

Bel-Robinson tensor and dominant energy property in the Bianchi type I Universe

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Within the framework of Bianchi type-I space-time we study the Bel-Robinson tensor and its impact on the evolution of the Universe. We use different definitions of the Bel-Robinson tensor existing in the literature and compare the results. Finally we investigate the so called "dominant super-energy property" for the Bel-Robinson tensor as a generalization of the usual dominant energy condition for energy momentum tensors.

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1. Introduction

The lack of a well-posed definition of *local energy-momentum tensor* is the consequence of the Principle of Equivalence¹, which lies at the heart of Einstein's theory of general relativity. Nevertheless, quest for the local tensors describing the strength of gravitational field has long been going on. One of the first successful attempt to address this problem was taken by Bel^{2,3,4} and independently Robinson⁵. In their works, in analogy with the electromagnetic energy-momentum tensor, they constructed a four-index tensor for the gravitational field in vacuum. The properties of the now famous Bel-Robinson (BR) tensor are similar to the traditional energy-momentum tensor and following Senovilla^{6,7} can be formulated as follows: (i) it possesses a positive-definite time-like component and a "causal" momentum vector; (ii) its divergence vanishes (in vacuum); (iii) the tensor is zero if and only if the curvature of the space-time vanishes; (iv) it has positivity property similar to the electromagnetic one; and some others. Construction of BR and the study of its properties were widely considered by a number of authors, e.g., Desher *et. al.*^{8,9}, Teyssandier¹⁰, Senovilla⁶, Bergqvist¹¹, Andersson¹², Wingbrant¹³,

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Choquet-Bruhat et al.¹⁴ etc.

It should be noted the authors of the papers mentioned above considered the BR and established its properties in general. On the other hand, in general relativity there exists a number of interesting and widely studied models of space-time. Therefore, in our view it is interesting to consider the BR within the scope of some concrete metric. In a recent paper¹⁵ we studied the BR within the framework of Bianchi type I (BI) universe using two different definitions. The purpose of this paper is to extend that study for some other definitions and analyze the dominant energy property (DEP) and dominant super-energy property (DSEP) within this model.

2. Bianchi I Universe: a brief description

A Bianchi type-I (BI) universe is the straightforward generalization of the flat Robertson-Walker (RW) universe and is one of the simplest models of an anisotropic universe that describes a homogeneous and spatially flat universe. It has the agreeable property that near the singularity it behaves like a Kasner universe, even in the presence of matter, and consequently falls within the general analysis of the singularity given by Belinskii et al.¹⁶ Also in a universe filled with matter for $p = \zeta \varepsilon$, $\zeta < 1$, it has been shown that any initial anisotropy in a BI universe quickly dies away and a BI universe eventually evolves into a Friedmann-RW (FRW) universe¹⁷. Since the present-day universe is surprisingly isotropic, this feature of the BI universe makes it a prime candidate for studying the possible effects of an anisotropy in the early universe on present-day observations. In light of the importance mentioned above, several authors have studied BI universe from different aspects.

A diagonal BI space-time is a spatially homogeneous space-time, which admits an Abelian group G_3 , acting on spacelike hypersurfaces, generated by the spacelike Killing vectors $\mathbf{x}_1 = \partial_1$, $\mathbf{x}_2 = \partial_2$, and $\mathbf{x}_3 = \partial_3$. In synchronous coordinates, the metric is^{18,19}:

$$ds^2 = dt^2 - \sum_{i=1}^3 a_i^2(t) dx_i^2. \quad (2.1)$$

If the three scale factors are equal (i.e., $a_1 = a_2 = a_3$), Eq. (2.1) describes an isotropic and spatially flat FRW universe. The BI universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. Thus, a BI universe, being the straightforward generalization of the flat FRW universe, is one of the simplest models of an anisotropic universe that describes a homogeneous and spatially flat universe. When two of the metric functions are equal (e.g., $a_2 = a_3$) the BI space-time is reduced to the important class of plane symmetric space-time (a special class of the locally rotational symmetric space-times^{20,21}), which admits a G_4 group of isometries acting multiply transitively on the spacelike hypersurfaces of homogeneity generated by the Killing vectors \mathbf{x}_1 , \mathbf{x}_2 , \mathbf{x}_3 , and $\mathbf{x}_4 = x^2 \partial_3 - x^3 \partial_2$.

The BI has the agreeable property that near the singularity it behaves like a Kasner universe, given by

$$a_1(t) = a_1^0 t^{p_1}, \quad a_2(t) = a_2^0 t^{p_2}, \quad a_3(t) = a_3^0 t^{p_3},$$

with p_j being the parameters of the BI space-time which measure the relative anisotropy between any two asymmetry axes and satisfy the constraints

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1. \quad (2.2)$$

As one sees, p_1 , p_2 and p_3 cannot be equal. Only two of them can be equal, and only in two special cases, namely, $(0, 0, 1)$ and $(-1/3, 2/3, 2/3)$. In all other cases p_1 , p_2 and p_3 are different, moreover, one of them is negative, while the two others are positive. If it is supposed that $p_1 < p_2 < p_3$, then their values are confined in the following intervals:

$$-1/3 \leq p_1 \leq 0, \quad 0 \leq p_2 \leq 2/3, \quad 2/3 \leq p_3 \leq 1.$$

The solutions of the algebraic equations (2.2) can be presented as

$$p_1 = \frac{-p}{p^2 + p + 1}, \quad p_2 = \frac{p(p+1)}{p^2 + p + 1}, \quad p_3 = \frac{p+1}{p^2 + p + 1}.$$

Thus instead of three, we have now one parameter p , which lies in the interval $0 \leq p \leq 1$.

Another particular parametrization can be given using an angle on the unit circle, since Eqs. (2.2) describe the intersection of a sphere with a plane in the parameter space (p_1, p_2, p_3) :

$$p_1 = \frac{1 + \cos\vartheta + \sqrt{3}\sin\vartheta}{3}, \quad p_2 = \frac{1 + \cos\vartheta - \sqrt{3}\sin\vartheta}{3}, \quad p_3 = \frac{1 - 2\cos\vartheta}{3}.$$

Although ϑ ranges over the unit circle, the labeling of each p_j is quite arbitrary. Thus the unit circle can be divided into six equal parts, each of which span 60° , and the choice of p_j is unique within each section separately. For $\vartheta = 0$, $p_1 = p_2 = \frac{2}{3}$ and $p_3 = -\frac{1}{3}$ while for $\vartheta = \pi/3$, $p_1 = 1$ and $p_2 = p_3 = 0$.

For later convenience we list the Christoffel symbol, scalar curvature, Ricci, Riemann and Weyl tensors for the BI space-time. The non-trivial Christoffel symbols are the following:

$$\Gamma_{i0}^i = \frac{\dot{a}_i}{a_i}, \quad \Gamma_{ii}^0 = a_i \dot{a}_i, \quad i = 1, 2, 3,$$

and the non-trivial components of Riemann tensors are

$$R^{0i}{}_{0i} = -\frac{\ddot{a}_i}{a_i}, \quad R^{ij}{}_{ij} = -\frac{\dot{a}_i \dot{a}_j}{a_i a_j}, \quad (2.3)$$

$$i, j = 1, 2, 3, \quad i \neq j.$$

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Finally the nontrivial components of the Ricci tensors for the BI metric are

$$R_0^0 = -\sum_{i=1}^3 \frac{\ddot{a}_i}{a_i}, \quad R_i^i = -\left[\frac{\ddot{a}_i}{a_i} + \frac{\dot{a}_i}{a_i} \left(\frac{\dot{a}_j}{a_j} + \frac{\dot{a}_k}{a_k} \right) \right],$$

$$i, j, k = 1, 2, 3, \quad i \neq j \neq k.$$

and the scalar curvature is

$$R = -2 \left(\sum_{i=1}^3 \frac{\ddot{a}_i}{a_i} + \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} + \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} + \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} \right).$$

It is convenient to separate the Riemann tensor into a trace-free part and a "Ricci" part. This gives the Weyl tensor

$$C_{ijkl} = R_{ijkl} - \frac{1}{(n-2)} \left(g_{ik} R_{jl} + g_{jl} R_{ik} - g_{jk} R_{il} - g_{il} R_{jk} \right)$$

$$+ \frac{1}{(n-1)(n-2)} \left(g_{ik} g_{jl} - g_{il} g_{jk} \right) R.$$

This tensor has manifestly all the symmetries of the Riemann tensor; but contrary to the Riemann tensor while it gives rise to Ricci tensor, the Weyl tensor gives

$$g^{ik} C_{ijkl} \equiv 0. \quad (2.4)$$

A further distinction is that while the Riemann tensor can be defined in a manifold endowed only with a connection, the Weyl tensor can be defined only when a metric is also defined. In 4 dimensions the Riemann tensor has 20 distinct components, while the Weyl and the Ricci have 10 components each. The non-trivial components of the Weyl tensor for the BI space-time are

$$C_{0i0i} = \frac{a_i}{6a_j a_k} \{ 2\ddot{a}_i a_j a_k - \ddot{a}_j a_k a_i - \ddot{a}_k a_i a_j - \dot{a}_i \dot{a}_j a_k - \dot{a}_k \dot{a}_i a_j + 2\dot{a}_j \dot{a}_k a_i \}$$

$$C_{j k j k} = -\frac{a_j a_k}{6a_i} \{ 2\ddot{a}_i a_j a_k - \ddot{a}_j a_k a_i - \ddot{a}_k a_i a_j - \dot{a}_i \dot{a}_j a_k - \dot{a}_k \dot{a}_i a_j + 2\dot{a}_j \dot{a}_k a_i \},$$

$$i, j, k = 1, 2, 3, \quad i \neq j \neq k. \quad (2.5)$$

From (2.5) one easily finds the following relation:

$$C_{0i0i} = -\frac{a_i^2}{a_j^2 a_k^2} C_{j k j k}, \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k. \quad (2.6)$$

Now having all the non-trivial components of Ricci and Riemann tensors, one can easily write the invariants of gravitational field which we need to study the space-time singularity. Moreover now we can construct the BR tensor that is defined differently by different authors.

3. Einstein equations and their solutions

In this section we study the Einstein equation. In doing so let us first write the Einstein equation for the BI metric governing the evolution of the Universe. In presence of a cosmological constant Λ the Einstein equation has the form

$$\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} = \kappa T_1^1 + \Lambda, \quad (3.1a)$$

$$\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} = \kappa T_2^2 + \Lambda, \quad (3.1b)$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} = \kappa T_3^3 + \Lambda, \quad (3.1c)$$

$$\frac{\dot{a}_1 \dot{a}_2}{a_1 a_2} + \frac{\dot{a}_2 \dot{a}_3}{a_2 a_3} + \frac{\dot{a}_3 \dot{a}_1}{a_3 a_1} = \kappa T_0^0 + \Lambda. \quad (3.1d)$$

Here over-dot means differentiation with respect to t and T_μ^ν is the energy-momentum tensor of the matter field which we choose in the form:

$$T_\mu^\nu = (\varepsilon + p)u_\mu u^\nu - p\delta_\mu^\nu, \quad (3.2)$$

where u^μ is the flow vector satisfying

$$g_{\mu\nu}u^\mu u^\nu = 1.$$

Here ε is the total energy density of a perfect fluid and/or dark energy density, while p is the corresponding pressure. p and ε are related by an equation of state which will be studied below in detail. In a co-moving system of coordinates from (3.2) one finds

$$T_0^0 = \varepsilon, \quad T_1^1 = T_2^2 = T_3^3 = -p. \quad (3.3)$$

In view of (3.3) from (3.1a) - (3.1d) one immediately obtains ²²

$$a_i(t) = D_i[\tau(t)]^{1/3} \exp[X_i \int [\tau(t')]^{-1} dt'], \quad i = 1, 2, 3.$$

Here D_i and X_i are some arbitrary constants obeying

$$D_1 D_2 D_3 = 1, \quad X_1 + X_2 + X_3 = 0,$$

and τ is a function of t defined to be

$$\tau = a_1 a_2 a_3.$$

From (3.1a) - (3.1d) for τ one find

$$\frac{\ddot{\tau}}{\tau} = \frac{3\kappa}{2}(\varepsilon - p) + 3\Lambda. \quad (3.4)$$

On the other hand the conservation law for the energy-momentum tensor gives

$$\dot{\varepsilon} = -\frac{\dot{\tau}}{\tau}(\varepsilon + p). \quad (3.5)$$

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After a little manipulations from (3.4) and (3.5) we find

$$\dot{\tau}^2 = 3(\kappa\varepsilon + \Lambda)\tau^2 + C_1, \quad (3.6)$$

with C_1 being an arbitrary constant. Let us now, in analogy with Hubble constant, define

$$\frac{\dot{\tau}}{\tau} = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = 3H. \quad (3.7)$$

On account of (3.7) from (3.6) one derives

$$\kappa\varepsilon = 3H^2 - \Lambda - C_1/(3\tau^2). \quad (3.8)$$

It should be noted that the energy density of the Universe is a positive quantity. It is believed that at the early stage of evolution when the volume scale τ was close to zero, the energy density of the Universe was infinitely large. On the other hand with the expansion of the Universe, i.e., with the increase of τ , the energy density ε decreases and an infinitely large τ corresponds to a ε close to zero. Say at some stage of evolution ε is too small to be ignored. In that case from (3.8) follows

$$3H^2 - \Lambda \rightarrow 0. \quad (3.9)$$

As it is seen from (3.9) in this case Λ is essentially non-negative. We can also conclude from (3.9) that in absence of a Λ term beginning from some value of τ the evolution of the Universe comes stand-still, i.e., τ becomes constant, since H becomes trivial, whereas in case of a positive Λ the process of evolution of the Universe never comes to a halt. Moreover it is believed that the presence of the dark energy (which can be explained with a positive Λ as well) results in the accelerated expansion of the Universe. As far as negative Λ is concerned, its presence imposes some restriction on ε , namely, ε can never be small enough to be ignored. In case of the perfect fluid given by $p = \zeta\varepsilon$ there exists some upper limit for τ as well (note that τ is essentially nonnegative, i.e. bound from below). In our previous papers we came to the same conclusion^{22,23} (with a positive Λ which in the present paper appears to be negative). A suitable choice of parameters in this case may give rise to an oscillatory mode of expansion, whereas in case of a Van der Waals fluid the highly nonlinear equation of state may result in an exponential expansion as well.

Inserting (3.7) and (3.8) into (3.4) one now finds

$$\dot{H} = -\frac{1}{2}(3H^2 - \Lambda + \frac{C_1}{3\tau^2} + \kappa p) = -\frac{\kappa}{2}(\varepsilon + p) - \frac{C_1}{3\tau^2}. \quad (3.10)$$

In view of (3.8), from (3.10), it follows that if the perfect fluid is given by a stiff matter where $p = \varepsilon$, the corresponding solution does not depend on the constant C_1 .

Let us now go back to the Eq. (3.6). It is in fact the first integral of (3.4) and can be written as

$$\dot{\tau} = \pm\sqrt{C_1 + 3(\kappa\varepsilon + \Lambda)\tau^2} \quad (3.11)$$

On the other hand, rewriting (3.5) in the form

$$\frac{\dot{\varepsilon}}{\varepsilon + p} = \frac{\dot{\tau}}{\tau},$$

and taking into account that p is a function of ε , one concludes that the right hand side of the Eq. (3.4) is a function of τ only, i.e.,

$$\ddot{\tau} = \frac{3\kappa}{2}(\varepsilon - p)\tau + 3\Lambda\tau = \mathcal{F}(\tau). \quad (3.12)$$

From a mechanical point of view Eq. (3.12) can be interpreted as an equation of motion of a single particle with unit mass under the force $\mathcal{F}(\tau)$. Then the following first integral exists ²³:

$$\dot{\tau} = \sqrt{2[\mathcal{E} - \mathcal{U}(\tau)]}. \quad (3.13)$$

Here \mathcal{E} can be viewed as energy and $\mathcal{U}(\tau)$ is the potential of the force \mathcal{F} . Comparing the Eqs. (3.11) and (3.13) one finds $\mathcal{E} = C_1/2$ and

$$\mathcal{U}(\tau) = -\frac{3}{2}(\kappa\varepsilon + \Lambda)\tau^2. \quad (3.14)$$

Let us finally write the solution to the Eq. (3.4) in quadrature:

$$\int \frac{d\tau}{\sqrt{C_1 + 3(\kappa\varepsilon + \Lambda)\tau^2}} = t + t_0, \quad (3.15)$$

where the integration constant t_0 can be taken to be zero, since it only gives a shift in time. The Eqs. (3.4) and (3.5) for perfect fluid obeying different equations of state has been thoroughly studied by us ^{22,23}.

4. Bel-Robinson tensors

BR tensor first appeared in the endless search for a covariant version of gravitational energy. In general relativity, the energetic content of an electromagnetic field propagating in a region free of charge is described by the well-known symmetric trace-less tensor

$$T_{\text{el}}^{\alpha\beta} = -\frac{1}{4\pi}(F^{\alpha\lambda}F_{\lambda}^{\beta} - \frac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}), \quad (4.1)$$

where $F^{\alpha\beta}$ is the electromagnetic field tensor. This tensor satisfies:

$$T_{\text{el};\alpha}^{\alpha\beta} = 0$$

as a consequence of Maxwell equations with $j^{\mu} = 0$. The tensor $T_{\text{el}}^{\alpha\beta}$ enables us to define a local density of electromagnetic energy as measured by an observer moving with the unit 4-velocity u :

$$w_{\text{el}}(u) = T_{\text{el}}^{\alpha\beta}u_{\alpha}u_{\beta}.$$

It follows from (4.1) that the energy density is positive definite for any time-like vector u .

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Within the scope of general relativity, however, it is well known that the concept of local energy density is meaningless for a gravitational field. To overcome this difficulty led to introduce the notion of super-energy tensor constructed with the curvature tensor $R_{\mu\nu\alpha\beta}$. The first example of such a tensor was exhibited by Bel², that was further generalized to the case of an arbitrary gravitational field³. Note that a similar tensor was also introduced by Robinson⁵. This tensor is now commonly known as the BR tensor as well. Since we are going to compare some distinct definition of BR in this paper, before defining them let us see what kind of properties they should have. In general, the BR tensor has the following symmetry properties:

$$B_{\mu\nu\alpha\beta} = B_{\nu\mu\alpha\beta}, \quad (4.2a)$$

$$B_{\mu\nu\alpha\beta} = B_{\mu\nu\beta\alpha}, \quad (4.2b)$$

$$B_{\mu\nu\alpha\beta} = B_{\alpha\beta\mu\nu}. \quad (4.2c)$$

The symmetry property leads to the fact that in n -dimensional case there are $n(n+1)[n(n+1)+2]/8$ independent components of the BR tensor. In case of $n = 4$ out of 256 components only 55 are linearly independent.

In literature there are a few definitions of BR. Here we mention only three.

I. By analogy with the tensor (4.1) which may be written as

$$T_{\mu\nu} = F_{\mu\alpha}F_{\nu}^{\alpha} + *F_{\mu\alpha} * F_{\nu}^{\alpha},$$

the BR tensor is defined as⁸:

$$B_{\mu\nu\alpha\beta} = R_{\mu}^{\rho}{}_{\alpha}{}^{\sigma} R_{\rho\nu\sigma\beta} + *R_{\mu}^{\rho}{}_{\alpha}{}^{\sigma} * R_{\rho\nu\sigma\beta}. \quad (4.3)$$

Here the dual curvature is $*R^{\mu\nu}{}_{\lambda\sigma} \equiv (1/2)\epsilon^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\sigma}$. It should be noted that this definition is adequate only in 4 dimensions and in vacuum. Otherwise this tensor cannot satisfy the DEP²⁴ and therefore this expression should not be used in other dimensions or in non-Ricci-flat spacetimes.

Using the definition of dual curvature, from (4.3) we find

$$B_{\mu\nu\alpha\beta} = R_{\mu}^{\rho}{}_{\alpha}{}^{\sigma} R_{\rho\nu\sigma\beta} + R_{\mu}^{\rho}{}_{\beta}{}^{\sigma} R_{\rho\nu\sigma\alpha} - \frac{1}{2}g_{\mu\nu}R_{\alpha}{}^{\rho\sigma\tau}R_{\beta\rho\sigma\tau}. \quad (4.4)$$

The properties (4.2a) and (4.2b) follow immediately from (4.3) thanks to the symmetry property of Riemann tensor. The property (4.2c) is straightforward from (4.3), but for (4.4) it requires

$$g_{\mu\nu}R_{\alpha}{}^{\rho\sigma\tau}R_{\beta\rho\sigma\tau} = g_{\alpha\beta}R_{\mu}{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau}. \quad (4.5)$$

In view of (2.3) for the BR tensor in this case we obtain the following non-trivial

components:

$$\begin{aligned}
 B_{0000} &= \sum_{i=1}^3 \frac{\ddot{a}_i^2}{a_i^2}, \\
 B_{iiii} &= \ddot{a}_i^2 + \dot{a}_i^2 \left\{ \frac{\dot{a}_j^2}{a_j^2} + \frac{\dot{a}_k^2}{a_k^2} \right\}, \\
 B_{0i0i} &= a_i \ddot{a}_i \left\{ \frac{\ddot{a}_j}{a_j} + \frac{\ddot{a}_k}{a_k} \right\}, \\
 B_{ijij} &= \ddot{a}_i \ddot{a}_j + a_i \dot{a}_i a_j \dot{a}_j \frac{\dot{a}_k^2}{a_k^2}, \\
 B_{00ii} &= \ddot{a}_i^2 - \dot{a}_i^2 \left\{ \frac{\dot{a}_j^2}{a_j^2} + \frac{\dot{a}_k^2}{a_k^2} \right\}, \\
 B_{iijj} &= \dot{a}_i^2 \dot{a}_j^2 - a_i^2 \ddot{a}_j^2 - a_i^2 \dot{a}_j^2 \frac{\dot{a}_k^2}{a_k^2}, \\
 & \quad i, j, k = 1, 2, 3 \quad i \neq j \neq k.
 \end{aligned}$$

Inserting (2.3) into (4.5) we obtain following additional relations:

$$\left(\frac{\ddot{a}_i}{a_i} \right)^2 \pm \left(\frac{\ddot{a}_j}{a_j} \right)^2 = \left(\frac{\dot{a}_k}{a_k} \right)^2 \left[\left(\frac{\dot{a}_i}{a_i} \right)^2 \pm \left(\frac{\dot{a}_j}{a_j} \right)^2 \right], \quad i, j, k = 1, 2, 3 \quad i \neq j \neq k \quad (4.6)$$

Among the six constrains in (4.6) only three are linearly independent. After a little manipulations with them finally obtains the following relations between the metric functions:

$$\frac{\ddot{a}_i}{a_i} = \frac{\dot{a}_j \dot{a}_k}{a_j a_k}, \quad i, j, k = 1, 2, 3 \quad i \neq j \neq k. \quad (4.7)$$

As one sees, in account of (4.7) the Einstein equation (3.1a) - (3.1d) leads to $T_0^0 = T_1^1 = T_2^2 = T_3^3$, which can be realized only when the source field satisfies the following equation of state:

$$p = -\varepsilon. \quad (4.8)$$

It is well known that only vacuum satisfies the state of equation given by (4.8). Thus we see that if we are to define BR tensor given by (4.3) or (4.4) we should deal with the Einstein equations with the source field given by a vacuum.

II. The restriction that arises above is due to the fact that in defining the BR tensor we used the dual term with the duality operator acting on the left pair only. To avoid this restrictions the BR tensor can be defined by ^{10,25}

$$\begin{aligned}
 2B_{\mu\nu\alpha\beta} &= R^\rho{}_\mu{}^\sigma{}_\alpha R_{\rho\nu\sigma\beta} + *R^\rho{}_\mu{}^\sigma{}_\alpha *R_{\rho\nu\sigma\beta} \\
 &+ R *^\rho{}_\mu{}^\sigma{}_\alpha R *_{\rho\nu\sigma\beta} + *R *^\rho{}_\mu{}^\sigma{}_\alpha *R *_{\rho\nu\sigma\beta}, \quad (4.9)
 \end{aligned}$$

where the duality operator acts on the left or on the right pair of indices according to its position. Nowadays this is known as the Bel tensor and was introduced by Bel ³ in a slightly different form.

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From (4.9) one easily finds

$$\begin{aligned} B_{\mu\nu\alpha\beta} &= R^\rho{}_\mu{}^\sigma{}_\alpha R_{\rho\nu\sigma\beta} + R^\rho{}_\mu{}^\sigma{}_\beta R_{\rho\nu\sigma\alpha} - \frac{1}{2}g_{\mu\nu}R_\alpha{}^{\rho\sigma\tau}R_{\beta\rho\sigma\tau} \\ &\quad - \frac{1}{2}g_{\alpha\beta}R_\mu{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau} + \frac{1}{8}g_{\mu\nu}g_{\alpha\beta}R^{\rho\sigma\tau\eta}R_{\rho\sigma\tau\eta}. \end{aligned}$$

Under the new definition the symmetry properties (4.2a), (4.2b) and (4.2c) follow immediately, without any restriction to the metric functions.

Let us now write the non-trivial components of the BR tensor for the BI metric. In view of (2.3) we now find

$$\begin{aligned} B_{0000} &= \frac{1}{2} \left\{ \sum_{i=1}^3 \frac{\ddot{a}_i^2}{a_i^2} + \frac{\dot{a}_1^2 \dot{a}_2^2}{a_1^2 a_2^2} + \frac{\dot{a}_2^2 \dot{a}_3^2}{a_2^2 a_3^2} + \frac{\dot{a}_3^2 \dot{a}_1^2}{a_3^2 a_1^2} \right\}, \\ B_{iiii} &= a_i^4 B_{0000}, \\ B_{0i0i} &= -a_i \dot{a}_i \left\{ \frac{\dot{a}_j \ddot{a}_j}{a_j} + \frac{\dot{a}_k \ddot{a}_k}{a_k} \right\}, \\ B_{ijij} &= a_i a_j \left\{ \ddot{a}_i \ddot{a}_j + \dot{a}_i \dot{a}_j \frac{\dot{a}_k^2}{a_k^2} \right\}, \\ B_{00ii} &= \frac{1}{2} \left\{ -\ddot{a}_i^2 + \dot{a}_i^2 \left(\frac{\dot{a}_j^2}{a_j^2} + \frac{\dot{a}_k^2}{a_k^2} \right) + a_i^2 \left(\frac{\ddot{a}_j^2}{a_j^2} + \frac{\ddot{a}_k^2}{a_k^2} - \frac{\dot{a}_j^2 \dot{a}_k^2}{a_j^2 a_k^2} \right) \right\}, \\ B_{iijj} &= \frac{1}{2} \left\{ \dot{a}_i^2 \dot{a}_j^2 - \ddot{a}_j^2 a_j^2 - a_i^2 \ddot{a}_j^2 - \frac{\dot{a}_k^2}{a_k^2} (\dot{a}_i^2 a_j^2 + a_i^2 \dot{a}_j^2) + a_i^2 a_j^2 \frac{\dot{a}_k^2}{a_k^2} \right\}, \\ &\quad i, j, k = 1, 2, 3 \quad i \neq j \neq k. \end{aligned}$$

But the BR tensor defined in this way is not trace-free and is not completely symmetric. It is achieved if and only if the manifold is Ricci flat, i.e., $R_{ij} = 0$. Since for the BI universe we have non-trivial components of Ricci tensor, we give an alternative definition of BR where it is totally symmetric and trace-free.

III. Here we give another definition that gives rise to BR tensor, that is traceless and totally symmetric. It can be achieved by constructing BR by means of Weyl tensor^{11,26}.

$$B_{\mu\nu\alpha\beta} = C^\rho{}_\mu{}^\sigma{}_\alpha C_{\rho\nu\sigma\beta} + *C^\rho{}_\mu{}^\sigma{}_\alpha *C_{\rho\nu\sigma\beta}. \quad (4.10)$$

It can be shown that this BR is totally symmetric, i.e.,

$$B_{ijkl} = B_{(ijkl)},$$

Moreover, as one can easily find from (2.4), the BR defined through Weyl tensor is trace-free, i.e.,

$$g^{jl} B_{ijkl} \equiv 0. \quad (4.11)$$

Let us study this case in detail. Using the properties of Levi-Civita tensor we first rewrite (4.10) in the form

$$B_{\mu\nu\alpha\beta} = C^\rho{}_\mu{}^\sigma{}_\alpha C_{\rho\nu\sigma\beta} + C^\rho{}_\mu{}^\sigma{}_\beta C_{\rho\nu\sigma\alpha} - \frac{1}{2}g_{\mu\nu}C_\alpha{}^{\rho\sigma\tau}C_{\beta\rho\sigma\tau}.$$

After a little manipulation one easily finds the following nontrivial components of the BR tensor.

$$B_{0000} = \left(g^{11}C_{1010}\right)^2 + \left(g^{22}C_{2020}\right)^2 + \left(g^{33}C_{3030}\right)^2,$$

which on account of (2.6) gives

$$B_{0000} = \left(\frac{1}{a_2^2 a_3^2} C_{2323}\right)^2 + \left(\frac{1}{a_3^2 a_1^2} C_{3131}\right)^2 + \left(\frac{1}{a_1^2 a_2^2} C_{1212}\right)^2. \quad (4.12)$$

On the other hand for we find

$$B_{1111} = \left(C_{1010}\right)^2 + \left(g^{22}C_{1212}\right)^2 + \left(g^{33}C_{1313}\right)^2,$$

which in view of (2.6) can be rewritten as

$$\begin{aligned} B_{1111} &= \left(\frac{a_1^2}{a_2^2 a_3^2} C_{2323}\right)^2 + \left(\frac{1}{a_3^2} C_{3131}\right)^2 + \left(\frac{1}{a_2^2} C_{1212}\right)^2 \\ &= a_1^4 \left[\left(\frac{1}{a_2^2 a_3^2} C_{2323}\right)^2 + \left(\frac{1}{a_3^2 a_1^2} C_{3131}\right)^2 + \left(\frac{1}{a_1^2 a_2^2} C_{1212}\right)^2 \right] = a_1^4 B_{0000}. \end{aligned} \quad (4.13)$$

In view of (4.12) and (4.13) symbolically we can write

$$B_{0000} = \left(\frac{1}{a_j^2 a_k^2} C_{jkjk}\right)^2 + \left(\frac{1}{a_k^2 a_i^2} C_{kiki}\right)^2 + \left(\frac{1}{a_i^2 a_j^2} C_{ijij}\right)^2 \quad (4.14a)$$

$$\begin{aligned} &= \frac{1}{6} \left\{ \sum_{i=1}^3 \frac{\ddot{a}_i^2}{a_i^2} + \frac{\dot{a}_1^2 \dot{a}_2^2}{a_1^2 a_2^2} + \frac{\dot{a}_2^2 \dot{a}_3^2}{a_2^2 a_3^2} + \frac{\dot{a}_3^2 \dot{a}_1^2}{a_3^2 a_1^2} \right. \\ &\quad - \left(\frac{\ddot{a}_1 \ddot{a}_2}{a_1 a_2} + \frac{\ddot{a}_2 \ddot{a}_3}{a_2 a_3} + \frac{\ddot{a}_3 \ddot{a}_1}{a_3 a_1} \right) - \frac{\dot{a}_1 \ddot{a}_2}{a_1 a_2} \left(\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} - 2 \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_3^2}{a_3^2} \right) \\ &\quad \left. - \frac{\dot{a}_2 \ddot{a}_3}{a_2 a_3} \left(\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} - 2 \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_1^2}{a_1^2} \right) - \frac{\dot{a}_3 \ddot{a}_1}{a_3 a_1} \left(\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} - 2 \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2} \right) \right\}, \end{aligned}$$

$$B_{iiii} = a_i^4 B_{0000}, \quad (4.14b)$$

$$B_{ijij} = 2C_{0i0i}C_{0j0j} \quad (4.14c)$$

$$\begin{aligned} &= \frac{1}{18a_k^2} \left(2\dot{a}_i \dot{a}_j \dot{a}_k - a_i \dot{a}_j \dot{a}_k - \dot{a}_i \dot{a}_j \dot{a}_k - a_i a_j \ddot{a}_k - \ddot{a}_i a_j a_k + 2a_i \ddot{a}_j a_k \right) \\ &\quad \times \left(2a_i \dot{a}_j \dot{a}_k - \dot{a}_i a_j \dot{a}_k - \dot{a}_i \dot{a}_j a_k - a_i a_j \ddot{a}_k - a_i \ddot{a}_j a_k + 2\ddot{a}_i a_j a_k \right), \end{aligned}$$

$$B_{0k0k} = -\frac{a_k^2}{a_i^2 a_j^2} B_{ijij}. \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k. \quad (4.14d)$$

Thus we have used three different definition of BR. The first one defined in ⁸ imposes some restriction on the metric functions, namely for the BI spacetime is coincides with vacuum solution of Einstein equations. The second definition removes this restriction, but since BI metric admits non-trivial Ricci tensor, the BR in this case is not totally symmetric. Finally we gave the definition used by Bergqvist and Senovilla. It satisfies all the properties of BR and it is totally symmetric. In what follows we study the DEP and DSEP for BR in BI Universe.

5. Cosmological singularity and the dominant energy condition

Recalling that a timelike geodesic is a world line for a particle moving without acceleration in the proper reference system we define the following:

A spacetime is nonsingular if any timelike geodesics, or null geodesics, can be continued into the past and the future without bound, i.e., to infinite proper length for the timelike geodesics and to an infinite value of an affine parameter for the null geodesics. Such a spacetime is termed "causally, geodesically complete". The requirements on the completeness are the minimum necessary so that the spacetime does not contain a singularity. It is necessary to point out that a spacetime not satisfying these requirements, however, one with a singularity, does not necessarily contain points with infinite curvature or with small hole.

From physical point of view, of course, one ought to take as singular any spacetime in which the geodesic world line of a particle cannot be continued without bound with respect to the proper time of this particle, for such a singular spacetime would lead to a violation of conservation laws.

As applied to the cosmological problem, the Hawking-Penrose theorem reads as follows²⁷:

Theorem. *A space-time \mathcal{M} cannot satisfy causal geodesic completeness if the GTR (General Theory of Relativity) equations hold and if the following conditions are fulfilled:*

- (i) *The space-time \mathcal{M} does not contain closed time-like curves.*
- (ii) *The energy condition (DEP) is satisfied at every point.*

The energy condition may be expressed as

$$t^\alpha t_\alpha = 1 \quad \text{implies} \quad R_{\alpha\beta} t^\alpha t^\beta \leq 0. \quad (5.1)$$

With Einstein's equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\kappa T_{\alpha\beta},$$

(5.1) becomes

$$t^\alpha t_\alpha = 1 \quad \text{implies} \quad T_{\alpha\beta} t^\alpha t^\beta \geq \frac{1}{2} T^\mu_\mu. \quad (5.2)$$

If, in an eigentetrad of $T_{\mu\nu}$, ε denotes the energy density and p_1, p_2, p_3 denote the three principal pressure, then (5.2) can be written as

$$\begin{aligned} \varepsilon + \sum_{\alpha} p_{\alpha} &\geq 0; \\ \varepsilon + p_{\alpha} &\geq 0, \quad \alpha = 1, 2, 3. \end{aligned}$$

The weak energy condition is

$$\ell^\alpha \ell_\alpha = 0 \quad \text{implies} \quad R_{\alpha\beta} \ell^\alpha \ell^\beta \leq 0,$$

which is a consequence of (5.1).

(iii) On each time-like or null geodesic γ , there is at least one point for which

$$K_{[a}R_{b]cd[e}K_{f]}K^cK^d \neq 0, \quad (5.3)$$

where K_a is the tangent to the curve γ at the given point and where the brackets on the subscripts imply antisymmetrization. If γ is timelike, we can rewrite (5.3) as

$$R_{abcd}K^cK^d \neq 0.$$

(iv) The space-time \mathcal{M} contains either (a) a trapped surface, (b) a point P for which the convergence of all the null geodesics through P changes sign somewhere to the past of P , or (c) a compact space-like hypersurface.

The DEP for the BI metric can be written in the form:

$$T_0^0 \geq T_1^1 a^2 + T_2^2 b^2 + T_3^3 c^2, \quad (5.4a)$$

$$T_0^0 \geq T_1^1 a^2, \quad (5.4b)$$

$$T_0^0 \geq T_2^2 b^2, \quad (5.4c)$$

$$T_0^0 \geq T_3^3 c^2. \quad (5.4d)$$

In analogy with the Hawking-Penrose theorem stated above, Senovilla and others introduced DEP for the higher dimensional tensors. A detailed description of singularity theorems and their consequences can be found in the review paper by Senovilla ²⁸.

The DSEP as defined by Senovilla reads: ⁶

Theorem. A rank- s tensor $T_{\mu_1 \dots \mu_s}$, is said to satisfy the DSEP if

$$T_{\mu_1 \dots \mu_s} k_1^{\mu_1} \dots k_s^{\mu_s} \geq 0 \quad (5.5)$$

for any future-pointing causal vectors $k_1^{\mu_1} \dots k_s^{\mu_s}$.

To justify its name the dominant DSEP obeys the following Lemma.

Lemma. If a tensor $T_{\mu_1 \dots \mu_s}$ satisfies the DSEP, then

$$T_{0 \dots 0} \geq |T_{\mu_1 \dots \mu_s}|, \quad \forall \mu_1, \dots, \mu_s = 0, \dots, n-1 \quad (5.6)$$

in any orthonormal basis $\{\vec{e}_\nu\}$.

It was also established in ²⁹, that any tensor satisfying the DEP possesses the following property:

Property. $T_{\mu_1 \dots \mu_s}$ satisfies DEP if and only if

$$T_{\mu_1 \dots \mu_s} l_1^{\mu_1} \dots l_s^{\mu_s} \geq 0 \quad (5.7)$$

for any set $l_1^{\mu_1} \dots l_s^{\mu_s}$ of future-pointing null vectors.

Let us now back to the 4-rank BR tensor and to check the DSEP for it. Since BR defined as (4.10) is a completely symmetric, trace-free 4-rank tensor, then it satisfies the DEP ³⁰. Therefore from the foregoing theorem, lemma and property for B_{ijkl} we can write:

$$B_{abcd} k_1^a k_2^b k_3^c k_4^d \geq 0,$$

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$$B_{0\dots 0} \geq |B_{abcd}|, \quad \forall a, b, c, d = 0, 1, 2, 3 \quad (5.8)$$

and

$$B_{abcd}l_1^a l_2^b l_3^c l_4^d \geq 0$$

Apparently, in view of (4.14b), Eq. (5.8) imposes some restriction on the metric functions, e.g., $a_i^4 \leq 1$. But this is not the case, since we have used a coordinate basis to compute the BR tensor components (4.14a)-(4.14d) and Eq. (5.6) refers to an orthonormal basis. We recall that in an orthonormal basis the components of the Weyl tensor obey the following relation:

$$C_{0i0i} = -C_{jkjk}, \quad i, j, k = 1, 2, 3 \text{ and } i \neq j \neq k. \quad (5.9)$$

In view of (5.9) the expressions (4.14a), (4.14b), (4.14c) and (4.14d) now read

$$B_{0000} = (C_{jkjk})^2 + (C_{kiki})^2 + (C_{ijij})^2 \quad (5.10a)$$

$$B_{iiii} = B_{0000}, \quad (5.10b)$$

$$B_{ijij} = 2C_{0i0i}C_{0j0j} \quad (5.10c)$$

$$B_{0k0k} = -B_{ijij}, \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k. \quad (5.10d)$$

Thus in connection with the above Lemma relative to DSEP, Eq. (5.8) is fulfilled without any restrictions on the metric functions.

6. Conclusions

In view of the importance of the BI model in the study of the present day Universe we considered the most simple model with a perfect fluid as a source field. The corresponding solutions to the Einstein equations have been obtained. Three alternative definitions of Bel-Robinson tensor are considered. It is shown that the definition used by Deser *et. al.* is consistent with the Einstein equations when the source field is given by a vacuum only. The second definition used by Teyssandier is free from this restriction, but BR defined in this way is not totally symmetric. Definition used by Senovilla and Bergqvist does not suffer from this shortcomings, i.e., it has all the symmetries and the DSEP is satisfied.

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