

Exact Self-consistent Particle-like Solutions to the Equations of Nonlinear Scalar Electrodynamics in General Relativity

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Exact self-consistent particle-like solutions with spherical and/or cylindrical symmetry to the equations governing the interacting system of scalar, electromagnetic and gravitational fields have been obtained. As a particular case it is shown that the equations of motion admit a special kind of solutions with sharp boundary known as droplets. For these solutions, the physical fields vanish and the space-time is flat outside of the critical sphere or cylinder. Therefore, the mass and the electric charge of these configurations are zero.

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1 Introduction

Since the early history of elementary particle physics the attempts to construct divergence-free theory had been being undertaken. In 1912 G. Mie [1] proposed the nonlinear modification of the Maxwell equations, with the nonlinear electric current of the form $j_\mu = (A_\nu A^\nu)^2 A_\mu$. Within the scope of this modification there do exist regular solutions approximating the electron structure.

In 1942 N. Rosen [2] considered the system of interacting electromagnetic and complex scalar fields that also admitted the existence of localized particle-like solutions. Nevertheless, these two models suffered one and the same defect: the mass of the localized object turned to be negative. Recently it was shown that this defect of nonlinear electrodynamics can be corrected within the framework of general relativity [3].

The aim of this paper is to consider self-consistent system of fields to obtain particle-like configurations in the framework of general relativity. We show that in the case of electromagnetic scalar and gravitation fields system with specific type of interactions there exist droplet-like solutions having zero electric charge and mass. It is noteworthy to underline that the effective potentials, raised in this case, possess confining property i.e. create a strong repulsion on certain surfaces in configurational space.

2 Fundamental Equations

As is known, there do not exist regular static spherically or cylindrically symmetric configurations within the framework of gauge invariant nonlinear electrodynamics [4]. One possible way to overcome this difficulty is the nonlinear generalization of electrodynamics, with the use of the Lagrangian explicitly containing 4-potential A_μ , $\mu = 0, 1, 2, 3$, thus breaking the gauge invariance inside the small critical sphere or cylinder. The introduction of the terms explicitly depending on potentials in electromagnetic equations presents the possibility to give an alternative explanation of the processes like inelastic photon-photon interaction [5], galactic red-shift anomalies [6], [7], [8], electric screening at low temperature in the limit of indirect interaction of photon with thermal neutrino background [9], the excess of high-energy photons in the isotropic flux [10], avoiding the Big Bang singularity [11], origin of self-focused beam in the effective nonlinear vector field theory [12]. The corresponding terms appear in our scheme due to the interaction between the electromagnetic and scalar fields. This interaction being negligible at large distances, the Maxwellian structure of the electromagnetic equations (and therefore the gauge invariance) is reinstated far from the center of the system.

We choose the Lagrangian in the form [4]

$$L = \frac{R}{2\kappa} - \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{8\pi} \varphi_{,\alpha} \varphi^{,\alpha} \Psi(I), \quad (2.1)$$

where $\kappa = 8\pi G$ is the Einstein's gravitational constant and the function $\Psi(I)$ of the invariant $I = A_\mu A^\mu$ characterizes the interaction between the scalar φ and electromagnetic A_μ fields. In the sequel the function $\Psi(I)$ will be viewed as an arbitrary one, thus the Lagrangian (2.1) defines the class of models parameterized by $\Psi(I)$. In 1951 J. Schwinger [13] used the special method to compute the effective coupling between a zero spin neutral meson and the electromagnetic field using some functions of electromagnetic field. Thus

our approach to generate an effective Lagrangian generalizes the one proposed by Schwinger. The particular choice of $\Psi(I)$ will be made to obtain droplet-like configurations. The field equations corresponding to the Lagrangian (2.1) read

$$\mathcal{G}_\mu^\nu = -\kappa T_\mu^\nu, \quad (2.2)$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g} g^{\alpha\beta} \varphi_{,\beta} \Psi) = 0, \quad (2.3)$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\beta} (\sqrt{-g} F^{\alpha\beta}) - (\varphi_{,\beta} \varphi^{,\beta}) \Psi_I A^\alpha = 0, \quad (2.4)$$

where $\Psi_I = d\Psi/dI$ and $\mathcal{G}_\mu^\nu = R_\mu^\nu - \delta_\mu^\nu R/2$ is the Einstein tensor. One can write the energy-momentum tensor of the interacting matter fields in the form:

$$\begin{aligned} T_\mu^\nu &= (1/4\pi) [\varphi_{,\mu} \varphi^{,\nu} \Psi(I) - F_{\mu\alpha} F^{\nu\alpha} + \varphi_{,\alpha} \varphi^{,\alpha} \Psi_I A_\mu A^\nu] \\ &- \delta_\mu^\nu \left[\frac{1}{8\pi} \varphi_{,\beta} \varphi^{,\beta} \Psi(I) - \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} \right]. \end{aligned} \quad (2.5)$$

3 Configurations with spherical symmetry

Searching for the static spherically-symmetric solutions to the system of equations (2.2)-(2.4), we consider the metric in the form [14]:

$$ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} d\xi^2 - e^{2\beta} [d\theta^2 + \sin^2\theta d\phi^2], \quad (3.1)$$

with ξ being the radial variable. Let us now formulate the requirements to be fulfilled by particle-like solutions (PLS). These are [15]

(a) Stationarity [applied to the metric (3.1)] i.e.

$$\alpha = \alpha(\xi), \quad \beta = \beta(\xi), \quad \gamma = \gamma(\xi);$$

(b) regularity of the metric and the matter fields in the whole space-time;

(c) asymptotically Schwarzschild metric and corresponding behavior of the field functions.

In view of requirement (a) it is convenient to choose the harmonic ξ coordinate ($\square\xi = 0$) in (3.1) to satisfy the subsidiary condition [16]:

$$\alpha = 2\beta + \gamma. \quad (3.2)$$

The corresponding coordinate in flat space-time is just $\xi = 1/r$. With the constraint (3.2) the system of Einstein equations (2.2) reads:

$$e^{-2\alpha} (2\beta'' - U) - e^{-2\beta} = -\kappa T_0^0, \quad (3.3)$$

$$e^{-2\alpha} U - e^{-2\beta} = -\kappa T_1^1, \quad (3.4)$$

$$e^{-2\alpha} (\beta'' + \gamma'' - U) = -\kappa T_2^2 = -\kappa T_3^3, \quad (3.5)$$

where $U = \beta'^2 + 2\beta'\gamma'$, and prime ($'$) denotes differentiation with respect to x . Note that the field functions, as well as the components of the metric tensor depend on the single spatial variable ξ . Assuming the electromagnetic field to be determined by the time

component $A_0 = A(\xi)$ of the 4-potential one finds the unique non-trivial component of the field tensor $F_{10} = A'$, and the invariant I reduces to $I = e^{-2\gamma} A^2(\xi)$.

One can write the non-zero components of the energy-momentum tensor (2.5) as follows:

$$T_0^0 = (1/8\pi) e^{-2\alpha} [A'^2 e^{-2\gamma} + \varphi'^2 (\Psi - 2A^2 e^{-2\gamma} \Psi_I)], \quad (3.6)$$

$$T_1^1 = -T_2^2 = -T_3^3 = (1/8\pi) e^{-2\alpha} [A'^2 e^{-2\gamma} - \varphi'^2 \Psi]. \quad (3.7)$$

Adding together the equations (3.4) and (3.5) and using the property $T_1^1 + T_2^2 = 0$, one obtains the differential equation

$$\beta'' + \gamma'' - e^{2(\beta+\gamma)} = 0,$$

with the solution [17]

$$e^{-(\beta+\gamma)} = \mathcal{S}(k, \xi) = \begin{cases} k^{-1} \text{sh } k\xi, & k > 0, \\ \xi, & k = 0, \\ k^{-1} \sin k\xi, & k < 0, \end{cases} \quad (3.8)$$

depending on the constant k . Notice that another constant of integration is trivial, so that $\xi = 0$ corresponds to the spatial infinity, where $e^\gamma = 1$ and $e^\beta = \infty$. Without loss of generality one can choose $\xi > 0$.

The scalar field equation (2.3) has the evident solution

$$\varphi' = C P(I), \quad (3.9)$$

where $P(I) = 1/\Psi(I)$ and C is the integration constant. Putting (3.9) into (2.4) one gets the equation for the electromagnetic field

$$(e^{-2\gamma} A')' - C^2 P_I e^{-2\gamma} A = 0, \quad (3.10)$$

where the second term could be naturally interpreted as the induced nonlinearity. In view of (3.9) one rewrites the Einstein equation (3.4) and the result of adding the equations (3.3) and (3.4) as follows :

$$\gamma'^2 = -G(C^2 P - A'^2 e^{-2\gamma}) + K, \quad K = k^2 \text{sign} k, \quad (3.11)$$

$$\gamma'' = G e^{-2\gamma} (A'^2 + C^2 A^2 P_I). \quad (3.12)$$

One can easily check that the equation (3.11) is the first integral of the equations (3.10) and (3.12). Eliminating the term $(P_I A)$ between (3.10) and (3.12) one gets the equation:

$$\gamma'' = G (A A' e^{-2\gamma})', \quad (3.13)$$

with the evident first integral:

$$\gamma' = G A A' e^{-2\gamma} + C_1, \quad C_1 = \text{const}. \quad (3.14)$$

Let us consider the simple case $C_1 = 0$. Then from (3.14) we get

$$e^{2\gamma} = G A^2 + H, \quad H = \text{const}. \quad (3.15)$$

Substituting γ' and $e^{2\gamma}$ from (3.14) and (3.15) into (3.10), we find for $A(\xi)$ the differential equation:

$$A'^2 (G A^2 + H)^{-2} = (G C^2 P - K)/G H, \quad (3.16)$$

which can be solved by quadrature:

$$\int \frac{dA}{(G A^2 + H) \sqrt{G C^2 P - K}} = \pm (1/\sqrt{G H}) (\xi - \xi_0), \quad \xi_0 = \text{const.} \quad (3.17)$$

It is clear that the configuration obtained has a center if and only if $e^\beta = 0$ at some $\xi = \xi_c$. One can show [16] that the conditions for the center $\xi_c = \infty$ to be regular imply $K = 0$ and the following behavior of the field quantities in the vicinity of the point $\xi_c = \infty$:

$$\begin{aligned} \gamma' &= O(\xi^{-2}), \quad A' \rightarrow A_c \neq \infty, \quad A' \rightarrow 0, \\ \xi^4 P(I) &\rightarrow 0, \quad |\xi^4 I P_I| < \infty. \end{aligned} \quad (3.18)$$

In view of (3.18) we deduce from (3.14) that $C_1 = 0$ in accordance with the earlier supposition.

Now we can write the boundary conditions on the surface of the critical sphere $\xi = \xi_0$:

$$T_\mu^\nu = A = A' = 0, \quad e^\gamma = 1, \quad e^\beta = 1/\xi_0 > 0. \quad (3.19)$$

Due to (3.19) and (3.15) we infer that $H = 1$. The condition $K = 0$ leads to $k = 0$ in (3.8) and the space-time (3.1) that fulfills the regularity conditions (3.18) takes the form

$$ds^2 = (G A^2 + 1) dt^2 - \frac{1}{\xi^2 (G A^2 + 1)} \left(\frac{d\xi^2}{\xi^2} + [d\theta^2 + \sin^2 \theta d\phi^2] \right). \quad (3.20)$$

We can finally write A and φ as follows:

$$\int \frac{dA}{(G A^2 + 1) \sqrt{P}} = \pm C (\xi - \xi_0), \quad (3.21)$$

$$\varphi = C \int P d\xi = \int \sqrt{P} e^{-2\gamma} dA = \int \frac{\sqrt{P} dA}{G A^2 + 1}. \quad (3.22)$$

Let us now calculate the matter field energy density:

$$T_0^0 = (C^2/8\pi) e^{-2\alpha} [P(1 + e^{2\gamma}) + 2 I P_I(I)]. \quad (3.23)$$

One can readily derive from (3.23) the energy E_f of the matter fields:

$$E_f = \int d^3x \sqrt{-^3g} T_0^0 = (C/2) \int_{A(\xi=0)}^{A(\xi \rightarrow \infty)} dA e^{-3\gamma} [\sqrt{P}(1 + e^{2\gamma}) + 4 I (\sqrt{P})_I]. \quad (3.24)$$

Thus the equations to the scalar and electromagnetic fields are completely integrated. As one sees, to write the scalar (φ) and vector (A) functions, as well as the energy density (T_0^0) and energy of the material fields (E_f), explicitly, one has to give $P(I)$ in explicit form. Here we will give the detail analysis for some concrete forms of $P(I)$.

I. Let us consider $P(I)$ in the form

$$P(I) = P_0(\lambda I - N)^s R(\lambda I), \quad 2 \leq s \leq 3, \quad (3.25)$$

where $R(\lambda I)$ is some arbitrary, continuous, positive defined function, having non-trivial value at the center; λ is the coupling parameter; $N > 0$ is some dimensionless constant that is equal to the value of λI at the center. The other constant P_0 is defined from the condition $P = 1$ at spatial infinity $\xi = 0$. For $R = \text{const.}$ one gets the most simple form of $P(I)$ that leads to regular solutions. In this case the energy density is positive if $\lambda I \geq N$.

a) Choosing $P(I)$ in the form

$$P(I) = P_0(\lambda I - N)^2, \quad (3.26)$$

we get

$$A(\xi) = \sqrt{\frac{N}{\lambda - GN}} \text{cth}\Lambda(\xi + \xi_1), \quad (3.27)$$

where $\Lambda = \sqrt{C^2 NP_0(\lambda - GN)}$, the integration constant ξ_1 is defined from $A(0) = m/q$ with m and q being the mass and the charge of the system. In this case we get

$$P_0 = (\lambda m^2/q^2 - N)^{-2}, \quad \lambda m^2/q^2 > N.$$

Inasmuch $\sqrt{\lambda}m/|q| > \sqrt{N}$, then $\delta = \sqrt{G}m/|q| > \sqrt{GN/\lambda} = \sigma$. Taking $\delta < 1$ and $\sigma < 1$ we get the inequality:

$$0 < \sigma < \delta < 1.$$

Now we can rewrite P_0 in the form

$$P_0 = \frac{G^2}{\lambda^2} (\delta^2 - \sigma^2)^{-2}.$$

The metric function $e^{2\gamma}$, electric field and the total energy of the material field system can be written as

$$e^{2\gamma} = GA^2 + 1 = \frac{C^2}{q^2} \left[\frac{\sigma^2}{1 - \sigma^2} \text{cth}^2\Lambda(\xi + \xi_1) + 1 \right], \quad (3.28)$$

$$|\mathbf{E}| = (-F_{10}F^{10})^{1/2} = \frac{\Lambda\sqrt{N}}{\sqrt{\lambda(1 - \sigma^2)}} \frac{\xi^2}{\text{sh}^2\Lambda(\xi + \xi_1)}, \quad (3.29)$$

$$\begin{aligned} E_f &= \frac{q}{2\sqrt{G}} \left[\frac{\delta - \sigma}{\delta + \sigma} \frac{\delta + 2\sigma}{3} + \frac{4(\delta^2 + \delta\sigma + \sigma^2) - 3}{3(\delta + \sigma)} \right. \\ &\quad \left. + \frac{1 - \sigma^2}{2(\delta^2 - \sigma^2)} \ln \frac{(1 + \delta)(1 - \sigma)}{(1 - \delta)(1 + \sigma)} \right]. \end{aligned} \quad (3.30)$$

As one sees

$$E_f|_{\delta \rightarrow \sigma} \rightarrow \frac{q\delta}{\sqrt{G}} = m, \quad E_f|_{\delta \rightarrow 1} \rightarrow \infty.$$

The infinite value of E_f can be interpreted as the physical reason of existence of limitation $\delta < 1$.

b) Let us consider the case with $I_c = 0$, choosing

$$P(I) = \lambda I. \quad (3.31)$$

On the spatial infinity, where $I = I_0 = m^2/q^2$, $P = 1$, that leads to $\lambda = q^2/m^2$, i.e. the coupling constant is connected with mass and charge. In this case we get

$$A(\xi) = \frac{1}{\sqrt{G} \operatorname{sh} m(\xi + \xi_1)}, \quad (3.32)$$

where as in previous case ξ_1 is defined from $A(0) = m/q$. The metric function $e^{2\gamma}$, electric field and the total energy of the material field system can be written as

$$e^{2\gamma} = \frac{C^2}{q^2} \operatorname{cth}^2 mC(\xi + \xi_1)/q, \quad (3.33)$$

$$|\mathbf{E}| = \frac{mC^2}{q^2 \sqrt{G}} \frac{\xi^2 \operatorname{ch} mC(\xi + \xi_1)/q}{\operatorname{sh}^2 mC(\xi + \xi_1)/q}, \quad (3.34)$$

$$E_f = \frac{q}{4\sqrt{G}} \left[3\delta \frac{1}{\delta} \ln(1 - \delta^2) \right]. \quad (3.35)$$

As one sees

$$E_f|_{\delta \ll 1} \approx m, \quad E_f|_{\delta \rightarrow 1} \rightarrow \infty.$$

II. A specific type of solutions to the nonlinear field equations in flat space-time were obtained in a series of interesting articles [18], [19], [20], [21]. These solutions are known as droplet-like solutions or simply droplets. Distinguishable property of these solutions is the availability of some sharp boundary, defining the space domain, in which the material field happens to be located i.e. the field is zero beyond this area. As was found the solutions mentioned exist in field theory with specific interactions that can be considered as effective, generated by initial interactions of the unknown origin. Contrary to the widely known soliton-like solutions, with field functions and energy density asymptotically tending to zero at spatial infinity, the solutions in question vanish at a finite distance from the center of the system (in the case of spherical symmetry) or from the axis (in the case of cylindrical symmetry). Thus, there exists the sphere or cylinder with critical radius r_0 , outside of which the fields disappear. Therefore the field configurations have the droplet-like structure [18], [22], [23].

Let us now choose the function $P(I)$ as follows [24] [see Figure 2]:

$$P(J) = J^{(1-2/\sigma)} [(1 - J)^{1/\sigma} - J^{1/\sigma}]^2 (1 - J), \quad (3.36)$$

where $J = GI$; $\sigma = 2n + 1$; $n = 1, 2, 3 \dots$. Then on account of $K = 0$ and $H = 1$ we get from (3.17) the following expression for $A(\xi)$ [see Figure 1]:

$$A(\xi \leq \xi_0) = 0, \quad A(\xi \geq \xi_0) = (1/\sqrt{G}) \left[1 - \exp\left(-\frac{2C\sqrt{G}}{\sigma}(\xi - \xi_0)\right) \right]^{\sigma/2}. \quad (3.37)$$

As one can see from (3.37), the conditions (3.18) for the center to be regular and the matching conditions (3.19) on the surface of the critical sphere are fulfilled if $\sigma > 2$. It is

also obvious from (3.37) that for $\xi < \xi_0$ the value of square bracket turns to be negative one and $A(\xi)$ becomes imaginary since σ is an odd number. Since we are interested in the real $A(\xi)$ only, without loss of generality we may assume the value of $A(\xi)$ be zero for $\xi \leq \xi_0$, the matching at $\xi = \xi_0$ being smooth.

Recalling that $J = G A^2 / (G A^2 + 1)$, we get from (3.37) that $J(\infty) = 1/2$ and $J(\xi_0) = 0$, thus implying:

$$P(I) |_{\xi=\infty} = P(I) |_{\xi=\xi_0} = 0. \quad (3.38)$$

It means that at $\xi = \xi_c = \infty$ and $\xi = \xi_0$, the interaction function $\Psi(I) = 1/P(I)$ is singular. It turns out nevertheless that the energy density T_0^0 is regular at these points due to the fact that it contains $\Psi(I)$ as a multiplier in the form:

$$e^{-2\alpha} \varphi'^2 \Psi = C^2 e^{-2\alpha} P(I), \quad (3.39)$$

which tends to zero as $\xi \rightarrow \xi_c$ or $\xi \rightarrow \xi_0$. As follows from (3.37), for the limiting case $\xi_0 = 0$, when the critical sphere goes to the spatial infinity and the solution in question is defined at $0 \leq \xi \leq \infty$, it appears that at spatial infinity ($\xi = 0$) $A = 0$ and $P(I) = 0$. In this case we obtain the usual soliton-like configuration not possessing any sharp boundary.

It should be emphasized that at spatial infinity ($\xi = 0$) one can compare the metric found with the Schwarzschild one and the electrical field with the Coulomb one, thus determining the total mass m and the charge q of the system:

$$Gm = -\gamma'(0), \quad q = -A'(0).$$

Taking into account that $e^{2\gamma} = G A^2 + 1$, one can find through the use of (3.37) that for $\xi_0 = 0$, $A'(0) = -q = 0$ and $\gamma'(0) = -Gm = 0$. Therefore, the total energy of the soliton-like system, defined as the sum of the material fields energy and that of the gravitational field, vanishes. If now one chooses the integration constant $\xi_0 > 0$, then the field configuration with the sharp boundary (droplet) appears. In this case for $\xi \leq \xi_0$ one obtains $A(\xi) = 0$ and $e^{2\gamma} = 1$, i.e. outside of the droplet gravitational and electromagnetic fields disappear, that implies the vanishing of the total mass and the charge of the system. This unusual property makes the droplet-like object poorly visible for the outer observer.

It should be emphasized that the field energy is localized in the region ($\xi_0 \leq \xi < \infty$):

$$T_0^0(\xi) |_{\xi \rightarrow \infty} \rightarrow 0, \quad T_0^0(\xi) |_{\xi \rightarrow \xi_0} \rightarrow 0, \quad (3.40)$$

namely, inside the critical sphere with the radius

$$R = \int_0^{\infty} d\xi e^{\alpha(\xi)} = \int_0^{\infty} d\xi / \xi^2 \{ [1 - e^{-2C\sqrt{G}(\xi-\xi_0)/\sigma}]^\sigma + 1 \}^{(1/2)} < \infty.$$

Taking into account that $e^{2\gamma} = 1/(1 - J)$ and $e^{-3\gamma} dA = dJ/2\sqrt{GJ}$, we rewrite total energy of the material fields in terms of J :

$$E_f = (C/4\sqrt{G}) \int_0^{1/2} \left\{ 4 \frac{d\sqrt{JP}}{dJ} + \frac{\sqrt{PJ}}{1-J} \right\} dJ.$$

Contribution of the first term of the foregoing equality is trivial for the choice of $P(I)$ in the form (3.36) as in this case $P(I)|_0 = P(I)|_{\lambda/2} = 0$. As $P(I)$ is positive and J lies in the interval $(0, 1/2)$, one estimates

$$E_f = \frac{C}{4\sqrt{G}} \int_0^{1/2} \frac{\sqrt{PJ}}{1-J} dJ > 0.$$

Let us note that we consider the constant C to be a positive one. Knowing that the total energy of the droplet-like object is zero this inequality implies the negativity of its gravitational energy. Thus the droplet-like configuration of the fields obtained is totally regular with zero total energy (including the energy of proper gravitational field) and null electric charge and remains unobservable to one located outside the sphere with radius R [24], [25]. In order to clarify the fact that the role of the gravitational field in forming the droplet-like configuration is not decisive it is worthwhile to compare the solution obtained with that in the flat space-time, described by the interval

$$ds^2 = dt^2 - dr^2 - r^2 [d\theta^2 + \sin^2\theta d\phi^2].$$

In the latter case the equation (2.3) admits the solution

$$\varphi'(r) = -C P(I)/r^2. \quad (3.41)$$

Substituting (3.41) into (2.4), one finds that the equation for the electromagnetic field can be solved by quadrature:

$$\int dA/\sqrt{P} = \pm C \left(\frac{1}{r} - \frac{1}{r_0} \right), \quad r_0 = \text{const}. \quad (3.42)$$

Note that the droplet-like configuration $A(r)$ will be similar to (3.37) if one chooses the function $P(I)$ more simple than (3.36):

$$P(I) = J^{1-2/\sigma} (1 - J^{1/\sigma})^2, \quad J = \lambda I, \quad (3.43)$$

where $\lambda = \text{const}$; $\sigma = 2n + 1$; $n = 1, 2, 3, \dots$. Then substituting (3.43) into (3.42) one gets the solution

$$A(r) = (1/\sqrt{\lambda}) [1 - \exp(-\frac{2C\lambda}{\sigma} (\frac{1}{r} - \frac{1}{r_0}))]^{\sigma/2}. \quad (3.44)$$

One can see from (3.44) that $A(r) = 0$ as $r \geq r_0$, i.e. the charge of the flat space-time droplet configuration also vanishes. For this solution the regularity conditions at the center $r = 0$ and on the surface of the critical sphere $r = r_0$ are evidently fulfilled. It similarly appears that for $r = \infty$ one finds the usual soliton-like structure with field vanishing as $r \rightarrow \infty$. The field energy E_f is defined as follows:

$$E_f = C \int_{A(r_0)}^{A(0)} dA (\sqrt{P} + I P_I / \sqrt{P}) = C \sqrt{P I} \Big|_{A(r_0)}^{A(0)}. \quad (3.45)$$

Inspecting that $P I = 0$ both at $r = 0$ and $r = r_0$, we arrive through (3.45) at $E_f = 0$. Thus in the flat space-time as well as for the self-gravitating system, the total energy and charge of the droplet-like configuration vanish.

4 Configurations with cylindrical symmetry

Obviously, in view of physics, the most interesting case is the spherically symmetric one, nevertheless in some cases it is necessary to study the two-dimensional cylindrically symmetric regular solutions in the vicinity of symmetry axis (vortex [26], string-like solutions [27]). These solutions can describe realistic objects like fluxion [28], light-beam [29] and can serve as the logical approximation to the objects with toroidal structure [30]. Let us now search for static cylindrically-symmetric solutions to the equations (2.2)-(2.4). In this case the metric can be chosen as follows [31], [32]:

$$ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} dx^2 - e^{2\beta} d\phi^2 - e^{2\mu} dz^2. \quad (4.1)$$

The requirements to be fulfilled by soliton-like solutions in this case are [34]

(a) Stationarity [applied to the metric (4.1)] i.e.

$$\alpha = \alpha(x), \quad \beta = \beta(x), \quad \gamma = \gamma(x), \quad \mu = \mu(x).$$

It means for (4.1) that all the components of the metrical tensor depend on the single spatial coordinate $x \in [x_0, x_a]$, where x_a is the value of x on the axis of symmetry, defined by the condition $\exp[\beta(x_a)] = 0$, and x_0 is the value of x on the surface of the critical cylinder. The coordinates z and ϕ take their standard values: $z \in [-\infty, \infty]$, $\phi \in [0, 2\pi]$.

(b) regularity of the metric and the matter fields in the whole space-time;

(c) localized in space-time (with finite field energy):

$$E_f = \int T_0^0 \sqrt{-^3g} dV < \infty.$$

Requirement (c) assumes the rapid decreasing of energy density of material field at spatial infinity, which together with (b) guaranties the finiteness of E_f . Let us note that E_f may be finite even for singular solutions on the axis. Requirement (b) means the regularity of material fields as well as the regularity of metric functions that entails the demand of finiteness of energy-momentum tensor of material fields all over the space. If the system considered contains scalar φ and electric \mathbf{E} (or magnetic \mathbf{H}) fields, the regularity conditions on $x = x_a$ take the form [31]:

$$\begin{aligned} e^\beta = 0; \quad |\gamma| < \infty; \quad |\mu| < \infty; \quad e^{2(\beta-\alpha)}(\beta')^2 = 1; \quad e^{-2\alpha}(\gamma')^2 = 0; \\ \{|\mathbf{E}| = 0; \quad |\mathbf{H}_\parallel| < \infty; \quad |\mathbf{H}_\perp| = 0\}; \quad |T_\mu^\nu| < \infty, \end{aligned} \quad (4.2)$$

where \mathbf{H}_\parallel and \mathbf{H}_\perp are the longitudinal and transverse magnetic fields defined as chronometric invariants [33]. In view of requirement (a) it is convenient to choose the coordinate x in (4.1) to satisfy the subsidiary condition [32]:

$$\alpha = \beta + \gamma + \mu,$$

that permits to present the system of the Einstein equations in the form:

$$\mu'' + \beta'' - V = -\kappa T_0^0 e^{2\alpha}, \quad (4.3)$$

$$\mu' \beta' + \beta' \gamma' + \gamma' \mu' = V = -\kappa T_1^1 e^{2\alpha}, \quad (4.4)$$

$$\gamma'' + \beta'' - V = -\kappa T_2^2 e^{2\alpha}, \quad (4.5)$$

$$\mu'' + \gamma'' - V = -\kappa T_3^3 e^{2\alpha}. \quad (4.6)$$

As in the preceding section, the electromagnetic field is described by the time component of the 4-potential $A_0(x) = A(x)$ and by the component $F_{10} = dA/dx = A'$ of the field strength tensor and the energy-momentum tensor of interacting fields is defined by the equations (3.6), (3.7).

Adding together the equations (4.4) and (4.5) and using (3.7), one obtains the simple equation:

$$\gamma'' + \beta'' = 0, \quad (4.7)$$

with the solution

$$\beta(x) + \gamma(x) = C_2 x, \quad C_2 = \text{const.} \quad (4.8)$$

Notice that the second integration constant in (4.8) can be taken trivial, as it determines only the choice of scale.

In a similar way the addition of equations (4.4) and (4.6) leads to the equation:

$$\gamma'' + \mu'' = 0, \quad (4.9)$$

with the solution

$$\mu(x) + \gamma(x) = C_3 x, \quad C_3 = \text{const.} \quad (4.10)$$

whereas the subtraction of (4.5) and (4.6) gives

$$\beta'' - \mu'' = 0, \quad (4.11)$$

with the solution

$$\beta(x) - \mu(x) = C_4 x, \quad C_4 = \text{const.} \quad (4.12)$$

Solving the equation (2.2) in the metric (4.1), one gets the same result as in (3.9), i.e.

$$\varphi'(x) = C P(I). \quad (4.13)$$

Substituting (4.13) into (2.4), one finds the equation for the electromagnetic field, coincident with (3.10) i.e.

$$(e^{-2\gamma} A')' - C^2 P_I e^{-2\gamma} A = 0, \quad (4.14)$$

where the second term could be naturally interpreted as the induced nonlinearity. Now as in the previous case, we use the equation (4.4) and sum of equations (4.3) and (4.4) which in view of (4.8) and (4.10), take the form:

$$\gamma'^2 - C_2 C_3 = -G(C^2 P - A'^2 e^{-2\gamma}), \quad (4.15)$$

$$\gamma'' = G e^{-2\gamma} (A'^2 + C^2 A^2 P_I). \quad (4.16)$$

Elimination of $P_I A$ between the equations (4.14) and (4.16) gives the equation

$$\gamma'' = G (A A' e^{-2\gamma})', \quad (4.17)$$

with the evident first integral:

$$\gamma' = G A A' e^{-2\gamma} + C_1, \quad C_1 = \text{const.} \quad (4.18)$$

Integrating (4.18) under the choice $C_1 = 0$, one again obtains

$$e^{2\gamma} = G A^2 + H, \quad H = \text{const.} \quad (4.19)$$

Finally, substituting γ' from (4.18) and $e^{2\gamma}$ from (4.19) into (4.15), one gets the equation for $A(x)$:

$$A'^2 (G A^2 + H)^{-2} = (G C^2 P - C_2 C_3) / G H. \quad (4.20)$$

The equation (4.20) can be solved by quadrature:

$$\int \frac{dA}{(G A^2 + H) \sqrt{G C^2 P - C_2 C_3}} = \pm (1/\sqrt{G H}) (x - x_0). \quad (4.21)$$

Let us formulate regularity conditions to be satisfied by the solutions to the equations (2.2)-(2.4) on the axis of symmetry defined by the value $x = x_a$, where $\exp[\beta(x_a)] = 0$. As according to the regularity conditions formulated earlier $|\gamma(x_a)| < \infty$ and $|\beta(x_a)| < \infty$ from (4.8) and (4.12) one gets $\beta(x) \approx C_2 x \rightarrow -\infty$ (whereas $x_a = -\infty$ if $C_2 > 0$ and $x_a = +\infty$ if $C_2 < 0$); $\beta(x) \approx C_4 x \rightarrow -\infty$ (whereas $x_a = -\infty$ if $C_4 > 0$ and $x_a = +\infty$ if $C_4 < 0$). It leads to $C_2 = C_4$, $\gamma(x) \equiv -\mu(x)$ and $\alpha(x) \equiv \beta(x)$. As one sees, from $\gamma(x) \equiv -\mu(x)$ follows $C_3 = 0$. The regularity conditions are similar to (3.18) for the case of spherical symmetry, implying that the following relations hold as $x \rightarrow x_a = \infty$:

$$\begin{aligned} \gamma' &\rightarrow 0, & A' &\rightarrow A_c \neq \infty, & A' &\rightarrow 0, \\ e^{2|C_2|x} P(I) &\rightarrow 0, & e^{2|C_2|x} |I P_I| &< \infty. \end{aligned} \quad (4.22)$$

Boundary conditions on the surface of the critical cylinder $x = x_a$ can be written as follows:

$$T_\mu^\nu = A = A' = 0, \quad e^\gamma = 1, \quad e^\beta = e^{-|C_2|x} > 0. \quad (4.23)$$

The conditions (4.23) together with the relations $e^{2\gamma} = G A^2 + H$, imply that $H = 1$. Therefore the metric (4.1) that satisfies the regularity conditions reads:

$$ds^2 = (G A^2 + 1) dt^2 - \frac{1}{(G A^2 + 1)} [e^{2C_2 x} \{dx^2 + d\phi^2\} + dz^2]. \quad (4.24)$$

As in the previous case, we will study the system for different $P(I)$.

I. Note that some class of regular solutions can be obtained choosing $P(I)$ in the form

$$P(I) = P_0 (\lambda I - N)^s Q(\lambda I), \quad (4.25)$$

where $Q(\lambda I)$ is some arbitrary, continuous, positive defined function, having non-trivial value at the center; λ is the coupling parameter; $N > 0$ is some dimensionless constant that is equal to the value of λI at the center. The other constant P_0 is defined from the condition $P = 1$ at spatial infinity $\xi = 0$. For $R = \text{const.}$ one gets the most simple form of $P(I)$ that leads to regular solutions. As in the spherically-symmetric case, for the regular solutions $\lambda \geq GN$.

a) Choosing $P(I)$ in the form

$$P(I) = P_0 (\lambda I - N)^2, \quad (4.26)$$

we get

$$A(\xi) = \sqrt{\frac{N}{\lambda - GN}} \text{th } bx, \quad (4.27)$$

where $b = \sqrt{C^2 NP_0(\lambda - GN)}$, the integration constant x_1 is taken to be trivial. The regularity condition implies $b \geq 1$. The metric function $e^{2\gamma}$, radial electric field and the total energy of the material field system can be written as

$$e^{2\gamma} = \frac{\lambda}{\lambda - GN} \left[1 - \frac{GN}{\lambda \text{ch}^2 bx} \right], \quad (4.28)$$

$$|\mathbf{E}| = |C| e^{\gamma - \beta} \sqrt{P(I)}, \quad (4.29)$$

$$E_f = \frac{\lambda C}{2G\sqrt{G}} \left[\frac{\sigma}{\sqrt{1 - \sigma^2}} \frac{\sqrt{1 - \sigma^2}}{2} \ln \frac{1 + \sigma}{1 - \sigma} \right], \quad (4.30)$$

where $\sigma^2 = GN/\lambda < 1$. As one sees $|\mathbf{E}| \rightarrow 0$ as $x \rightarrow \pm\infty$. The solution obtained satisfies all the regularity conditions and is a solitonian one. The density of mass (ρ_m) and the density of effective charge (ρ_e) are

$$\begin{aligned} \rho_m|_{x \rightarrow -\infty} &\rightarrow \begin{cases} \text{const} & b = 1; \\ 0 & b > 1; \end{cases} \\ \rho_m|_{x \rightarrow +\infty} &\rightarrow 0 \quad b \geq 1; \\ \rho_e|_{x \rightarrow -\infty} &\rightarrow \begin{cases} 2C^2\sqrt{G}(1 - \sigma^2)/\pi\sigma & b = 1; \\ 0 & b > 1; \end{cases} \\ \rho_e|_{x \rightarrow +\infty} &\rightarrow 0 \quad b \geq 1. \end{aligned}$$

The total charge of the system is equal to zero.

b) Let us consider the case with $I_c = 0$, choosing

$$P(I) = \lambda I. \quad (4.31)$$

In this case we get

$$A(\xi) = \frac{1}{\sqrt{G} \text{sh}(\sqrt{\lambda C} x)}. \quad (4.32)$$

The metric function $e^{2\gamma}$ in this case reads

$$e^{2\gamma} = \text{cth}^2(\sqrt{\lambda C} x), \quad (4.33)$$

that gives

$$e^{2\gamma}|_{x \rightarrow \pm\infty} \rightarrow 1, \quad e^{2\gamma}|_{x \rightarrow \pm 0} \rightarrow \infty.$$

Inasmuch $e^{2\beta} = e^{-2\gamma + 2C_2 x}$, $x = x_1 = -\infty$ corresponds to one of the axes of the field configurations. This axis is regular if $\sqrt{\lambda C} > 1$ and $A(x_1) = 0$ and $e^{2\gamma(x_1)} = 1$. So for $e^{2\gamma}|_{x \rightarrow \pm 0} \rightarrow \infty$, one gets $e^{2\beta}|_{x \rightarrow \pm 0} \rightarrow 0$, i.e. $x = x_2 = 0$ corresponds to the second, singular axis. In this case the solution obtained is defined on $-\infty \leq x \leq 0$. At $x \rightarrow +\infty$ $e^{2\beta}|_{x \rightarrow +\infty} \rightarrow \infty$ and $A(x) \rightarrow 0$. It means that $x = +\infty$ defines the spatial infinity. In this case the solution is defined on $0 \leq x \leq \infty$ and possesses one singular axis corresponding $x = 0$.

II. Let us now obtain the droplet-like configuration. Choosing $P(I)$ in the form [see Figure 2]:

$$P(J) = J^{(1-2/\sigma)} [(1 - J)^{1/\sigma} - J^{1/\sigma}]^2 (1 - J), \quad (4.34)$$

where $J = GI$; $\sigma = 2n + 1$; $n = 1, 2, 3 \dots$ one can find the expression for $A(x)$ which is similar to the one in spherically-symmetrical case [see Figure 1]:

$$A(x) = (1/\sqrt{G}) [1 - \exp(-\frac{2C\sqrt{G}}{\sigma}(x - x_0))]^{\sigma/2}. \quad (4.35)$$

As one can readily see from (4.35), the conditions (4.22) and (4.23) are fulfilled if $|C_2| \leq C\sqrt{G}/\sigma$. It is noteworthy that at $x \leq x_0$, $A(x) \equiv 0$ and the space-time is flat, the gravitational field being absent [35].

There is a principal difference between solutions (3.37) and (4.35). For the case of spherical symmetry the droplet-like solution can be transformed to the soliton-like one if the boundary ξ_0 is removed by putting $\xi_0 = 0$ (as in this case $\exp[\beta(\xi_0)] = 1/\xi_0 = \infty$). On the contrary, for the case of cylindrical symmetry the removal of the boundary is equivalent to putting $x_0 = -\infty$, as in this case $\exp[\beta(x_0)] = \exp(-|C_2|x_0) = \infty$. Under this last choice the solution (4.35) takes constant value $A(x) = 1/\sqrt{G}$ and the soliton structure disappears. For the considered case, as well as for that of spherical symmetry, the density of the field energy is given by equation (3.23) and the linear density of energy is similar to (3.24):

$$E_f = (C/4) \int_0^{1/\sqrt{G}} dA e^{-3\gamma} [\sqrt{P}(1 + e^{2\gamma}) + 4I(\sqrt{P})_I], \quad (4.36)$$

Substituting $P(I)$ from (4.34) into (4.36), one can find that E_f is finite and the total energy $E_f + E_g$ turns out to be zero.

Let us now define the effective charge density ρ_e and total charge Q , corresponding to the unit length on z-axis. In generally from (2.4) one gets [34]

$$j^\alpha = \frac{1}{4\pi} (\varphi_{,\beta} \varphi^{,\beta}) \Psi_I A^\alpha, \quad (4.37)$$

that for static radial electric field leads to

$$j^0 = \frac{C^2}{4\pi} e^{-2(\alpha+\gamma)} P_I A. \quad (4.38)$$

Then for chronometric invariant electric charge density ρ_e we have

$$\rho_e = \frac{j^0}{\sqrt{g^{00}}} = \frac{C^2}{4\pi} e^{-(2\alpha+\gamma)} P_I A. \quad (4.39)$$

The total charge is defined from the equality

$$Q = 2\pi \int_{x_a}^{x_\infty} \rho_e \sqrt{-^3g} dx. \quad (4.40)$$

Putting the corresponding quantities into the foregoing equality after some simple calculations we obtain

$$Q = \frac{1}{2} e^{-2\gamma} A' \Big|_{x_a}^{x_\infty} = 0. \quad (4.41)$$

Now it is worthwhile to make again the comparison with the flat-space solutions of the equations (2.3) and (2.4), using the interval:

$$ds^2 = dt^2 - d\rho^2 - \rho^2 d\phi^2 - dz^2.$$

In this case the scalar field equation (2.3) admits the solution:

$$\varphi'(\rho) = C P(I)/\rho, \quad P(I) = 1/\Psi(I), \quad C = \text{const.} \quad (4.42)$$

Inserting (4.42) into (2.4), one can find the electromagnetic field equation which admits the solution in quadratures:

$$\int \frac{dA}{\sqrt{P(I)}} = \pm C \ln \frac{\rho}{\rho_0}, \quad \rho_0 = \text{const.} \quad (4.43)$$

Substituting $P(I)$ from (4.34) in (4.43), one gets the solution of the droplet-like form:

$$A(\rho) = (1/\sqrt{\lambda}) [1 - (\frac{\rho}{\rho_0})^{2C\sqrt{\lambda}/\sigma}]^{\sigma/2}. \quad (4.44)$$

One concludes from (4.44) that $A(\rho \geq \rho_0) \equiv 0$. It means that the electric charge of the system is zero. For the solution (4.34) the regularity conditions both on the axis $\rho = 0$ and on the surface of the critical cylinder $\rho = \rho_0$ are fulfilled if $C\sqrt{\lambda} \geq \sigma$. It is noteworthy that in the case of cylindrical symmetry, both in the flat space-time and with account of the proper gravitational field, there do not exist any soliton-like solutions, as for the choice $\rho_0 = \infty$ the solution (4.44) degenerates into the constant: $A(\rho) = 1/\sqrt{\lambda}$. The linear density of the field energy in flat space-time can be found from the expression similar to (3.23), and as well as in the case of spherical symmetry, it is equal to zero:

$$E_f = \frac{C}{2} \sqrt{P I} \Big|_{A(\rho_0)}^{A(0)} = 0,$$

as was expected.

5 Discussion

Exact regular static spherically- and/or cylindrically-symmetrical particle-like solutions to the equations of scalar nonlinear electrodynamics in General Relativity have been obtained. As a particular case we found a class of regular solutions with sharp boundary (droplet-like solutions or simply droplets). It is shown that outside the droplet gravitational and electromagnetic fields remain absent i.e. total energy and total charge of the configuration are zero. We underline once more the principal difference between the droplet-like solutions with spherical symmetry and those with cylindrical one. In the first case there exists a possibility of continuous transformation of the droplet-like configuration into the solitonic one by transporting the sharp boundary to the infinity. As for the second case, there is no such a possibility, and the soliton-like configuration disappears when the boundary is smoothed tending to the infinity. Further we intend to study the interaction processes of droplets with external electromagnetic and gravitational fields and also the scattering of photons and electrons on droplets.

References

- [1] G. Mie 1912 *Ann. d. Phys.* **37**, 511; **39** 1
- [2] N. Rosen 1939 *Phys. Rev.*, **55**, (1), 94
- [3] I. G. Chugunov, Yu. P. Rybakov, and G. N. Shikin 1996 *Int. J. Theor. Phys.* **35**, (7), 1493
- [4] K. A. Bronnikov, and G. N. Shikin 1985 *Proc. of Sir A. Eddington Centenary Symp. on the Relativity Theory, Singapore, WS 2* 196
- [5] M. Novello, and J. M. Salim 1979 *Phys. Rev., D* **20** 377
- [6] H. Schiff 1969 *Canad. J. Phys.* **47** (21) 2387
- [7] J. C. Peckev, A. P. Roberts, and J. P. Vigiier 1972 *Nature* **237** 227
- [8] Alfred S. Goldhaber, and M. M. Nieto 1971 *Rev. Mod. Phys.* **43** 277
- [9] R. M. Woloshyn 1983 *Phys. Rev. D* **27** (6) 1393
- [10] A. Ljubicic, K. Pisk, and B. A. Loggan 1979 *Phys. Rev. D* **20** (4) 1016
- [11] M. Novello, and H. Heintzmann 1983 *Phys. Lett. A* **98** (1,2) 10
- [12] F. Bisshopp 1972 *Inter. J. Theor. Phys.* **6** (3) 225
- [13] J. Schwinger 1951 *Phys. Rev.* **82** (5) 664
- [14] K. A. Bronnikov, and M. A. Kovalchuk 1980 *J. Phys. A: Math. Gen.* **3** 187
- [15] K. A. Bronnikov, Yu. P. Rybakov, and G. N. Shikin 1993 *Comm. in Theor. Phys.* **2** 19
- [16] K. A. Bronnikov, V. N. Mel'nikov, G. N. Shikin, and K. P. Stanukovich 1979 *Ann. Phys.* **118** (1) 84
- [17] K. A. Bronnikov 1973 *Acta Physica Polonica* **B 4** (2) 251
- [18] J. Werle 1977 *Phys. Lett. B* **71** (2) 367
- [19] J. Werle 1980 *Phys. Lett. B* **95** (3,4), 391
- [20] J. Werle 1981 *Acta Physica Polonica* **B 12** (6) 601
- [21] J. Werle 1988 *Acta Physica Polonica* **B 19** (3) 203
- [22] K. A. Bronnikov, Yu. P. Rybakov, and G. N. Shikin 1991 *Izvestiya Vuzov, Physics* (5) 24
- [23] Yu. P. Rybakov, B. Saha, and G. N. Shikin 1994 *Problems of Statistical Physics and Field Theory, Moscow, PFU* 18

- [24] Yu. P. Rybakov, B. Saha, and G. N. Shikin 1994 *Communications in Theor. Phys.* **3** (1) 67
- [25] Yu. P. Rybakov, B. Saha, and G. N. Shikin 1992 *GR 13, Cordoba, Argentina* 66
- [26] H. B. Nielsen and P. Olesen 1973 *Nucl. Phys. B* **51**, (1), 45
- [27] Ya. P. Terletsky *Docl. Acad. Nauk USSR* **236**, (4), 828
- [28] A. A. Abrikosov 1957 *Sov J. Exp. Theor. Phys.* **32**, (6), 1442
- [29] V. E. Zakharov, V. V. Sobolev and V. S. Synakh 1971 *Sov J. Exp. Theor. Phys.* **70**, (1), 136
- [30] H. J. de Vega 1978 *Phys. Rev. D* **18**, (8), 2945
- [31] K. A. Bronnikov 1979 *Prob. Theor. Grav. and Elem. Part. Moscow, Energoizdat* (10) 37
- [32] G. N. Shikin 1984 *Prob. Theor. Grav. and Elem. Part. Moscow, Energoizdat* (14) 85
- [33] N. V. Mitskevich 1969 "Physcal Fields in General Relativity" *Nauka, Moscow*
- [34] G. N. Shikin 1995 "Theory of Solitons in General Relativity" *URSS, Moscow*
- [35] Yu. P. Rybakov, B. Saha, and G. N. Shikin 1993 *8 Russian Gravitation Conf., RGA* 193

Caption of figures

Figure 1. Perspective view of droplet-like solution. The configurations are plotted for $\lambda = 1$, $x_0 = 2$ and σ takes the values 3, 5, 7, 9. (Note that in the figures illustrated here 3, 5, 7, 9 correspond to the value of σ).

Figure 2. Perspective view of the inverse function to the interaction one (i.e. $P(I)$) that provides us with the droplet-like configurations (Figure 1). As is seen from Figure 1, the stronger the interaction the more localized the corresponding droplet-like configuration.