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Interacting Spinor and Scalar Fields in Bianchi type-I Universe filled with Perfect Fluid: Exact Self-consistent Solutions

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Self-consistent solutions to interacting spinor and scalar field equations in General Relativity are studied for the case of Bianchi type-I space-time filled with perfect fluid. The initial and the asymptotic behavior of the field functions and the metric one has been thoroughly studied.

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1 Introduction

The quantum field theory in curved space-time has been a matter of great interest in recent years because of its applications to cosmology and astrophysics. The evidence of existence of strong gravitational fields in our Universe led to the study of the quantum effects of material fields in external classical gravitational field. After the appearance of Parker's paper on scalar fields [1] and spin-$\frac{1}{2}$ fields [2], several authors have studied this subject. Although the Universe seems homogenous and isotropic at present, there is no observational data that guaranties the isotropy in the era prior to the recombination. In fact, there are theoretical arguments that sustain the existence of an anisotropic phase that approaches an isotropic one [3]. Interest in studying Klein-Gordon and Dirac equations in anisotropic models has increased since Hu and Parker [4] have shown that the creation of scalar particles in anisotropic backgrounds can dissipate the anisotropy as the Universe expands.

A Bianchi type-I (B-I) Universe, being the straightforward generalization of the flat Robertson-Walker (RW) Universe, is one of the simplest models of an anisotropic Universe that describes a homogenous and spatially flat Universe. Unlike the RW Universe which has the same scale factor for each of the three spatial directions, a B-I Universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. It moreover has the agreeable property that near the singularity it behaves like a Kasner Universe even in the presence of matter and consequently falls within the general analysis of the singularity given by Belinskii et al [5]. And in a Universe filled with matter for $p = \gamma\,\varepsilon, \quad \gamma < 1$, it has been shown that any initial anisotropy in a B-I Universe quickly dies away and a B-I Universe eventually evolve into a RW Universe [6]. Since the present-day Universe is surprisingly isotropic, this feature of the B-I Universe makes it a prime candidate for studying the possible effects of an anisotropy in the early Universe on present-day observations. In light of the importance of mentioned above, several authors have studied linear spinor field equations [7], [8] and the behavior of gravitational waves (GW's) [9], [10], [11] in B-I Universe. Nonlinear spinor field (NLSF) in external cosmological gravitation field was first studied by G. N. Shikin in 1991 [12]. This study was extended by us for more general case where we consider nonlinear term as an arbitrary function of all possible invariants generated from spinor bilinear forms. In that paper we also studied the possibility of elimination of initial singularity specially for Kasner Universe [13]. In a recent paper [14] we studied the behavior of self-consistent NLSF in B-I Universe that was followed by the papers [15], [16] where we studied the self-consistent system of interacting spinor and scalar fields. The purpose of this paper is to extend our study for different kinds of interacting term in presence of perfect fluid. In the section 2 we derive fundamental equations corresponding to the Lagrangian for the self-consistent system of spinor, scalar and gravitational fields in presence of perfect fluid and seek their general solutions. In section 3 we give a detail analysis of the solutions obtained for different kinds of interacting term. In section 4 we sum up the results obtained.

2 Fundamental equations and general solutions

The Lagrangian for the self-consistent system of spinor and gravitation fields in presence of perfect fluid is

$$L \;=\; L_g + L_{sp} + L_{sc} + L_m + L_{int}, \tag{2.1}$$

where L_g, L_{sp}, L_{sc}, corresponding to gravitational, free spinor and free scalar fields read

$$\begin{aligned} L_g &= R/2\kappa, \\ L_{sp} &= (i/2)[\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma^\mu\psi] - m\bar{\psi}\psi, \\ L_{sc} &= (1/2)\varphi_{,\mu}\varphi^{,\mu}, \end{aligned}$$

with R being the scalar curvature and κ being the Einstein's gravitational constant and L_m is the Lagrangian of perfect fluid. As interaction Lagrangian we consider the following cases [15], [16], [17] :

$$\begin{aligned} 1.\ L_{int} &= (\lambda/2)\,\varphi_{,\alpha}\varphi^{,\alpha}F, \\ 2.\ L_{int} &= \lambda\bar{\psi}\gamma^\mu\psi\varphi_{,\mu}, \\ 3.\ L_{int} &= i\lambda\bar{\psi}\gamma^\mu\gamma^5\psi\varphi_{,\mu}, \end{aligned}$$

where λ is the coupling constant and F can be presented as some arbitrary functions of invariants generated from the real bilinear forms of spinor field having the form:

$$S \;=\; \bar{\psi}\psi, \quad P \;=\; i\bar{\psi}\gamma^5\psi, \quad v^\mu \;=\; (\bar{\psi}\gamma^\mu\psi), \quad A^\mu \;=\; (\bar{\psi}\gamma^5\gamma^\mu\psi), \quad T^{\mu\nu} \;=\; (\bar{\psi}\sigma^{\mu\nu}\psi),$$

where $\sigma^{\mu\nu} \;=\; (i/2)[\gamma^\mu\gamma^\nu \;-\; \gamma^\nu\gamma^\mu]$. Invariants, corresponding to the bilnear forms, look

$$I = S^2, \quad J = P^2, \quad I_A = A_\mu\, A^\mu \;=\; (\bar{\psi}\gamma^5\gamma^\mu\psi)\, g_{\mu\nu}(\bar{\psi}\gamma^5\gamma^\nu\psi),$$

$$I_v = v_\mu\, v^\mu \;=\; (\bar{\psi}\gamma^\mu\psi)\, g_{\mu\nu}(\bar{\psi}\gamma^\nu\psi), \quad I_T = T_{\mu\nu}\, T^{\mu\nu} \;=\; (\bar{\psi}\sigma^{\mu\nu}\psi)\, g_{\mu\alpha}g_{\nu\beta}(\bar{\psi}\sigma^{\alpha\beta}\psi).$$

According to the Pauli-Fierz theorem [18] among the five invariants only I and J are independent as all other can be expressed by them:

$$I_v = -I_A = I + J, \qquad I_T = I - J.$$

Therefore we choose $F = F(I, J)$.
We choose B-I space-time metric in the form

$$ds^2 \;=\; dt^2 - \gamma_{ij}(t)\, dx^i\, dx^j. \tag{2.2}$$

As it admits no rotational matter, the spatial metric $\gamma_{ij}(t)$ can be put into diagonal form. Now we can rewrite the B-I space-time metric in the form [19]:

$$ds^2 = dt^2 - a^2(t)\, dx^2 - b^2(t)\, dy^2 - c^2(t)\, dz^2, \tag{2.3}$$

where the velocity of light c is taken to be unity. Let us now write the Einstein equations for $a(t), b(t)$ and $c(t)$ corresponding to the metric (2.3) and Lagrangian (2.1) [19]:

$$\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\Big(\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\Big) = -\kappa\Big(T_1^1 - \frac{1}{2}T\Big), \tag{2.4}$$

$$\frac{\ddot{b}}{b} + \frac{\dot{b}}{b}\Big(\frac{\dot{a}}{a} + \frac{\dot{c}}{c}\Big) = -\kappa\Big(T_2^2 - \frac{1}{2}T\Big), \tag{2.5}$$

$$\frac{\ddot{c}}{c} + \frac{\dot{c}}{c}\Big(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\Big) = -\kappa\Big(T_3^3 - \frac{1}{2}T\Big), \tag{2.6}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = -\kappa\Big(T_0^0 - \frac{1}{2}T\Big), \tag{2.7}$$

where points denote differentiation with respect to t, and T_μ^ν is the energy-momentum tensor of material fields and perfect fluid.
The scalar and the spinor field equations and the energy-momentum tensor of material fields and perfect fluid corresponding to (2.1) are

$$\partial_\alpha\left[\sqrt{-g}\,(g^{\alpha\beta}\varphi_{,\beta} + \partial L_{int}/\partial\varphi_{,\alpha})\right] \,=\, 0, \tag{2.8}$$

$$\begin{aligned} i\gamma^\mu\nabla_\mu\psi - m\psi + \partial L_{int}/\partial\bar{\psi} &= 0, \\ i\nabla_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} - \partial L_{int}/\partial\psi &= 0. \end{aligned} \tag{2.9}$$

$$\begin{aligned} T_\mu^\rho &= \frac{i}{4}g^{\rho\nu}\Big(\bar{\psi}\gamma_\mu\nabla_\nu\psi + \bar{\psi}\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma_\nu\psi - \nabla_\nu\bar{\psi}\gamma_\mu\psi\Big) \\ &+ \varphi_{,\mu}\varphi^{,\rho} + 2\frac{\delta L_{int}}{\delta g^{\mu\rho}}g^{\rho\nu} - \delta_\mu^\rho(L_{sp} + L_{sc} + L_{int}) + T_{\mu\,(m)}^\rho. \end{aligned} \tag{2.10}$$

Here $T_{\mu\,(m)}^\rho$ is the energy-momentum tensor of perfect fluid. For a Universe filled with perfect fluid, in the concomitant system of reference ($u^0 = 1$, $u^i = 0, i = 1, 2, 3$) we have

$$T_{\mu(m)}^\nu \,=\, (p+\varepsilon)u_\mu u^\nu - \delta_\mu^\nu p \,=\, (\varepsilon,\, -p,\, -p,\, -p), \tag{2.11}$$

where energy ε is related to the pressure p by the equation of state $p = \gamma\,\varepsilon$, the general solution has been derived by Jacobs [6]. γ varies between the interval $0 \leq \gamma \leq 1$, whereas $\gamma = 0$ describes the dust Universe, $\gamma = \frac{1}{3}$ presents radiation Universe, $\frac{1}{3} < \gamma < 1$ ascribes hard Universe and $\gamma = 1$ corresponds to the stiff matter. As one sees changes in the solutions performed by perfect fluid carried out through Einstein equations, namely through $\tau = a(t)b(t)c(t)$. So, let us first see how the quantities ε and p connected with τ. In doing this we use the well-known equality $T_{\mu;\nu}^\nu = 0$, that leads to

$$\frac{d}{dt}(\tau\varepsilon) + \dot{\tau}p \,=\, 0, \tag{2.12}$$

with the solution

$$\ln\tau \,=\, -\int\frac{d\varepsilon}{(\varepsilon+p)}. \tag{2.13}$$

Recalling the equation of state $p = \xi\varepsilon,\ 0 \le \xi \le 1$ finally we get

$$T^0_{0(m)} = \varepsilon = \frac{\varepsilon_0}{\tau^{1+\xi}},\ T^1_{1(m)} = T^2_{2(m)} = T^3_{3(m)} = -p = -\frac{\varepsilon_0\xi}{\tau^{1+\xi}}, \tag{2.14}$$

where ε_0 is the integration constant.
Note that we consider space-independent field only. Under this assumption and with regard to spinor field equations, the components of the energy-momentum tensor read:

$$T^0_0 = mS + \frac{1}{2}\dot{\varphi}^2 + L_{int} + \varepsilon, \quad T^1_1 = T^2_2 = T^3_3 = \frac{1}{2}(\bar{\psi}\frac{\partial L_{int}}{\partial\bar{\psi}} + \frac{\partial L_{int}}{\partial\psi}\psi) - L_{sc} - L_{int} - p. \tag{2.15}$$

In (2.8) and (2.10) ∇_μ denotes the covariant derivative of spinor, having the form [20]:

$$\nabla_\mu\psi = \frac{\partial\psi}{\partial x^\mu} - \Gamma_\mu\psi, \tag{2.16}$$

where $\Gamma_\mu(x)$ are spinor affine connection matrices. $\gamma^\mu(x)$ matrices are defined for the metric (2.3) as follows. Using the equalities [21], [22]

$$g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab}, \qquad \gamma_\mu(x) = e^a_\mu(x)\bar{\gamma}^a,$$

where $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$, $\bar{\gamma}_\alpha$ are the Dirac matrices of Minkowski space and $e^a_\mu(x)$ are the set of tetradic 4-vectors, we obtain the Dirac matrices $\gamma^\mu(x)$ of curved space-time

$$\gamma^0 = \bar{\gamma}^0, \quad \gamma^1 = \bar{\gamma}^1/a(t), \quad \gamma^2 = \bar{\gamma}^2/b(t), \quad \gamma^3 = \bar{\gamma}^3/c(t),$$
$$\gamma_0 = \bar{\gamma}_0, \quad \gamma_1 = \bar{\gamma}_1 a(t), \quad \gamma_2 = \bar{\gamma}_2 b(t), \quad \gamma_3 = \bar{\gamma}_3 c(t).$$

$\Gamma_\mu(x)$ matrices are defined by the equality

$$\Gamma_\mu(x) = \frac{1}{4}g_{\rho\sigma}(x)\Big(\partial_\mu e^b_\delta e^\rho_b - \Gamma^\rho_{\mu\delta}\Big)\gamma^\sigma\gamma^\delta,$$

which gives

$$\Gamma_0 = 0, \quad \Gamma_1 = \frac{1}{2}\dot{a}(t)\bar{\gamma}^1\bar{\gamma}^0, \quad \Gamma_2 = \frac{1}{2}\dot{b}(t)\bar{\gamma}^2\bar{\gamma}^0, \quad \Gamma_3 = \frac{1}{2}\dot{c}(t)\bar{\gamma}^3\bar{\gamma}^0. \tag{2.17}$$

Flat space-time matrices we choose in the form, given in [23]:

$$\bar{\gamma}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \bar{\gamma}^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\gamma}^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\gamma}^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Defining γ^5 as follows

$$\begin{aligned}
\gamma^5 &= -\frac{i}{4}E_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho, \quad E_{\mu\nu\sigma\rho} = \sqrt{-g}\varepsilon_{\mu\nu\sigma\rho}, \quad \varepsilon_{0123} = 1, \\
\gamma^5 &= -i\sqrt{-g}\gamma^0\gamma^1\gamma^2\gamma^3 \;=\; -i\bar{\gamma}^0\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3 = \bar{\gamma}^5,
\end{aligned}$$

we obtain

$$\bar{\gamma}^5 \;=\; \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let us now solve the Einstein equations. With respect to (2.15) summation of Einstein equations (2.4), (2.5) and (2.6) leads to the equation

$$\frac{\ddot{\tau}}{\tau} = -\kappa(T_1^1 + T_2^2 + T_3^3 - \frac{3}{2}T) = \frac{3\kappa}{2}\,(T_0^0 + T_1^1). \tag{2.18}$$

In case if the right hand side of (2.18) be the function of $\tau(t) \;=\; a(t)b(t)c(t)$, this equation takes the form

$$\ddot{\tau} + \Phi(\tau) = 0. \tag{2.19}$$

As is known this equation possesses exact solutions for arbitrary function $\Phi(\tau)$. Giving the explicit form of L_{int}, from (2.18) one can find concrete function $\tau(t) = abc$. Once the value of τ is obtained, one can get expressions for components $V_\alpha(t), \quad \alpha = 1, 2, 3, 4$. Let us express a, b, c through τ. For this we notice that subtraction of Einstein equations (2.4) - (2.5) leads to the equation

$$\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{c}}{ac} - \frac{\dot{b}\dot{c}}{bc} = \frac{d}{dt}\Big(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\Big) + \Big(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\Big)\Big(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\Big) = 0. \tag{2.20}$$

Equation (2.20) possesses the solution

$$\frac{a}{b} = D_1 \exp\Big(X_1 \int \frac{dt}{\tau}\Big), \quad D_1 = \text{const.}, \quad X_1 = \text{const.} \tag{2.21}$$

Subtracting equations (2.4) - (2.6) and (2.5) - (2.6) one finds the equations similar to (2.20), having solutions

$$\frac{a}{c} = D_2 \exp\Big(X_2 \int \frac{dt}{\tau}\Big), \quad \frac{b}{c} = D_3 \exp\Big(X_3 \int \frac{dt}{\tau}\Big), \tag{2.22}$$

where D_2, D_3, X_2, X_3 are integration constants. There is a functional dependence between the constants D_1, D_2, D_3, X_1, X_2, X_3:

$$D_2 = D_1\, D_3, \qquad X_2 = X_1 \,+\, X_3.$$

Using the equations (2.21) and (2.22), we rewrite $a(t), b(t), c(t)$ in the explicit form:

$$\begin{aligned} a(t) &= (D_1^2 D_3)^{\frac{1}{3}} \tau^{\frac{1}{3}} \exp\Big[\frac{2X_1 + X_3}{3} \int \frac{dt}{\tau(t)}\Big], \\ b(t) &= (D_1^{-1} D_3)^{\frac{1}{3}} \tau^{\frac{1}{3}} \exp\Big[-\frac{X_1 - X_3}{3} \int \frac{dt}{\tau(t)}\Big], \\ c(t) &= (D_1 D_3^2)^{-\frac{1}{3}} \tau^{\frac{1}{3}} \exp\Big[-\frac{X_1 + 2X_3}{3} \int \frac{dt}{\tau(t)}\Big]. \end{aligned} \tag{2.23}$$

Thus the previous system of Einstein equations is completely integrated. In this process of integration only first three of the complete system of Einstein equations have been used. General solutions to these three second order equations have been obtained. The solutions contain six arbitrary constants: D_1, D_3, X_1, X_3 and two others, that were obtained while solving equation (2.19). Equation (2.7) is the consequence of first three of Einstein equations. To verify the correctness of obtained solutions, it is necessary to put a, b, c into (2.7). It should lead either to identity or to some additional constraint between the constants. Putting a, b, c from (2.23) into (2.7) one can get the following equality:

$$\frac{\ddot{\tau}}{\tau} - \frac{2}{3}\frac{\dot{\tau}^2}{\tau^2} + \frac{2}{9\tau^2}\mathcal{X} = -\frac{\kappa}{2}(T_0^0 - 3T_1^1), \quad \mathcal{X} := X_1^2 + X_1 X_3 + X_3^2, \tag{2.24}$$

that guaranties the correctness of the solutions obtained. This together with (2.18) gives the equation for τ with the solution in quadrature:

$$\int \frac{d\tau}{\sqrt{3\kappa\tau^2 T_0^0 + \mathcal{X}/3}} = t. \tag{2.25}$$

It should be emphasized that we are dealing with cosmological problem and our main goal is to investigate the initial and the asymptotic behavior of the field functions and the metric ones. As one sees, all these functions are in some functional dependence with τ. Therefore in our further investigation we mainly look for τ, though in some particular cases we write down field and metric functions explicitly.

3 Analysis of the solutions obtained for some special choice of interaction Lagrangian

Let us now study the system for some special choice of L_{int}. We first study the solution to the system of field equations with minimal coupling when the direct interaction between the spinor and scalar fields remains absent. The reason to get the solution to the self-consistent system of equations for the fields with minimal coupling is the necessity of comparing this solution with that for the system of equations for the interacting spinor, scalar and gravitational fields that permits to clarify the role of interaction terms in the evolution of the cosmological model in question.

In this case from the scalar and spinor field equations one finds $\dot{\varphi} = C/\tau$ and $\bar{\psi}\psi = S = C_0/\tau$ with C and C_0 being the constants of integration. Therefore the components of the energy-momentum tensor look:

$$T_0^0 = \frac{mC_0}{\tau} + \frac{C^2}{2\tau^2}, \quad T_1^1 = T_2^2 = T_3^3 = -\frac{C^2}{2\tau^2}. \tag{3.1}$$

Note that as the energy density T_0^0 should be a quantity positively defined, the equation (3.1) leads to $C_0 > 0$. The inequality $C_0 > 0$ will also be preserved for the system with direct interaction between the fields as in this case the correspondence principle should be fulfilled: for $\lambda = 0$ the field system with direct interaction turns into that with minimal coupling.

The components of spinor field functions in this case read

$$\psi_{1,2}(t) = (C_{1,2}/\sqrt{\tau})\, e^{-imt}, \quad \psi_{3,4}(t) = (C_{3,4}/\sqrt{\tau})\, e^{imt}. \tag{3.2}$$

Taking into account (3.1) equation (2.25) writes

$$\int \frac{d\tau}{\sqrt{3\kappa m C_0 \tau + 3\kappa C^2/2 + \mathcal{X}/3}} = t. \tag{3.3}$$

with the solution

$$\tau|_{t\to 0} \approx \sqrt{3\kappa C^2/2 + \mathcal{X}/3}\, t \to 0,$$

and

$$\tau|_{t\to\infty} \approx \sqrt{3\kappa m C_0}\, t^2.$$

Thus one concludes that the solutions obtained are initially singular and the space-time is asymptotically isotropic.

Let us now study the case with different kinds of interactions.

Case 1. For the case when $L_{int} = (\lambda/2)\varphi_{,\mu}\varphi^{,\mu} F(I,J)$ one writes the scalar field equation as

$$\frac{\partial}{\partial t}(\tau\dot{\varphi}(1+\lambda F)) = 0, \tag{3.4}$$

with the solution

$$\dot{\varphi} = C/\tau(1+\lambda F). \tag{3.5}$$

In this case the first equation of the system (2.9) now reads

$$i\bar{\gamma}^0\Big(\frac{\partial}{\partial t} + \frac{\dot{\tau}}{2\tau}\Big)\psi - m\psi + \mathcal{D}\psi + i\mathcal{G}\gamma^5\psi = 0, \tag{3.6}$$

where $\mathcal{D} := \varphi_{,\alpha}\varphi^{,\alpha} S\, F_I$ and $\mathcal{G} := \varphi_{,\alpha}\varphi^{,\alpha} P\, F_J$. For the components $\psi_\rho = V_\rho(t)$, where $\rho = 1,2,3,4$, from (3.6) one deduces the following system of equations:

$$\begin{aligned}
\dot{V}_1 + \frac{\dot{\tau}}{2\tau}V_1 + i(m-\mathcal{D})V_1 - \mathcal{G}V_3 &= 0,\\
\dot{V}_2 + \frac{\dot{\tau}}{2\tau}V_2 + i(m-\mathcal{D})V_2 - \mathcal{G}V_4 &= 0,\\
\dot{V}_3 + \frac{\dot{\tau}}{2\tau}V_3 - i(m-\mathcal{D})V_3 + \mathcal{G}V_1 &= 0,\\
\dot{V}_4 + \frac{\dot{\tau}}{2\tau}V_4 - i(m-\mathcal{D})V_4 + \mathcal{G}V_2 &= 0.
\end{aligned} \tag{3.7}$$

Let us now define the equations for

$$\begin{aligned} P &= i(V_1V_3^* - V_1^*V_3 + V_2V_4^* - V_2^*V_4), \\ R &= (V_1V_3^* + V_1^*V_3 + V_2V_4^* + V_2^*V_4), \\ S &= (V_1^*V_1 + V_2^*V_2 - V_3^*V_3 - V_4^*V_4). \end{aligned} \tag{3.8}$$

After a little manipulation one finds

$$\begin{aligned} \frac{dS_0}{dt} - 2\mathcal{G}\, R_0 &= 0, \\ \frac{dR_0}{dt} + 2(m-\mathcal{D})\, P_0 + 2\mathcal{G} S_0 &= 0, \\ \frac{dP_0}{dt} - 2(m-\mathcal{D})\, R_0 &= 0, \end{aligned} \tag{3.9}$$

where $S_0 = \tau S, \quad P_0 = \tau P, \quad R_0 = \tau R$. From this system one can easily find

$$S_0\dot{S}_0 + R_0\dot{R}_0 + P_0\dot{P}_0 = 0,$$

that gives

$$S^2 + R^2 + P^2 = A^2/\tau^2, \qquad A^2 = \text{const.} \tag{3.10}$$

Let us go back to the system of equations (3.7). It can be written as follows if one defines $W_\alpha = \sqrt{\tau}\, V_\alpha$:

$$\begin{aligned} \dot{W}_1 + i\Phi W_1 - \mathcal{G}W_3 &= 0, \quad \dot{W}_2 + i\Phi W_2 - \mathcal{G}W_4 = 0, \\ \dot{W}_3 - i\Phi W_3 + \mathcal{G}W_1 &= 0, \quad \dot{W}_4 - i\Phi W_4 + \mathcal{G}W_2 = 0, \end{aligned} \tag{3.11}$$

where $\Phi = m - \mathcal{D}$. Defining $U(\sigma) = W(t)$, where $\sigma = \int \mathcal{G}dt$, we rewrite the foregoing system as:

$$\begin{aligned} U_1' + i(\Phi/\mathcal{G})U_1 - U_3 &= 0, \qquad U_2' + i(\Phi/\mathcal{G})U_2 - U_4 = 0, \\ U_3' - i(\Phi/\mathcal{G})U_3 + U_1 &= 0, \qquad U_4' - i(\Phi/\mathcal{G})U_4 + U_2 = 0, \end{aligned} \tag{3.12}$$

where prime ($'$) denotes differentiation with respect to σ. One can now define V_α giving the explicit value of L_{int}.

I. Let us consider the case when $F = I^n = S^{2n}$. It is clear that in this case $\mathcal{G} = 0$. From (3.9) we find

$$S = C_0/\tau, \quad C_0 = \text{const.} \tag{3.13}$$

As in the considered case F depends only on S, from (3.13) it follows that $\mathcal{D}$ is a functions of $\tau = abc$. Taking this fact into account, integration of the system of equations (3.11) leads to the expressions

$$V_r(t) = (C_r/\sqrt{\tau})\, e^{-i\Omega}, \quad r = 1,2, \quad V_l(t) = (C_l/\sqrt{\tau})\, e^{i\Omega}, \quad l = 3,4, \tag{3.14}$$

where C_r and C_l are integration constants and $\Omega = \int \Phi(t)dt$. Putting this solution into (3.8) one gets

$$S = (C_1^2 + C_2^2 - C_3^2 - C_4^2)/\tau. \tag{3.15}$$

Comparing it with (3.13) we find $C_0 = C_1^2 + C_2^2 - C_3^2 - C_4^2$. The equation (2.25) in this case reads

$$\int \frac{d\tau}{\sqrt{3\kappa(mC_0\tau + \frac{C^2}{2(1+\lambda C_0^{2n}/\tau^{2n})} + \varepsilon_0\tau^{1-\xi}) + \mathcal{X}/3}} = t. \tag{3.16}$$

As one sees

$$\tau(t)|_{t\to\infty} \approx \frac{3}{4}\kappa m C_0 t^2 \to \infty,$$

$$\tau(t)|_{t\to 0} \approx \sqrt{\mathcal{X}/3}\, t \to 0.$$

Thus in the case considered, the asymptotical isotropization of the expansion process of initially anisotropic Bianchi type-I space-time takes place without the influence of scalar field. For a detail analysis of this case see [15].

II. We study the system when $F = J^n = P^{2n}$, which means in the case considered $\mathcal{D} = 0$. Let us note that, the interaction between the fields inevitably leads to the appearnce of nonlinear terms in the field equations. As is known, in the unified nonlinear spinor theory of Heisenberg the massive term remains absent, as according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter [24]. In nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in linear one, as by no means it defines total energy (or mass) of nonlinear field system. Thus without losing the generality we can consider massless spinor field putting $m = 0$ that leads to $\Phi = 0$. Then from (3.9) one gets

$$P(t) = D_0/\tau, \; D_0 = \text{const}. \tag{3.17}$$

The system of equations (3.12) in this case reads

$$\begin{aligned} U_1' - U_3 &= 0, \qquad U_2' - U_4 = 0, \\ U_3' + U_1 &= 0, \qquad U_4' + U_2 = 0. \end{aligned} \tag{3.18}$$

Differentiating the first equation of system (3.18) and taking into account the third one we get

$$U_1'' + U_1 = 0, \tag{3.19}$$

which leads to the solution

$$U_1 = D_1 e^{i\sigma} + iD_3 e^{-i\sigma}, \quad U_3 = iD_1 e^{i\sigma} + D_3 e^{-i\sigma}. \tag{3.20}$$

Analogically for U_2 and U_4 one gets

$$U_2 = D_2 e^{i\sigma} + iD_4 e^{-i\sigma}, \quad U_4 = iD_2 e^{i\sigma} + D_4 e^{-i\sigma}, \tag{3.21}$$

where D_i are the constants of integration. Finally, we can write

$$\begin{aligned} V_1 &= (1/\sqrt{\tau})(D_1 e^{i\sigma} + iD_3 e^{-i\sigma}), \quad V_2 = (1/\sqrt{\tau})(D_2 e^{i\sigma} + iD_4 e^{-i\sigma}), \\ V_3 &= (1/\sqrt{\tau})(iD_1 e^{i\sigma} + D_3 e^{-i\sigma}), \quad V_4 = (1/\sqrt{\tau})(iD_2 e^{i\sigma} + D_4 e^{-i\sigma}). \end{aligned} \tag{3.22}$$

Putting (3.22) into (3.8) one finds

$$P = 2\,(D_1^2 + D_2^2 - D_3^2 - D_4^2)/\tau. \tag{3.23}$$

Comparison of (3.17) with (3.23) gives $D_0 = 2\,(D_1^2 + D_2^2 - D_3^2 - D_4^2)$. Let us now estimate τ using the equation

$$\int \frac{d\tau}{\sqrt{3\kappa(\frac{C^2}{2(1+\lambda\, C_0^{2n}/\tau^{2n})} + \varepsilon_0 \tau^{1-\xi}) + \mathcal{X}/3}} = t. \tag{3.24}$$

In this case we obtain

$$\tau|_{t\to\infty} \approx ([\sqrt{\varepsilon_0}(\xi+1)/2]\, t)^{2/(\xi+1)},$$

$$\tau|_{t\to 0} \approx \sqrt{\mathcal{X}/3}\, t,$$

i.e. The solutions obtained are initially singular and the space-time is asymptotically isotropic if $\xi < 1$ and anisotropic if $\xi = 1$.

III. In this case we study $F = F(I,\, J)$. Choosing

$$F = F(K_\pm), \quad K_+ = I + J = I_v = -I_A, \quad K_- = I - J = I_T, \tag{3.25}$$

in case of massless spinor field we find

$$\mathcal{D} = \varphi_{,\mu}\varphi^{,\mu} S F_{K_\pm}, \quad \mathcal{G} = \pm\varphi_{,\mu}\varphi^{,\mu} S F_{K_\pm}, \quad F_{K_\pm} = dF/dK_\pm.$$

Putting them into (3.9) we find

$$S_0^2 \pm P_0^2 = D_\pm. \tag{3.26}$$

Choosing $F = K_\pm^n$ from (2.25) one comes to the conclusion similar to that of previous case (**II**).

Case 2. In this case the scalar and spinor field equations read

$$\frac{\partial}{\partial t}[\tau\,(\dot\varphi + \lambda\bar\psi\gamma^0\psi)] = 0, \tag{3.27}$$

$$\begin{aligned} i\gamma^0(\frac{\partial}{\partial t} + \frac{\dot\tau}{2\tau})\psi - m\psi + \lambda\dot\varphi\gamma^0\psi &= 0, \\ i(\frac{\partial}{\partial t} + \frac{\dot\tau}{2\tau})\bar\psi\gamma^0 + m\bar\psi - \lambda\dot\varphi\bar\psi\gamma^0 &= 0. \end{aligned} \tag{3.28}$$

Using the spinor field equations one finds: $\bar\psi\gamma^0\psi = C_1/\tau$, and $S = \bar\psi\psi = C_0/\tau$ with C_1 and C_0 being the constant of integration. Putting it in the scalar field equation one obtains

$$\dot\varphi = (C - \lambda C_1)/\tau, \quad C = \text{const}. \tag{3.29}$$

In account with all these the spinor field equation can be written as

$$\gamma^0(\frac{\partial}{\partial t}+\frac{\dot{\tau}}{2\tau})\psi+im\psi-\frac{i\lambda(C-\lambda C_1)}{\tau}\gamma^0\psi=0, \tag{3.30}$$

with the solution

$$\begin{aligned}\psi_{1,2}(t) &= \frac{D_{1,2}}{\sqrt{\tau}}\exp\,[-i\{mt-\lambda(C-\lambda C_1)\int\tau^{-1}dt\}],\\ \psi_{3,4}(t) &= \frac{D_{3,4}}{\sqrt{\tau}}\exp\,[i\{mt+\lambda(C-\lambda C_1)\int\tau^{-1}dt\}].\end{aligned} \tag{3.31}$$

The components of energy-momentum tensor in this case read

$$T_0^0=\frac{mC_0}{\tau}+\frac{\mathcal{C}}{\tau^2}+\frac{\varepsilon_0}{\tau^{1+\xi}},\quad T_1^1=T_2^2=T_3^3=-\frac{(C-\lambda C_1)^2}{2\tau^2}-\frac{\varepsilon_0\xi}{\tau^{1+\xi}},$$

where $\mathcal{C}:=(C^2-\lambda^2C_1^2)/2$ and $0<\xi<1$. Putting this into (2.25) one gets

$$\int\frac{d\tau}{\sqrt{3\kappa(mC_0\tau+\varepsilon_0\tau^{1-\xi}+\mathcal{C})+\mathcal{X}/3}}=t. \tag{3.32}$$

As one sees

$$\tau|_{t\to 0}\approx\sqrt{\mathcal{X}/3-3\kappa\mathcal{C}}\,t\to 0,$$

and

$$\tau|_{t\to\infty}\approx\sqrt{3\kappa mC_0}\,t^2,$$

which means the solution obtained is initially singular and the isotropization process of the initially anisotropic Universe takes place as $t\to\infty$.

Case 3. In this case the scalar and spinor field equations read

$$\frac{\partial}{\partial t}[\tau\,(\dot{\varphi}+i\lambda\bar{\psi}\gamma^0\gamma^5\psi)]\;=\;0, \tag{3.33}$$

$$\begin{aligned}i\gamma^0(\frac{\partial}{\partial t}+\frac{\dot{\tau}}{2\tau})\psi-m\psi+i\lambda\dot{\varphi}\gamma^0\gamma^5\psi &= 0,\\ i(\frac{\partial}{\partial t}+\frac{\dot{\tau}}{2\tau})\bar{\psi}\gamma^0+m\bar{\psi}-\lambda\dot{\varphi}\bar{\psi}\gamma^0\gamma^5 &= 0.\end{aligned} \tag{3.34}$$

We consider the massless spinor field. In this case from the spinor field equations one finds: $i\bar{\psi}\gamma^0\gamma^5\psi=C_2/\tau$, with C_2 being the constant of integration. Putting it in the scalar field equation one obtains

$$\dot{\varphi}=(C-\lambda C_2)/\tau,\quad C=\text{const}. \tag{3.35}$$

In account with all these the spinor field equation can be written as

$$\gamma^0(\frac{\partial}{\partial t}+\frac{\dot{\tau}}{2\tau})\psi-\frac{i\lambda(C-\lambda C_2)}{\tau}\gamma^0\gamma^5\psi=0. \tag{3.36}$$

Defining $W(t) = \sqrt{\tau}\,\psi(t)$ one writes the foregoing equations as

$$\begin{aligned} \dot{W}_1 - \lambda\dot{\varphi}W_3 &= 0, \quad \dot{W}_2 - \lambda\dot{\varphi}W_4 = 0, \\ \dot{W}_3 - \lambda\dot{\varphi}W_1 &= 0, \quad \dot{W}_1 - \lambda\dot{\varphi}W_3 = 0. \end{aligned} \tag{3.37}$$

Differentiating the fisrt equation of the foregoing system one gets

$$\ddot{W}_1 + \frac{\dot{\tau}}{\tau}\dot{W}_1 - [\lambda(C - \lambda C_2)]^2 \frac{1}{\tau^2} W_1 = 0, \tag{3.38}$$

where the third equation of the system as well as $\dot{\varphi}$ has been taken into account. The first integral of this equation reads

$$\tau W_1 = \lambda(C - \lambda C_2)W_1, \tag{3.39}$$

with the constant of integration be taken trivial. Procceding analogically one writes the solution of the system as

$$W_{1,3} = D_+ \exp\left[\lambda(C - \lambda C_2)\int \tau^{-1}dt, \quad W_{2,4} = D_- \exp\left[\lambda(C - \lambda C_2)\int \tau^{-1}dt. \right.\right. \tag{3.40}$$

The components of energy-momentum tensor in this case read

$$T_0^0 = \frac{mC_0}{\tau} + \frac{\mathcal{C}}{\tau^2} + \frac{\varepsilon_0}{\tau^{1+\xi}}, \quad T_1^1 = T_2^2 = T_3^3 = -\frac{(C - \lambda C_2)^2}{2\tau^2} - \frac{\varepsilon_0 \xi}{\tau^{1+\xi}},$$

where $\mathcal{C} := (C^2 - \lambda^2 C_2^2)/2$ and $0 < \xi < 1$. Putting this into (2.25) one gets

$$\int \frac{d\tau}{\sqrt{3\kappa(mC_0\tau + \varepsilon_0\tau^{1-\xi} + \mathcal{C}) + \mathcal{X}/3}} = t. \tag{3.41}$$

As one sees

$$\tau|_{t\to 0} \approx \sqrt{\mathcal{X}/3 - 3\kappa\mathcal{C}}\, t \to 0,$$

and

$$\tau|_{t\to\infty} \approx \sqrt{3\kappa m C_0}\, t^2,$$

which means the solution obtained is initially singular and the isotropization process of the initially anisotropic Universe takes place as $t \to \infty$.

4 Conclusions

Exact solutions to the self-consistent system of spinor and scalar field equations have been obtained for the B-I space-time filled with perfect fluid. It is shown that the solutions obtained are initially singular and the space-time is basically asymptotically isotropic independent to the choice of interacting term in the Lagrangian, though there are some special cases that occur initially regular (with breaking energy-dominent condition [15]) solutions and leave the space-time asymptotically anisotropic.

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