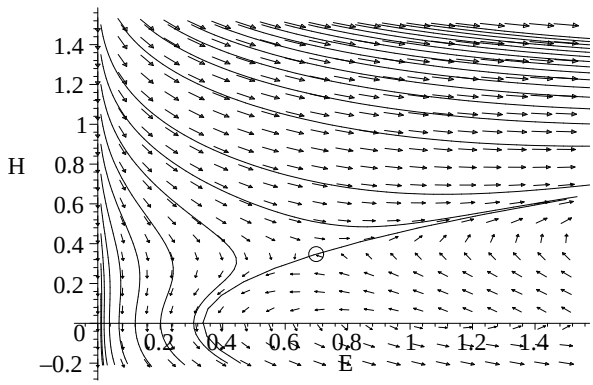

FIG. 7: Phase diagram on $H-\varepsilon$ plane for $\beta = .75,\ \Lambda = .337,\ B = 1.169$.

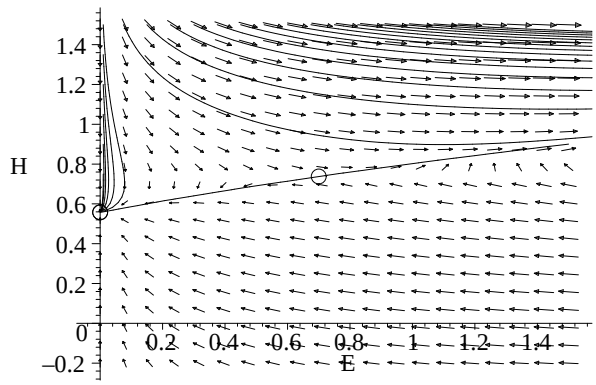

FIG. 8: Phase diagram on $H-\varepsilon$ plane for $\beta = 1.5,\ \Lambda = -.933,\ B = .720, A = 1,\ \alpha = 1$.

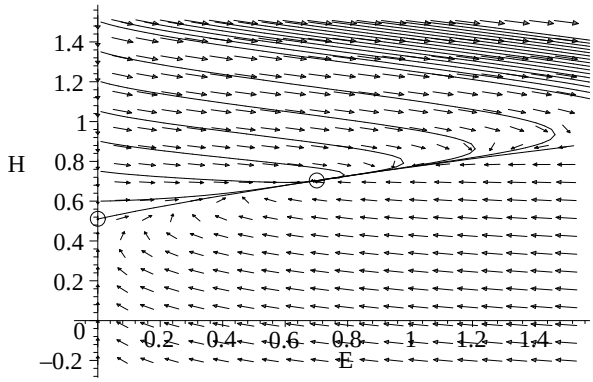

FIG. 9: Phase diagram on $H-\varepsilon$ plane for $\beta = .05,\ \Lambda = -.785,\ B = .451, A = 1,\ \alpha = 1,\ \kappa = 1$.

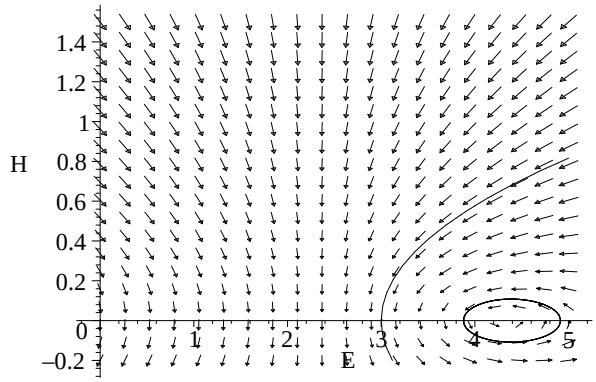

FIG. 10: Phase diagram on $H-\varepsilon$ plane for $\Lambda = 3,\ \zeta = 0.333,\ C_2 = 1,\ C_3 = 1$.

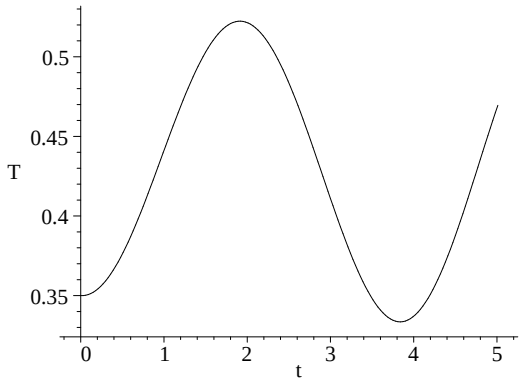

FIG. 11: Evolution of the BI Universe corresponding to the phase diagram given in Fig. 10. As one sees, the BI Universe in this case undergoes an oscillatory mode of expansion

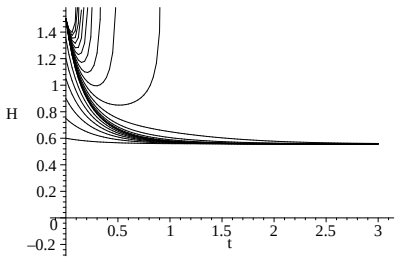

FIG. 12: Evolution of the Hubble constant H with parameters as in Fig. 1.

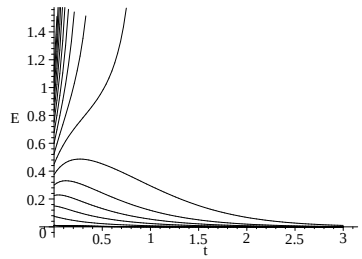

FIG. 13: Evolution of the energy density ε with parameters as in Fig. 1.

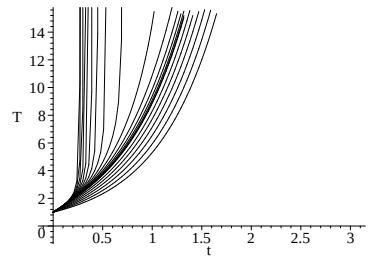

FIG. 14: Evolution of the volume scale τ with parameters as in Fig. 1.

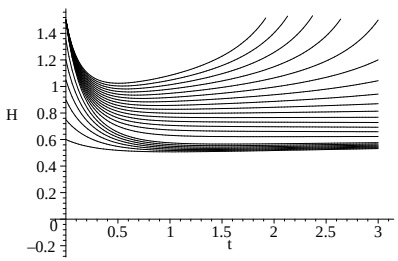

FIG. 15: Evolution of the Hubble constant H with parameters as in Fig. 2.

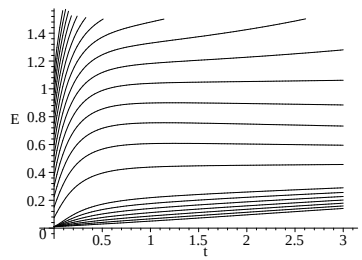

FIG. 16: Evolution of the energy density ε with parameters as in Fig. 2.

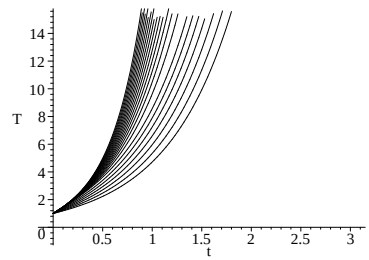

FIG. 17: Evolution of the volume scale τ with parameters as in Fig. 2.

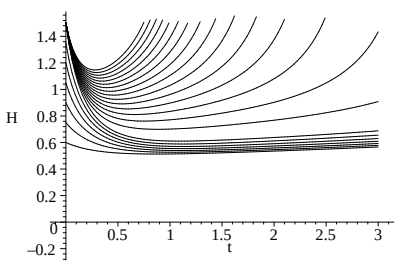

FIG. 18: Evolution of the Hubble constant H with parameters as in Fig. 3.

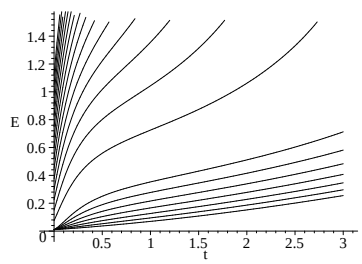

FIG. 19: Evolution of the energy density ε with parameters as in Fig. 3.

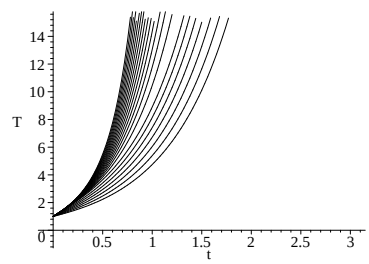

FIG. 20: Evolution of the volume scale τ with parameters as in Fig. 3.

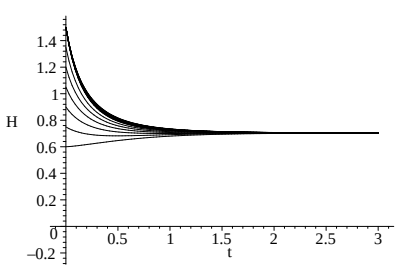

FIG. 21: Evolution of the Hubble constant H with parameters as in Fig. 4.

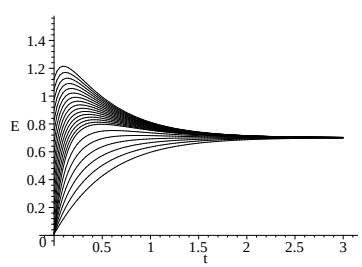

FIG. 22: Evolution of the energy density ε with parameters as in Fig. 4.

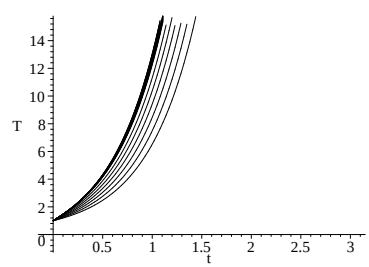

FIG. 23: Evolution of the volume scale τ with parameters as in Fig. 4.

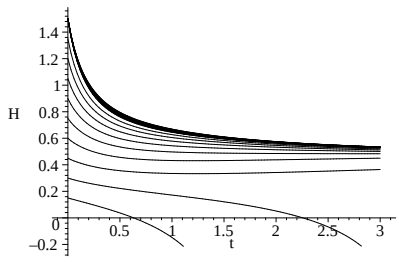

FIG. 24: Evolution of the Hubble constant H with parameters as in Fig. 5.

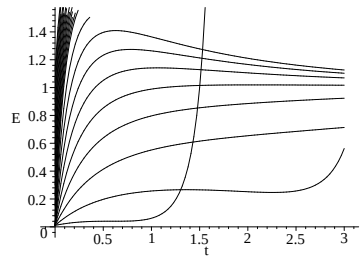

FIG. 25: Evolution of the energy density ε with parameters as in Fig. 5.

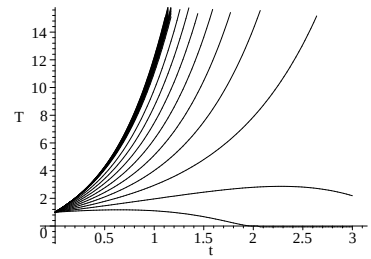

FIG. 26: Evolution of the volume scale τ with parameters as in Fig. 5.

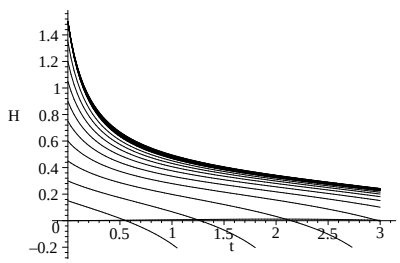

FIG. 27: Evolution of the Hubble constant H with parameters as in Fig. 6.

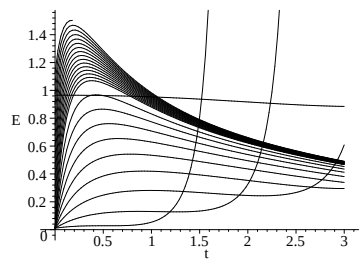

FIG. 28: Evolution of the energy density ε with parameters as in Fig. 6.

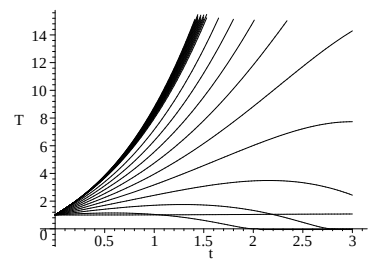

FIG. 29: Evolution of the volume scale τ with parameters as in Fig. 6.

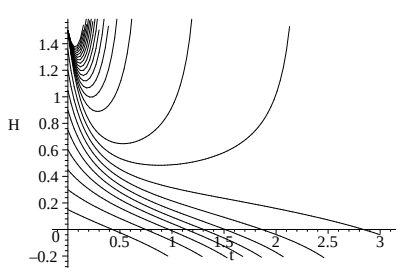

FIG. 30: Evolution of the Hubble constant H with parameters as in Fig. 7.

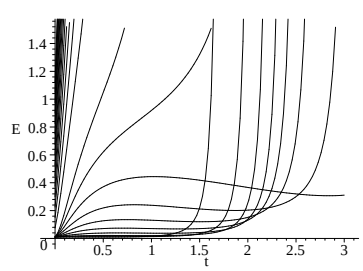

FIG. 31: Evolution of the energy density ε with parameters as in Fig. 7.

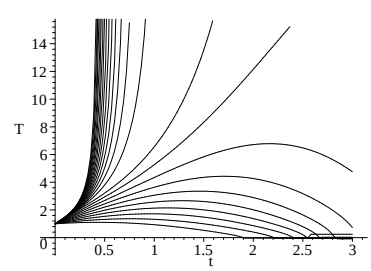

FIG. 32: Evolution of the volume scale τ with parameters as in Fig. 7.