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GEOMETRIC REPRESENTATIONS OF $\mathrm{GL}(n, R)$, CELLULAR HECKE ALGEBRAS AND THE EMBEDDING PROBLEM

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Abstract. We study geometric representations of $GL(n, R)$ for a ring R. The structure of the associated Hecke algebras is analyzed and shown to be cellular. Multiplicities of the irreducible constituents of these representations are linked to the embedding problem of pairs of R-modules $x \subset y$.

1. Introduction

Let R be a ring and let $F = R^n$ be the free module of rank n. In this paper we study families of representations of $G = GL(n, R) = \mathrm{Aut}_R(F)$ which arise from its action on the lattice of submodules of F. More precisely, let ${}_R\mathcal{M}$ denote the category of finitely generated left R-modules, and let $\lambda \in \mathrm{Iso}({}_R\mathcal{M})$ be an isomorphism type of a submodule of F. Let $X_\lambda = \mathrm{Gr}(\lambda, F)$ be the Grassmannian of submodules of type λ in F. Let $\mathcal{F}_\lambda$ be the vector space of $\mathbb{Q}$-valued functions on X_λ. We define a family of representations of G

$$\begin{aligned} \rho_\lambda : G &\longrightarrow \mathrm{Aut}_{\mathbb{Q}}(\mathcal{F}_\lambda) \\ g &\longmapsto [\rho_\lambda(g)f](x) = f(g^{-1}x). \end{aligned}$$

For each representation $\mathcal{F}_\lambda$, let $\mathcal{H}_\lambda = \mathrm{End}_G(\mathcal{F}_\lambda)$ be the Hecke algebra associated to it. The aim of this paper is two fold:

First, we analyze the structure of the Hecke algebras $\{\mathcal{H}_\lambda\}$ for a distinguished family of λ's and establish their cellular structure, which is a consequence of the rich underlying geometrical/combinatorial structure. This is a generalization of [5] in which the ring in question is a discrete valuation ring. Moreover, the technics used here are valid in a much broader setting, when F is an object in a category $\mathcal{C}$ satisfying some axioms, and the group G is $\mathrm{Aut}_{\mathcal{C}}(F)$.

Second, an equality is established between multiplicities of a family of irreducible representations $\{\mathcal{U}_\mu\}_{\mu \in \mathrm{Iso}({}_R\mathcal{M})}$ in the $\mathcal{F}_\lambda$'s and the cardinality of non-equivalent pairs $x \subset y$ of R-modules. As a result we give a representation theoretic view on the embedding problem [7, 9].

1.1. Description of results.

To make things concrete, we explain our results in the special case $R = \mathcal{O}$, a discrete valuation ring with finite residue field. Let $\mathcal{O}$ be such a ring, let $\wp$ be its maximal ideal and let $\mathcal{O}_k = \mathcal{O}/\wp^k$ for some $k \in \mathbb{N}$. By the principal divisors theorem, any finite $\mathcal{O}$-module is of the form $M = \oplus_{i=1}^{l} \mathcal{O}/\wp^{\lambda_i}$ with decreasing exponents and its isomorphism type is the partition $\lambda = (\lambda_1, \ldots, \lambda_l)$. The set $\mathcal{T} = \mathrm{Iso}({}_R\mathcal{M})$ of isomorphism types of R-modules carries a natural structure of a poset, the partial order defined by: $\lambda \leq \nu$ if and only if a module of type λ can be embedded in a module of type ν. Denote by $\{\lambda \hookrightarrow \nu\}$ the set of arrow types with source of type λ and range of type ν, and fix $\phi = k^m = (k, \ldots, k)$ where $m \leq n/2$. Then the following theorem is a specialization to $R = \mathcal{O}$ of Theorem 3.

Theorem 1. *There exists a collection of non-equivalent irreducible G-representations $\{\mathcal{U}_\lambda\}_{\lambda \leq \phi}$ such that:*

(1) $\mathcal{F}_\phi = \bigoplus_{\lambda \leq \phi} \mathcal{U}_\lambda$.

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(2) *For every* $\lambda, \nu \leq \phi$*:*

$$\langle \mathcal{U}_\lambda, \mathcal{F}_\nu \rangle = |\{\lambda \hookrightarrow \nu\}|$$

I.e., the multiplicity of $\mathcal{U}_\lambda$ *in* $\mathcal{F}_\nu$ *is the number of non-equivalent embeddings of a module of type* λ *in a module of type* ν*. In particular* $\mathcal{U}_\lambda$ *appears in* $\mathcal{F}_\lambda$ *with multiplicity one and does not appear in* $\mathcal{F}_\mu$ *unless* $\lambda \leq \nu$*.*

For a general ring R, we isolate the properties of the type $\phi = k^m$ in the above theorem, and show that it is still valid for a general ring for types satisfying these properties (see Definitions 2.1 and 2.2).

Apart from the representations, another central object in this work is the Hecke algebra $\mathcal{H}_\phi = \mathrm{End}_G(\mathcal{F}_\phi)$. This algebra is commutative and its irreducible representations correspond bijectively to the irreducibles of G which occur in $\mathcal{F}_\phi$. We show that this algebra has a cellular structure, intimately related to the poset of isomorphism types. One of the main results concerning the Hecke algebra is the existence of two families of geometrically defined ideals of $\mathcal{H}_\phi$, $\{\mathcal{H}_\phi^\lambda\}_{\lambda \leq \phi}$ and $\{\mathcal{H}_\phi^{\lambda^-}\}_{\lambda \leq \phi}$, such that for each λ, $\mathcal{K}_\lambda = \mathcal{H}_\phi^\lambda / \mathcal{H}_\phi^{\lambda^-}$ is one dimensional representation of $\mathcal{H}_\phi$, and

Theorem 2.

(1) $\{\mathcal{K}_\lambda\}_{\lambda \leq \phi}$ *forms a complete set of irreducible representations of* $\mathcal{H}_\phi$*.*
(2) *As* $\mathcal{H}_\phi$*-modules,*

$$\mathcal{H}_\phi^\lambda \simeq \bigoplus_{\mu \leq \lambda} \mathcal{K}_\mu$$

1.2. **About the organization of this paper.** The paper is organized as follows. In Section 2 we give notations, definitions and basic results on incidence algebras. These are important ingredients in the dictionary between combinatorial data, encoded in the poset structure, and the algebraic or representation theoretic structure we seek to expose. Section 3 is the core of this manuscript. There we define and study the Hecke algebra and Hecke modules associated to the various representations under consideration. In section 4 we open a discussion on the explicit computation of the idempotents of the Hecke algebra. They are expressed in terms of combinatorial invariants of the lattice of submodules. As an illustration we give a new proof of the Fourier decomposition for the case $R = \mathbb{F}_q$, the finite field with q elements. In a sequel paper [1] we push these results further and give the Fourier decomposition for the case $R = \mathcal{O}$, a discrete valuation ring with finite residue field. In Section 5 we show how to general most of the results in this paper to a much general framework where the category ${}_R\mathcal{M}$ is replaced with a general category.

1.3. **Relevant references.** Some of the examples considered here (see 5.2) have already appeared in the literature. Using different methods, Grassmannians of sets and vector spaces were studied in [3], [4] respectively. This study was carried on in [10] to somewhat more general Grassmannians related to finite Chevalley groups. One of the main themes in these papers concerns the relations between representations of the groups and orthogonal polynomials. In a different perspective, the study of Grassmannians of finite modules over discrete valuation rings was initiated by G. Hill in [5]. Hill was motivated by the classification of irreducible representations of $\mathrm{GL}_n(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a non-archimedean local field. The main theorem in [5] turns out to be a special case of Theorem 4. Furthermore, the explicit computation of the idempotents in this case will be carried out in a subsequent paper [1], using a delicate version of the method developed in Section 4.

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2. The setup

2.1. **Grassmannians.** Let R be a ring and $\mathcal{M}$ the category of finitely generated R-modules. We denote by $\vec{\mathcal{M}}$ the category whose objects are injections in $\mathcal{M}$, and the set of morphisms between two injections $x \overset{i}{\hookrightarrow} y$ and $x' \overset{i'}{\hookrightarrow} y'$ is the set of commutative squares

$$\begin{array}{ccc} x & \overset{i}{\hookrightarrow} & y \\ \downarrow & & \downarrow \\ x' & \overset{i'}{\hookrightarrow} & y' \end{array} \tag{1}$$

There is a natural notion of subobjects in $\vec{\mathcal{M}}$, namely, i is a subobject of i', denoted $i \leq i'$, if the vertical arrows in (1) are inclusions. Let $\mathcal{T} = \pi_0(\mathcal{M})$ be the set of *types* (isomorphism classes) in $\mathcal{M}$; Similarly, let $\vec{\mathcal{T}}$ be the set of types in $\vec{\mathcal{M}}$. $\mathcal{T}$ and $\vec{\mathcal{T}}$ have natural poset structure: $\xi \leq \eta$ if an object of type ξ can be a subobject of an object of type η. Let $\tau : \mathcal{M} \to \mathcal{T}$ and $\vec{\tau} : \vec{\mathcal{M}} \to \vec{\mathcal{T}}$ be the *type maps*.

For $y \in \mathrm{Ob}(\mathcal{M})$ let $\mathcal{M}_y$ be the lattice of submodules of y. Denote $\eta = \tau(y)$. We define the *Grassmannian of submodules of type ξ in y* and the *Grassmanniann of submodules with embedding type ι*

$$\binom{y}{\xi} = \{x \in \mathcal{M}_y \mid \tau(x) = \xi\}\ , \qquad \binom{y}{\iota} = \{x \in \mathcal{M}_y \mid \vec{\tau}(x \subset y) = \iota\}$$

their cardinality are denoted $\binom{\eta}{\xi}$ and $\binom{\eta}{\iota}$ respectively. There is a natural map $\vec{\mathcal{T}} \to \mathcal{T}^2$ taking a morphism type to its source and range types. We denote by $\{\xi \hookrightarrow \eta\}$ the preimage of (ξ, η) under this map.

Note that $\binom{y}{\xi}$ and $\binom{y}{\iota}$ are an $\mathrm{Aut}(y)$-spaces. There are $|\{\xi \hookrightarrow \eta\}|$ many $\mathrm{Aut}(y)$-orbits in $\binom{y}{\xi}$, thus

$$\binom{\eta}{\xi} = \sum_{\iota \in \{\xi \hookrightarrow \eta\}} \binom{\eta}{\iota}.$$

Of special interest is the case where the action is transitive. If y is a symmetric object, then the action of $\mathrm{Aut}(y)$ is transitive.

Definition 2.1. *We say that a type ϕ is symmetric if for any f and f' of type ϕ, $x \subseteq f$ and $x' \subseteq f'$ with an isomorphism $h : x \xrightarrow{\sim} x'$, the following diagram could be completed:*

$$\begin{array}{ccc} f & \overset{\sim}{\dashrightarrow} & f' \\ \cup & & \cup \\ x & \xrightarrow{h} & x' \end{array}$$

An object of symmetric type will be called a symmetric object.

In the category of finite $\mathcal{O}$-modules, a module f is symmetric if and only if it is free over $\mathcal{O}/\mathrm{Ann}(f)$. Furthermore, for any pair $f \leq F$ of such modules, of types ϕ and Φ respectively, with $2 \cdot \mathrm{rank}(f) \leq \mathrm{rank}(F)$, the couple (Φ, ϕ) is symmetric.

Definition 2.2. *Let $\phi \leq \Phi$ be symmetric types. The couple (Φ, ϕ) is called a* symmetric couple *if for every object F of type Φ, every $f \subseteq F$ of type ϕ and for every $\lambda \leq \phi$,*

1. *There exist $f' \subseteq F$ such that $\tau(f') = \phi$ and $\tau(f \wedge f') = \lambda$.*
2. *For every $x \subseteq F$ such that $\tau(x) \leq \phi$ the square*

$$\begin{array}{ccc} x & \subseteq & x \vee f \\ \cup & & \cup \\ x \wedge f & \subseteq & f \end{array} \tag{$\diamondsuit$}$$

is cartesian in $\mathcal{M}$.

Remark 2.3.

(1) *The symbols* $\wedge$ *and* $\vee$ *stand for the meet and the join in the lattice of submodules in* F.
(2) *Explicitly, the second part of definition* (2.2) *is equivalent to* $x \vee f = x + f \simeq x \oplus f/x \cap f$, *with* $x \wedge f = x \cap f$ *embedded in* $x \oplus f$ *via the diagonal inclusion maps.*

The properties of being symmetric and symmetric couple will be used in the sequel to give a parametrization of couples of submodules up to automorphisms of the ambient module, and therefore parameterize bases of some Hecke algebras (see §3.4).

2.2. **Incidence Algebras.** Let $(\mathcal{P}, \leq)$ be a finite (more generally - locally finite) poset and let $\mathcal{P}^{(2)} \subseteq \mathcal{P} \times \mathcal{P}$ be the set of all (x, y) satisfying the relation $x \leq y$. Following [8], define the **incidence algebra** of $\mathcal{P}$ denoted by $I(\mathcal{P})$ to be the collection of $\mathbb{Q}$-valued functions on $\mathcal{P}^{(2)}$ with product given by:

$$[f * g](x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y), \qquad x \leq y$$

Endowed with this product, $I(\mathcal{P})$ becomes a unital associative $\mathbb{Q}$-algebra with unit element given by Kronecker's delta function. This algebra has two distinguished elements which will be used frequently in the sequel: the **zeta function** $\zeta_{\mathcal{P}} = \mathbf{1}_{\mathcal{P}^{(2)}}$, and its inverse the **Möbius function** denoted by $\mu_{\mathcal{P}}$ (cf. [8]).

The Möbius function can be calculated inductively - for all $x \in \mathcal{P}$ set $\mu_{\mathcal{P}}(x, x) = 1$ and assuming $\mu_{\mathcal{P}}(x, z)$ has been calculated for all $z \in [x, y)$, set:

$$\mu_{\mathcal{P}}(x, y) = -\sum_{x \leq z < y} \mu_{\mathcal{P}}(x, z) \tag{2}$$

In the special case of $x = 0$, we denote $\chi(y) = \mu_{\mathcal{P}}(0, y)$. This notation is justified since equation 2 shows that $\mu_{\mathcal{P}}(0, y)$ is the Euler characteristic of the flag complex associate to the poset $[0, y]$.

Observation 2.4. $\mu_{\mathcal{P}}(x, y)$ *depends merely on the poset* $[x, y]$. *More formally,*

$$\mu_{\mathcal{P}}(x, y) = \mu_{[x,y]}(0, 1)$$

The vector space of $\mathbb{Q}$-valued functions on $\mathcal{P}$, which we denote by $V(\mathcal{P})$, has a natural structure of a module over the incidence algebra $I(\mathcal{P})$. The action is given by:

$$f \cdot v(x) = \sum_{y \geq x} f(x, y) v(y) \qquad f \in I(\mathcal{P}), v \in V(\mathcal{P})$$

Let $(\mathcal{P}', \leq')$ be another poset, and let $\tau : \mathcal{P} \to \mathcal{P}'$ be a poset map. There exist a natural map from $V(\mathcal{P})$ to $V(\mathcal{P}')$ given by summation along fibers:

$$\tau_\star v(\xi) = \sum_{\tau(x) = \xi} v(x) \qquad v \in V(\mathcal{P}),\ \xi \in \mathcal{P}'$$

We would have liked to define a similar map from the incidence algebras $I(\mathcal{P})$ to $I(\mathcal{P}')$ which will be a homomorphism. In general this is not possible. Nevertheless, such a morphism can be defined on a subalgebra of $I(\mathcal{P})$. Denote

$$J(\mathcal{P}) = \left\{ f \in I(\mathcal{P}) \mid \forall y_1, y_2 \in \mathcal{P} \text{ with } \tau(y_1) = \tau(y_2),\ \forall x' \in \mathcal{P}',\ \sum_{x \in \tau^{-1}(x')} f(x, y_1) = \sum_{x \in \tau^{-1}(x')} f(x, y_2) \right\}$$

It is easily verified that $J(\mathcal{P})$ is in fact a subalgebra of $I(\mathcal{P})$. Define $\tau_\star^{(1)} : J(\mathcal{P}) \to I(\mathcal{P}')$ by:

$$[\tau_\star^{(1)} f](x', y') = \sum_{x \in \tau^{-1}(x')} f(x, y) \qquad x' \in \mathcal{P}',\ y \in \mathcal{P},\ \tau(y) = y'$$

Proposition 2.5.

(1) *The map* $\tau_\star^{(1)}$ *is a well defined homomorphism of algebras.*

(2) *The following diagram is commutative:*

$$\begin{array}{ccc} J(\mathcal{P}) \times V(\mathcal{P}) & \to & V(\mathcal{P}) \\ \tau_\star^{(1)} \downarrow \quad \tau_\star \downarrow & & \tau_\star \downarrow \\ I(\mathcal{P}') \times V(\mathcal{P}') & \to & V(\mathcal{P}') \end{array}$$

(the horizontal maps are the module maps)

Proof. Direct calculation. □

Given $h \in J(\mathcal{P})$, we shall denote its image in $I(\mathcal{P}')$ by $\hat{h} = \tau_\star^{(1)}(h)$. Let us introduce the following property on the triple $(\mathcal{P}, \mathcal{P}', \tau)$.

(♣) *$\mathcal{P}$ has a 0 element, and for every $y_1, y_2 \in \mathcal{P}$ with $\tau(y_1) = \tau(y_2)$, the posets $[0, y_1]$ and $[0, y_2]$ are isomorphic over $\mathcal{P}'$, that is there is a poset isomorphism $\pi : [0, y_1] \to [0, y_2]$ with $\tau \circ \pi = \tau$.*

Whenever property (♣) is satisfied, one easily sees that both $\zeta_{\mathcal{P}}$ and $\mu_{\mathcal{P}}$ are in $J(\mathcal{P})$. For the two cases of interest for us, the posets of submodules $\mathcal{M}_z$ and the poset of subinjections $\vec{\mathcal{M}}_z$ with the type maps this is indeed the case. We now specify the discussion to them.

Claim 2.6. *Fix an R-module z and consider $\mathcal{M}_z$, the poset of its submodules. For every $x \leq y \in \mathcal{M}_z$, the interval $[x, y]$ is isomorphic as posets to the interval $[0, y/x]$. It follows that $\mu_{\mathcal{M}_z}(x, y) = \chi(y/x)$. Consider the type map $\tau : \mathcal{M}_z \to \mathcal{T}$. Let $\alpha \leq \beta$ be types, then*

$$\hat{\mu}(\alpha, \beta) = \sum_{\iota \in \{\alpha \hookrightarrow \beta\}} \sum_{x : \vec{\tau}(x \subseteq y) = \iota} \mu(x, y) = \sum_{\iota \in \{\alpha \hookrightarrow \beta\}} \binom{\beta}{\iota} \chi(\mathrm{coker}(\iota))$$

If furthermore β is symmetric, then for every $x \leq y$ of types α and β, the type of y/x is the same, and will be denote β/α. For a fixed module y of type β we have

$$\hat{\mu}(\alpha, \beta) = \sum_{x \in \tau^{-1}\alpha} \mu(x, y) = \sum_{x \in \tau^{-1}\alpha} \chi(\beta/\alpha) = \binom{\beta}{\alpha} \chi(\beta/\alpha)$$

Example 2.7. *Assume R is a finite field with q elements. The flag complex associated to the poset $[0, x]$ is nothing but the Tits building associated to x, and it is known [2, 8] that*

$$\chi(x) = (-1)^{\dim(x)} q^{\binom{\dim(x)}{2}},$$

therefore,

$$\mu_{\mathcal{M}_z}(x, y) = (-1)^{\dim(y) - \dim(x)} q^{\binom{\dim(y) - \dim(x)}{2}},$$

and

$$\hat{\mu}(m, n) = (-1)^{n-m} q^{\binom{n-m}{2}} \binom{n}{m}_q.$$

Claim 2.8. *Fix an R-module z and consider $\vec{\mathcal{M}}_z$, the poset of the subinclusions of the identity map $z \to z$. Consider the type map $\vec{\tau} : \vec{\mathcal{M}}_z \to \vec{\mathcal{T}}$. For a type $\alpha \leq \beta$ and an inclusion type $\iota \in \{\alpha \hookrightarrow \beta\}$ we have*

$$\hat{\mu}(\iota, \beta) = \binom{\beta}{\iota} \chi(\mathrm{coker}(\iota))$$

3. Representations

3.1. **Representations arising from Grassmannians.** Let F be a symmetric module in ${}_R\mathcal{M}$ of type Φ (recall Definition 2.1). For each type $\lambda \in \mathcal{T}$ let $X_\lambda = \binom{F}{\lambda}$ be the Grassmannian of submodules of F of type λ. Let $G = \mathrm{Aut}_R(F)$. As we mentioned in §2, F being symmetric implies that X_λ is a homogeneous G-space. Let $\mathcal{F}_\lambda$ stand for $\mathbb{Q}$-valued functions on X_λ. Then the $\mathcal{F}_\lambda$'s become a family of representations of G:

$$\begin{aligned} \rho_\lambda : G &\longrightarrow \mathrm{Aut}_{\mathbb{Q}}(\mathcal{F}_\lambda) \\ g &\longmapsto [\rho_\lambda(g)f](x) = f(g^{-1}x) \end{aligned}$$

$\mathcal{F}_\lambda$ is equipped with the standard G-invariant inner product:

$$(f,g) = \sum_{x \in X_\lambda} f(x)g(x) \qquad \forall f,g \in \mathcal{F}_\lambda$$

Fix once and for all $\phi \leq \Phi$ which forms a symmetric couple (Definition 2.2). Recall that $\{\lambda \hookrightarrow \mu\}$ stands for the set of arrow types with source of type λ and range of type μ. The main theorem in this section is:

Theorem 3. *Let F be a module in ${}_R\mathcal{M}$ of symmetric type Φ, let ϕ be a type such that $\phi \leq \Phi$ is a symmetric couple, and let $G = \mathrm{Aut}_R(F)$. Then, there exists a collection of non-equivalent irreducible G-representations $\{\mathcal{U}_\lambda\}_{\lambda \leq \phi}$ such that:*

1. *$\mathcal{F}_\phi = \bigoplus_{\lambda \leq \phi} \mathcal{U}_\lambda$.*
2. *For every $\lambda, \nu \leq \phi$:*
$$\langle \mathcal{U}_\lambda, \mathcal{F}_\nu \rangle = |\{\lambda \hookrightarrow \nu\}|$$
I.e., the multiplicity of $\mathcal{U}_\lambda$ in $\mathcal{F}_\nu$ is the number of non-equivalent embeddings of a module of type λ in a module of type ν. In particular $\mathcal{U}_\lambda$ appears in $\mathcal{F}_\lambda$ with multiplicity one and does not appear in $\mathcal{F}_\nu$ unless $\lambda \leq \nu$.

Proof. Postponed to § 3.6. □

3.2. **An equivalence of categories.** Let G be a group and denote by $\mathcal{M}_G$ the category of its finite dimensional representations. For any $V \in \mathcal{M}_G$ we can slice $\mathcal{M}_G$ and look at the full subcategory $\mathcal{M}_{G,V}$ which consists of representations generated by irreducibles appearing in V. Let $\mathcal{H}_V = \mathrm{End}_G(V)$ be the *Hecke algebra* associated with V and let $\mathcal{M}_{\mathcal{H}_V}$ be the category of finitely generated left $\mathcal{H}_V$-modules.

Proposition 3.1. *$\mathcal{M}_{G,V}$ and $\mathcal{M}_{\mathcal{H}_V}$ are equivalent categories. The irreducible representations in V are in one to one correspondence with the irreducible $\mathcal{H}_V$-modules.*

Proof. The functors:

$$\begin{aligned} \mathcal{M}_{G,V} &\longrightarrow \mathcal{M}_{\mathcal{H}_V} \\ U &\longmapsto \mathrm{Hom}_G(V,U) \end{aligned} \qquad\qquad \begin{aligned} \mathcal{M}_{\mathcal{H}_V} &\longrightarrow \mathcal{M}_{G,V} \\ M &\longmapsto V \otimes_{\mathcal{H}_V} M \end{aligned}$$

forms an equivalence of categories. □

For $\lambda, \mu \in \mathcal{T}$ set $\mathcal{N}_{\mu,\lambda} = \mathrm{Hom}_G(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ and $\mathcal{H}_\lambda = \mathcal{N}_{\lambda,\lambda}$. Composition gives paring:

$$\mathcal{N}_{\lambda,\mu} \times \mathcal{N}_{\mu,\nu} \longrightarrow \mathcal{N}_{\lambda,\nu}$$

which turns $\mathcal{H}_\lambda$ into an algebra and $\mathcal{N}_{\mu,\lambda}$ into an $\mathcal{H}_\lambda$-$\mathcal{H}_\mu$-bimodule. Our main goal is to study $\mathcal{H}_\phi$ and its modules. It turns out that it is fruitful to have a slightly broader picture.

3.3. **Notations for intertwining operators.**

- Given an embedding type $\lambda \underset{i}{\hookrightarrow} \mu$, attach to it the following operators (they are transpose of each other):

$$\begin{aligned}
\mathbf{T}_{\mu \underset{i}{\hookleftarrow} \lambda} :& \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\mu \\
& h \longmapsto \mathbf{T}_{\mu \underset{i}{\hookleftarrow} \lambda} h(y) = \sum_{x : \vec{\tau}(x \subseteq y) = i} h(x), \qquad y \in X_\mu \\
\mathbf{T}_{\lambda \underset{i}{\hookrightarrow} \mu} :& \mathcal{F}_\mu \longrightarrow \mathcal{F}_\lambda \\
& h \longmapsto \mathbf{T}_{\lambda \underset{i}{\hookrightarrow} \mu} h(x) = \sum_{y : \vec{\tau}(x \subseteq y) = i} h(y), \qquad x \in X_\lambda
\end{aligned}$$

- Given $\lambda \leq \mu$ we can define the maps:

$$\mathbf{T}_{\mu \succ \lambda} = \sum_{\lambda \underset{i}{\hookrightarrow} \mu} \mathbf{T}_{\mu \underset{i}{\hookleftarrow} \lambda} \qquad\qquad \mathbf{T}_{\lambda \prec \mu} = \sum_{\lambda \underset{i}{\hookrightarrow} \mu} \mathbf{T}_{\lambda \underset{i}{\hookrightarrow} \mu}$$

  i.e. averaging over all submodules (supmodules) of same type regardless of the embedding. If $\mu = \phi$ is a symmetric type these sums consist of a unique summand each, as there is unique embedding. Then the two notations degenerate: $\mathbf{T}_{\phi \succ \lambda} = \mathbf{T}_{\phi \hookleftarrow \lambda}$ and $\mathbf{T}_{\lambda \prec \phi} = \mathbf{T}_{\lambda \hookrightarrow \phi}$.

In order to minimize confusion, we follow the rule that whenever an operator is labeled with a diagram, it acts from the space indexed by the *right* type of the diagram to the space indexed by the *left* type.

3.4. **The Hecke algebra $\mathcal{H}_\phi$.**

3.4.1. *The geometric basis of $\mathcal{H}_\phi$.* Let $X_\phi \times_G X_\phi$ be the equivalence classes of the diagonal action of G on $X_\phi \times X_\phi$. The set $X_\phi \times_G X_\phi$ has a natural parametrization:

$$X_\phi \times_G X_\phi \simeq \{\lambda | \lambda \leq \phi\} = [0, \phi]$$

This isomorphism is given by the (G-invariant map) $X_\phi \times X_\phi \to [0, \phi]$, given by $(x, y) \mapsto \tau(x \wedge y)$. This map is onto by definition 2.2 part (1) and is one-to-one by part (2). Indeed, for (x, y) and (x', y') in $X_\phi \times X_\phi$, if $\tau(x \wedge y) = \tau(x' \wedge y')$ then the following diagram of inclusions could be completed to a commutative one using vertical isomorphisms:

$$\begin{array}{ccccc}
x & \supseteq & x \wedge y & \subseteq & y \\
|\wr & & |\wr & & |\wr \\
x' & \supseteq & x' \wedge y' & \subseteq & y'
\end{array} \tag{D1}$$

By the cartesianity of the diagram ($\diamondsuit$) in Definition 2.2, these isomorphisms extend to $x \vee y \simeq x' \vee y'$. Finally, since F is symmetric this isomorphism extends to an automorphism of F. By viewing functions as integration kernels $\mathcal{H}_\phi$ is seen to be isomorphic to $\mathcal{F}(X_\phi \times_G X_\phi)$, which in turn is isomorphic to $\mathcal{F}([0, \phi])$ (by the above discussion). Composing these isomorphisms we obtain the vector space isomorphism

$$\begin{aligned}
\mathcal{F}([0, \phi]) &\xrightarrow{\sim} \mathcal{H}_\phi \\
\mathbf{1}_\lambda &\mapsto \mathbf{g}_\lambda
\end{aligned}$$

where

$$[\mathbf{g}_\lambda h](x) = \sum_{\{y \in X_\phi | \tau(x \wedge y) = \lambda\}} h(y) \qquad x \in X_\phi,\ h \in \mathcal{F}_\phi$$

Remark 3.2. *It is useful to view* $\mathbf{g}$ *as a function* $\lambda \mapsto \mathbf{g}_\lambda$*, that is as an element of* $V([0, \phi]) \otimes \mathcal{H}_\phi$*, that is the incidence module over the subposet* $[0, \phi]$ *in* $\mathcal{T}$ *with coefficients in* $\mathcal{H}_\phi$*.* $V([0, \phi]) \otimes \mathcal{H}_\phi$ *is a module over* $I([0, \phi]) \otimes \mathcal{H}_\phi$ *by extension of scalars from* $\mathbb{Q}$ *to* $\mathcal{H}_\phi$*.*

Proposition 3.3. $\mathcal{H}_\phi$ *is a semisimple commutative* $\mathbb{Q}$*-algebra of dimension* $|[0, \phi]|$*.*

Proof. The only thing we have to show is commutativity. This follows from Gelfand's trick: transposition, which is an anti-isomorphism, is realized by the flip map on $X_\phi \times_G X_\phi$, which is a trivial operation. □

3.4.2. *The cellular structure of $\mathcal{H}_\phi$.* Define $\mathbf{c}_\lambda = \mathbf{T}_{\phi\succ\lambda}\mathbf{T}_{\lambda\prec\phi}$. Observe that up to constant $\mathbf{c}_\lambda$ is simply a composition of averaging operators $\mathcal{F}_\phi \to \mathcal{F}_\lambda \to \mathcal{F}_\phi$:

$$[\mathbf{c}_\lambda h](x) = \sum_{X_\lambda \ni y \subseteq x} \sum_{X_\phi \ni z \supset y} h(z) \qquad (h \in \mathcal{F}_\phi,\ x \in X_\phi)$$

Collecting terms according to intersection types and using the definition of $\mathbf{g}_\lambda$ gives:

$$\mathbf{c}_\lambda = \sum_{\kappa \geq \lambda} \binom{\kappa}{\lambda} \mathbf{g}_\kappa \tag{c-g}$$

where $\binom{\kappa}{\lambda} = \hat{\zeta}(\lambda, \kappa)$ is the number of submodules of type λ inside a module of type κ. Viewing $\mathbf{c}$ as an element of $V([0,\phi]) \otimes \mathcal{H}_\phi$, equation (**c-g**) becomes $\mathbf{c} = \hat{\zeta}\mathbf{g}$. Therefore $\mathbf{g} = \hat{\mu}\mathbf{c}$, and it follows that $\{\mathbf{c}_\lambda\}_{\lambda \leq \phi}$ forms a basis of $\mathcal{H}_\phi$. We call this basis the *cellular basis* of $\mathcal{H}_\phi$. For each $\lambda \leq \phi$ set:

$$\mathcal{H}_\phi^\lambda = \mathrm{span}_{\mathbb{Q}}\{\mathbf{c}_{\lambda'} | \lambda' \leq \lambda\} \qquad \mathcal{H}_\phi^{\lambda^-} = \mathrm{span}_{\mathbb{Q}}\{\mathbf{c}_{\lambda'} | \lambda' < \lambda\}$$

Theorem 4 (Cellular ideal structure)**.** *For every $\lambda, \mu \leq \phi$:*

1. *$\mathcal{H}_\phi^\lambda$ and $\mathcal{H}_\phi^{\lambda^-}$ are ideals. In particular, they contain a unit when considered as rings.*
2. *$\mathcal{H}_\phi^\lambda \cdot \mathcal{H}_\phi^\mu = \mathcal{H}_\phi^\lambda \cap \mathcal{H}_\phi^\mu$. If $\mathcal{T}$ is a lattice they both equal to $\mathcal{H}_\phi^{\lambda\wedge\mu}$.*
3. *$\left\{\mathcal{K}_\lambda = \mathcal{H}_\phi^\lambda / \mathcal{H}_\phi^{\lambda^-}\right\}_{\lambda \leq \phi}$ is a complete set of inequivalent irreducible $\mathcal{H}_\phi$-modules.*
4. *As a $\mathcal{H}_\phi$-modules,*

$$\mathcal{H}_\phi^\lambda \simeq \bigoplus_{\mu \leq \lambda} \mathcal{K}_\mu$$

Proof. Postponed to §3.6. □

Note that Theorem 4 can be used to characterize the irreducible representations of $\mathcal{H}_\phi$: $\mathcal{K}_\lambda$ is the *unique* representation which is **annihilated** by all $\left\{\mathcal{H}_\phi^\mu\right\}_{\mu<\lambda}$ and **not annihilated** by $\mathcal{H}_\phi^\lambda$. In view of the dictionary between representations of $\mathcal{H}_\phi$ and representations of G (section 3.2), we can now label the representations of G which occur in $\mathcal{F}_\phi$ by: $\mathcal{U}_\lambda \leftrightarrow \mathcal{K}_\lambda$.

Moreover, by the definition of $\mathbf{c}_\lambda$ as the composition $\mathbf{T}_{\phi\succ\lambda}\mathbf{T}_{\lambda\prec\phi}$, the annihilation criterion above translates to the fact that $\mathcal{U}_\lambda$ occurs in $\mathcal{F}_\lambda$ and does not occur in $\mathcal{F}_\mu$ for $\lambda \not\leq \mu$. The exact multiplicities in which the $\mathcal{U}_\lambda$'s appear in $\mathcal{F}_\mu$ for arbitrary μ is the subject of the next section.

3.5. **The Hecke modules $\mathcal{N}_{\phi,\nu}$.** In this section we consider the various bases, and the cellular structure of the $\mathcal{H}_\phi$-module $\mathcal{N}_{\phi,\nu}$, for $\nu \leq \phi$. To begin with, we give a simple Lemma.

Lemma 3.4. *Let $\lambda \leq \nu \leq \phi$ and $\lambda \overset{i}{\hookrightarrow} \nu$ an embedding type. Then there exist $x, f \subseteq F$, such that $\tau(f) = \phi$ and $\vec{\tau}(x \wedge f \subseteq x) = i$.*

Proof. The proof is an exercise with the definitions. As ϕ is symmetric, there exist f, f' of type ϕ with $\tau(f \wedge f') = \lambda$. Denote $y = f \wedge f'$. Fix a submodule $x' \leq f'$ of type ν. There exist also $y'' \leq x''$ with $\vec{\tau}(y'' \leq x'') = i$. Since F is symmetric there is $g \in G$ with $gx'' = x' \leq f'$. Denote gy'' by y'. Since f' is symmetric, there is some $h \in \mathrm{Aut}(f')$ such that $hy' = y$. We are done by letting $x = hx'$. □

3.5.1. *The geometric basis of $\mathcal{N}_{\phi,\nu}$.* We analyze the structure of $X_\phi \times_G X_\nu$ following a similar line to the analysis of $X_\phi \times_G X_\phi$, given in §3.4.1.

The set $X_\phi \times_G X_\nu$ has a natural parametrization:

$$X_\phi \times_G X_\nu \simeq \{\lambda \overset{i}{\hookrightarrow} \nu | \lambda \leq \nu\} = [0, \nu \overset{\mathrm{id}}{\to} \nu] \subset \vec{\mathcal{T}}$$

This isomorphism is given by the (G-invariant map) $X_\phi \times X_\nu \to [0, \nu \overset{\mathrm{id}}{\to} \nu]$, given by $(x, y) \mapsto \vec{\tau}(x \wedge y \subseteq x)$. This map is an isomorphism by the discussion in §3.4.1 together with Lemma 3.4. $\mathcal{N}_{\phi,\nu}$ is isomorphic to $\mathcal{F}(X_\phi \times_G X_\nu)$, which in turn is isomorphic to $\mathcal{F}([0, \nu \overset{\mathrm{id}}{\to} \nu])$. Composing these isomorphisms we obtain the vector space isomorphism

$$\begin{aligned} V([0, \nu \overset{\mathrm{id}}{\to} \nu]) &\overset{\sim}{\to} \mathcal{N}_{\phi,\nu} \\ \mathbf{1}_i &\mapsto \mathbf{G}_i = \mathbf{G}_{\phi \succ \lambda \underset{i}{\hookrightarrow} \nu} \end{aligned}$$

where

$$[\mathbf{G}_i h](x) = \sum_{\{y \in X_\nu | \vec{\tau}(x \wedge y \subseteq x) = i\}} h(y) \qquad x \in X_\phi,\ h \in \mathcal{F}_\nu$$

Remark 3.5. *It is useful to view* $\mathbf{G}$ *as a function* $i \mapsto \mathbf{G}_i$*, that is as an element of* $V([0, \nu \overset{id}{\to} \nu]) \otimes \mathcal{N}_{\phi,\nu}$.

3.5.2. *The cellular structure of $\mathcal{N}_{\phi,\nu}$.* For an embedding type $\lambda \overset{i}{\hookrightarrow} \nu$, define $\mathbf{C}_i = \mathbf{C}_{\phi \succ \lambda \underset{i}{\hookrightarrow} \nu} = \mathbf{T}_{\phi \succ \lambda} \mathbf{T}_{\lambda \underset{i}{\hookrightarrow} \nu}$. In analogy with (**c-g**), we have:

$$\textbf{(C-G)} \qquad \mathbf{C}_i = \sum_{(\eta' \underset{i'}{\hookrightarrow} \lambda) \geq (\eta \underset{i}{\hookrightarrow} \lambda)} \begin{pmatrix} \eta' \underset{i'}{\hookrightarrow} \lambda \\ \eta \underset{i}{\hookrightarrow} \lambda \end{pmatrix} \mathbf{G}_{i'}$$

where $\begin{pmatrix} \eta' \underset{i'}{\hookrightarrow} \lambda \\ \eta \underset{i}{\hookrightarrow} \lambda \end{pmatrix} = \hat{\zeta}(i, i')$. Viewing $\mathbf{C}$ as an element of $V([0, \nu \overset{\mathrm{id}}{\to} \nu]) \otimes \mathcal{H}_\phi$, equation (**C-G**) becomes $\mathbf{C} = \hat{\zeta}\mathbf{G}$. Therefore $\mathbf{G} = \hat{\mu}\mathbf{C}$, and it follows that $\{\mathbf{C}_i\}_{\lambda \underset{i}{\hookrightarrow} \nu}$ forms a basis of $\mathcal{N}_{\phi,\nu}$. We call this basis the *cellular basis* of $\mathcal{N}_{\phi,\nu}$. For each $\lambda \leq \nu \leq \phi$ set:

$$\mathcal{N}^{\lambda}_{\phi,\nu} = \mathrm{span}_{\mathbb{Q}}\{\mathbf{G}_i \mid \lambda' \leq \lambda,\ \lambda' \overset{i}{\hookrightarrow} \nu\}$$

$$\mathcal{N}^{\lambda^-}_{\phi,\nu} = \mathrm{span}_{\mathbb{Q}}\{\mathbf{G}_i | \lambda' < \lambda,\ \lambda' \overset{i}{\hookrightarrow} \nu\}$$

Theorem 5 (Cellular submodule structure)**.** *For every* $\lambda \leq \nu \leq \phi$*:*

(1) $\mathcal{N}^{\lambda}_{\phi,\nu}$ *and* $\mathcal{N}^{\lambda^-}_{\phi,\nu}$ *are submodules of* $\mathcal{N}_{\phi,\nu}$.
(2) *For every* $\mu \leq \phi$, $\mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda}_{\phi,\nu} = \mathcal{N}^{\mu}_{\phi,\nu} \cap \mathcal{N}^{\lambda}_{\phi,\nu}$. *In particular,* $\mathcal{H}^{\lambda}_{\phi}\mathcal{N}_{\phi,\nu} = \mathcal{N}^{\lambda}_{\phi,\nu}$ *and* $\mathcal{H}^{\lambda^-}_{\phi}\mathcal{N}_{\phi,\nu} = \mathcal{N}^{\lambda^-}_{\phi,\nu}$.
(3) $\mathcal{N}^{\lambda}_{\nu,\phi}/\mathcal{N}^{\lambda^-}_{\nu,\phi}$ *is isomorphic to the* $\mathcal{K}_\lambda$*-isotypic submodule of* $\mathcal{N}_{\phi,\nu}$.
(4) *As a* $\mathcal{H}_\phi$*-modules,*

$$\mathcal{N}^{\lambda}_{\phi,\nu} \simeq \bigoplus_{\mu \leq \lambda} (\mathcal{K}_\mu)^{|\{\mu \hookrightarrow \nu\}|}$$

Proof. Postponed to §3.6. □

3.6. **Proofs.**

3.6.1. *Some Lemmas.* Denote by $[\mu \prec \dot{\lambda}]_\phi$ the number of submodules of type λ which contain a given submodule of type μ and are contained in a given object of type ϕ (when ϕ is symmetric this is a well defined quantity).

Lemma 3.6. *Let $\mu \leq \lambda \leq \eta, \phi$ be types, assume ϕ is symmetric. Let $i : \mu \hookrightarrow \lambda$ and $j : \lambda \hookrightarrow \eta$ be given types of embeddings. Then*

1. $\mathbf{T}_{\eta \underset{j}{\hookleftarrow} \lambda} \mathbf{T}_{\lambda \underset{i}{\hookleftarrow} \mu} \in \mathrm{span}_{\mathbb{N}}\{\mathbf{T}_{\eta \underset{k}{\hookleftarrow} \mu} | k\}$

1′. $\mathbf{T}_{\mu \underset{i}{\hookrightarrow} \lambda} \mathbf{T}_{\lambda \underset{j}{\hookrightarrow} \eta} \in \mathrm{span}_{\mathbb{N}}\{\mathbf{T}_{\mu \underset{k}{\hookrightarrow} \eta} | k\}$

2. $\mathbf{T}_{\phi \succ \lambda} \mathbf{T}_{\lambda \underset{i}{\hookleftarrow} \mu} \in \mathbb{N} \cdot \mathbf{T}_{\phi \succ \mu}$

2′. $\mathbf{T}_{\mu \underset{i}{\hookrightarrow} \lambda} \mathbf{T}_{\lambda \prec \phi} \in \mathbb{N} \cdot \mathbf{T}_{\mu \prec \phi}$

3. $\mathbf{T}_{\phi \succ \lambda} \mathbf{T}_{\lambda \succ \mu} = [\mu \prec \dot{\lambda}]_\phi \mathbf{T}_{\phi \succ \mu}$

3′. $\mathbf{T}_{\mu \prec \lambda} \mathbf{T}_{\lambda \prec \phi} = [\mu \prec \dot{\lambda}]_\phi \mathbf{T}_{\mu \prec \phi}$

Proof. **1.** We fix $x_0 \in X_\mu$ and denote its characteristic function by $\delta_{x_0} \in \mathcal{F}_\mu$. Since this is a cyclic vector for the representation, everything is determined by the action on this element:

$$(\star) \qquad \begin{aligned} \mathbf{T}_{\eta \underset{j}{\hookleftarrow} \lambda} \mathbf{T}_{\lambda \underset{i}{\hookleftarrow} \mu} \delta_{x_0}(z) &= \sum_{\{y | y \underset{j}{\hookrightarrow} z\}} \sum_{\{x | x \underset{i}{\hookrightarrow} y\}} \delta_{x_0}(x) = \sum_{\{y | y \underset{j}{\hookrightarrow} z\}} \mathbf{1}_{\{y' | x_0 \hookrightarrow y'\}}(y) \\ &= \sum_{\mu \underset{k}{\hookrightarrow} \eta} \#\left\{ y \,\middle|\, \begin{matrix} x_0 & \subseteq & y & \subseteq & z \\ \wr & & \wr & & \wr \\ \mu & \underset{i}{\hookrightarrow} & \lambda & \underset{j}{\hookrightarrow} & \eta \end{matrix} \right\} \mathbf{T}_{\eta \underset{k}{\hookleftarrow} \mu} \delta_{x_0}(z) \end{aligned}$$

In particular $\mathbf{T}_{\eta \underset{j}{\hookleftarrow} \lambda} \mathbf{T}_{\lambda \underset{i}{\hookleftarrow} \mu} \in \mathrm{span}_{\mathbb{N}}\{\mathbf{T}_{\eta \underset{k}{\hookleftarrow} \mu} \mid k\}$.

2. Set $\eta = \phi$ in the first part. Then $(\star)$ reduces to:

$$(\star\star) \qquad \mathbf{T}_{\phi \succ \lambda} \mathbf{T}_{\lambda \underset{i}{\hookleftarrow} \mu} = \#\left\{ y \,\middle|\, \begin{matrix} x_0 & \subseteq & y & \subseteq & z \\ \wr & & \wr & & \wr \\ \mu & \underset{i}{\hookrightarrow} & \lambda & \hookrightarrow & \phi \end{matrix} \right\} \mathbf{T}_{\phi \succ \mu}$$

3. Follows from equation $(\star\star)$, as

$$\sum_i \#\left\{ y \,\middle|\, \begin{matrix} x_0 & \subseteq & y & \subseteq & z \\ \wr & & \wr & & \wr \\ \mu & \underset{i}{\hookrightarrow} & \lambda & \hookrightarrow & \phi \end{matrix} \right\} = [\mu \prec \dot{\lambda}]_\phi$$

The other three assertions follow by transposition. □

Lemma 3.7. *Let $\mu \leq \lambda, \omega \leq \phi$ be types, and assume that the couple (Φ, ω) is symmetric. Then the map*

$$\mathcal{N}_{\omega,\lambda} \to \mathcal{H}_\phi, \ C \mapsto \mathbf{T}_{\phi \succ \omega} \circ C \circ \mathbf{T}_{\lambda \prec \phi}$$

maps $\mathcal{N}^{\mu}_{\omega,\lambda}$ onto $\mathcal{H}^{\mu}_{\phi}$, and it maps $\mathcal{N}^{\mu^-}_{\phi,\lambda}$ onto $\mathcal{H}^{\mu^-}_{\phi}$.

Proof. Follows immediately from Lemma 3.6, part (2), using cellular bases. □

Lemma 3.8. *$\mathcal{N}^{\lambda}_{\phi,\nu}$ and $\mathcal{N}^{\lambda^-}_{\phi,\nu}$ are $\mathcal{H}_\phi$-submodules of $\mathcal{N}_{\phi,\nu}$ $\forall \lambda \in [0, \nu]$.*

Proof. Let $\mu \leq \phi$, $\lambda' \leq \lambda$ and $i : \lambda' \hookrightarrow \nu$.

$$\begin{aligned} \mathbf{c}_\mu \cdot \mathbf{C}_i &= (\mathbf{T}_{\phi \succ \mu} \mathbf{T}_{\mu \prec \phi} \mathbf{T}_{\phi \succ \lambda'}) \mathbf{T}_{\lambda' \underset{i}{\hookrightarrow} \nu} \\ &\in \mathrm{span}_{\mathbb{Q}}\{\mathbf{C}_j | \eta \leq \lambda', \ \eta \underset{j}{\hookrightarrow} \lambda'\} \circ \mathbf{T}_{\lambda' \underset{i}{\hookrightarrow} \nu} \\ &= \mathrm{span}_{\mathbb{Q}}\{\mathbf{T}_{\phi \succ \eta} \mathbf{T}_{\eta \underset{j}{\hookrightarrow} \lambda'} \mathbf{T}_{\lambda' \underset{i}{\hookrightarrow} \nu} | \eta \leq \lambda', \ \eta \underset{j}{\hookrightarrow} \lambda'\} \\ &\subseteq \mathrm{span}_{\mathbb{Q}}\{\mathbf{T}_{\phi \succ \eta} \mathbf{T}_{\eta \underset{k}{\hookrightarrow} \nu} | \eta \leq \nu, \ \eta \underset{k}{\hookrightarrow} \nu\} \\ &= \mathrm{span}_{\mathbb{Q}}\{\mathbf{C}_k | \eta \leq \nu, \ \eta \underset{k}{\hookrightarrow} \nu\} = \mathcal{N}^{\lambda}_{\phi,\nu} \end{aligned}$$

(the inclusion follows from Lemma 3.6)

This implies that $\mathcal{N}^{\lambda}_{\nu,\phi}$ is a module since the $\mathbf{c}_\mu$'s generate $\mathcal{H}_\phi$ and the $\mathbf{C}_i$'s generate $\mathcal{N}^{\lambda}_{\phi,\nu}$. Finally $\mathcal{N}^{\lambda^-}_{\phi,\nu}$ is a submodule as well, since $\mathcal{N}^{\lambda^-}_{\phi,\nu} = \sum_{\lambda'<\lambda} \mathcal{N}^{\lambda'}_{\phi,\nu}$. □

3.6.2. *Proof of Theorem 4.*

Proof. By Lemma 3.8, for every $\lambda \leq \phi$, $\mathcal{H}^{\lambda}_{\phi}$ is an ideal (substitute $\nu = \phi$). It follows that for every $\lambda \leq \phi$, $\mathcal{H}^{\lambda^-}_{\phi}$ is an ideal as well, by: $\mathcal{H}^{\lambda^-}_{\phi} = \sum_{\lambda'<\lambda} \mathcal{H}^{\lambda'}_{\phi}$. This proves part (1). To prove part (2), observe first that $\mathcal{H}^{\lambda}_{\phi} \cdot \mathcal{H}^{\mu}_{\phi} \subseteq \mathcal{H}^{\lambda}_{\phi} \cap \mathcal{H}^{\mu}_{\phi}$. The opposite inclusion follows from the fact that each ideal (considered as a ring) contains a unit, namely the sum of its minimal idempotents. Therefore:

$$\mathcal{H}^{\lambda}_{\phi} \cdot \mathcal{H}^{\mu}_{\phi} \supseteq (\mathcal{H}^{\lambda}_{\phi} \cap \mathcal{H}^{\mu}_{\phi}) \cdot (\mathcal{H}^{\lambda}_{\phi} \cap \mathcal{H}^{\mu}_{\phi}) = \mathcal{H}^{\lambda}_{\phi} \cap \mathcal{H}^{\mu}_{\phi}$$

Part (3) follows from the fact that $\{\mathcal{K}_\lambda\}_{\lambda\leq\phi}$ is a collection of $\dim(\mathcal{H}_\phi)$ distinct one dimensional $\mathcal{H}_\phi$-representations (they are indeed distinct by the characterization of $\mathcal{K}_\lambda$ given after Theorem 4). Part (4) is proved by induction with respect to the partial order on $\mathcal{T}$. □

3.6.3. *Proof of Theorem 5.* Before proving the theorem we shall need one more lemma.

Lemma 3.9. *Let $\lambda \leq \nu \leq \phi$ and $\mu \leq \phi$ be types.*

(1) *If $\mu \geq \lambda$ then*

$$\mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda}_{\phi,\nu} = \mathcal{N}^{\lambda}_{\phi,\nu}$$

(2) *If $\mu \not\geq \lambda$ then*

$$\mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda}_{\phi,\nu} \subseteq \mathcal{N}^{\lambda^-}_{\phi,\nu}$$

Proof. **Part 1:** By Lemma 3.8, $\mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda}_{\phi,\nu} \subseteq \mathcal{N}^{\lambda}_{\phi,\nu}$. We argue to show the reverse inclusion. Assume (by induction with respect to the partial order on $\mathcal{T}$) that for every $\lambda' < \lambda$, $\mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda'}_{\phi,\nu} = \mathcal{N}^{\lambda'}_{\phi,\nu}$. Then $\mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda}_{\phi,\nu} \supseteq \mathcal{N}^{\lambda^-}_{\phi,\nu}$, and we are left to show that for every inclusion type $\lambda \underset{i}{\hookrightarrow} \nu$, $\mathbf{C}_{\phi\succ\lambda\underset{i}{\hookrightarrow}\nu} \in \mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda}_{\phi,\nu}$. We first consider the case $\nu = \lambda$. Assume $\mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda}_{\phi,\lambda} = \mathcal{N}^{\lambda^-}_{\phi,\lambda}$. Composing on the right with $\mathbf{T}_{\lambda\prec\phi}$ we get, using Lemma 3.7, that $\mathcal{H}^{\mu}_{\phi}\mathcal{H}^{\lambda}_{\phi} = \mathcal{H}^{\lambda^-}_{\phi}$, which is an absurd, as $\mathcal{H}^{\lambda}_{\phi}$ has a unit. Therefore $\mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda}_{\phi,\lambda} = \mathcal{N}^{\lambda}_{\phi,\lambda}$. In particular we get that $\mathbf{T}_{\phi\succ\lambda} \in \mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda}_{\phi,\lambda}$. The general case follows by Lemma 3.6, because

$$\mathbf{C}_{\phi\succ\lambda\underset{i}{\hookrightarrow}\nu} = \mathbf{T}_{\phi\succ\lambda} \circ \mathbf{T}_{\lambda\underset{i}{\hookrightarrow}\nu} \in \mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda}_{\phi,\lambda}\mathbf{T}_{\lambda\underset{i}{\hookrightarrow}\nu} \subseteq \mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\nu}_{\phi,\lambda}$$

and we proved $\mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda}_{\phi,\nu} = \mathcal{N}^{\lambda}_{\phi,\nu}$.

Part 2: By Lemma 3.8, $\mathcal{H}^{\mu}_{\phi}\mathcal{N}^{\lambda^-}_{\phi,\nu} \subseteq \mathcal{N}^{\lambda^-}_{\phi,\nu}$. We have therefore reduced the proof to showing that $\mathbf{c}_{\mu'} \cdot \mathbf{C}_{\phi\succ\lambda\underset{i}{\hookrightarrow}\nu} = \mathbf{T}_{\phi\succ\mu'}\mathbf{T}_{\mu'\prec\phi}\mathbf{T}_{\phi\succ\lambda}\mathbf{T}_{\lambda\underset{i}{\hookrightarrow}\nu} \in \mathcal{N}^{\lambda^-}_{\phi,\nu}$ for all $\mu' \leq \mu$ and $i : \lambda \hookrightarrow \nu$. Express the map $\mathbf{T}_{\phi\succ\mu'}\mathbf{T}_{\mu'\prec\phi}\mathbf{T}_{\phi\succ\lambda}$ in the cellular basis of $\mathcal{N}_{\phi,\lambda}$:

$$\mathbf{T}_{\phi\succ\mu'}\mathbf{T}_{\mu'\prec\phi}\mathbf{T}_{\phi\succ\lambda} = \sum_{\lambda'\underset{j}{\hookrightarrow}\lambda} a_{\lambda'\underset{j}{\hookrightarrow}\lambda}\mathbf{T}_{\phi\succ\lambda'}\mathbf{T}_{\lambda'\underset{j}{\hookrightarrow}\lambda} \tag{3}$$

We claim that $a_{\lambda\underset{id}{\hookrightarrow}\lambda} = 0$. Indeed, assume this was not the situation. Compose the map $\mathbf{T}_{\lambda\prec\phi}$ on the right of both sides of equation (3), and present it with respect to the cellular basis of $\mathcal{H}_\phi$. Using Lemma 3.6, it is clear from the presentation of the r.h.s. that the coefficient of $\mathbf{c}_\lambda$ is non-trivial. On the other hand, the l.h.s. becomes $\mathbf{c}_{\mu'} \cdot \mathbf{c}_\lambda$. The latter is in $\mathcal{H}^{\lambda^-}_{\phi}$ by the assumption $\mu \not\geq \lambda$, using Theorem 4. This is a contradiction.

Composing with $\mathbf{T}_{\lambda \underset{i}{\hookrightarrow} \nu}$ on the right of both sides of equation (3) gives:

$$\begin{aligned}\mathbf{T}_{\phi \succ \mu'} \mathbf{T}_{\mu' \prec \phi} \mathbf{T}_{\phi \succ \lambda} \mathbf{T}_{\lambda \underset{i}{\hookrightarrow} \nu} &= \sum_{\lambda' \underset{j}{\hookrightarrow} \lambda, \lambda' < \lambda} a_{\lambda' \underset{j}{\hookrightarrow} \lambda} \mathbf{T}_{\phi \succ \lambda'} (\mathbf{T}_{\lambda' \underset{j}{\hookrightarrow} \lambda} \mathbf{T}_{\lambda \underset{i}{\hookrightarrow} \nu}) \\ &= \sum_{\lambda' \underset{k}{\hookrightarrow} \nu, \lambda' < \lambda} b_{\lambda' \underset{k}{\hookrightarrow} \nu} \mathbf{T}_{\phi \succ \lambda'} \mathbf{T}_{\lambda' \underset{k}{\hookrightarrow} \nu} \in \mathcal{N}^{\lambda^-}_{\phi,\nu}\end{aligned}$$

□

proof of Theorem 5. The main part of the theorem is part (2). Indeed, part (1) immidately follows from part (2). Also, assuming part (2), it follows that for every $\mu \leq \phi$, the $\mathcal{K}_\lambda$-isotypic component of $\mathcal{N}_{\phi,\nu}$ is

$$\mathcal{K}_\lambda \otimes_{\mathcal{H}_\phi} \mathcal{N}_{\phi,\nu} \simeq (\mathcal{H}^\lambda_\phi / \mathcal{H}^{\lambda^-}_\phi) \otimes_{\mathcal{H}_\phi} \mathcal{N}_{\phi,\nu} \simeq \mathcal{H}^\lambda_\phi \mathcal{N}_{\phi,\nu} / \mathcal{H}^{\lambda^-}_\phi \mathcal{N}_{\phi,\nu} \simeq \mathcal{N}^\lambda_{\phi,\nu} / \mathcal{N}^{\lambda^-}_{\phi,\nu}$$

and part (3) follows as well. Finally, using cellular bases, we see that $\dim(\mathcal{N}^\lambda_{\phi,\nu}/\mathcal{N}^{\lambda^-}_{\phi,\nu}) = |\{\lambda \hookrightarrow \nu\}|$, and part (4) follows.

We proceed to the proof of part (2). By part (1) lemma 3.9, for every $\alpha \leq \lambda, \mu$,

$$\mathcal{H}^\mu_\phi \mathcal{N}^\lambda_{\phi,\nu} \supset \mathcal{H}^\alpha_{\phi,\nu} \mathcal{N}^\alpha_{\phi,\nu} = \mathcal{N}^\alpha_{\phi,\nu}$$

hence

$$\mathcal{H}^\mu_\phi \mathcal{N}^\lambda_{\phi,\nu} \supset \sum_{\alpha \leq \lambda, \mu} \mathcal{N}^\alpha_{\phi,\nu} = \mathcal{N}^\mu_{\phi,\nu} \cap \mathcal{N}^\lambda_{\phi,\nu}$$

Also we have that $\mathcal{H}^\mu_\phi \mathcal{N}^\lambda_{\phi,\nu} \subset \mathcal{H}_\phi \mathcal{N}^\lambda_{\phi,\nu} = \mathcal{N}^\lambda_{\phi,\nu}$. We are left to show that $\mathcal{H}^\mu_\phi \mathcal{N}^\lambda_{\phi,\nu} \subset \mathcal{N}^\mu_{\phi,\nu}$. This follows by induction (with respect to λ) on the poset $\mathcal{T}$. Indeed, assume that for every $\lambda' < \lambda$, $\mathcal{H}^\mu_\phi \mathcal{N}^{\lambda'}_{\phi,\nu} \subset \mathcal{N}^\mu_{\phi,\nu}$. If $\mu \geq \lambda$ then we are done by part (1) of lemma 3.9. If $\mu \not\geq \lambda$, then recalling that $\mathcal{H}^\mu_\phi$ has a unit, and using part (2) of lemma 3.9,

$$\mathcal{H}^\mu_\phi \mathcal{N}^\lambda_{\phi,\nu} = \mathcal{H}^\mu_\phi \left(\mathcal{H}^\mu_\phi \mathcal{N}^\lambda_{\phi,\nu} \right) \subset \mathcal{H}^\mu_\phi \left(\mathcal{N}^{\lambda-}_{\phi,\nu} \right) = \mathcal{H}^\mu_\phi \left(\sum_{\lambda' < \lambda} \mathcal{N}^{\lambda'}_{\phi,\nu} \right) \subset \mathcal{N}^\mu_{\phi,\nu}.$$

□

3.6.4. *Proof of Theorem 3.*

Proof. By Proposition 3.1, The modules $\mathcal{K}_\lambda$ correspond to the irreducible representations of G, $\mathcal{U}_\lambda = \mathcal{F}_\phi \otimes \mathcal{K}_\lambda$. Part (2) of Theorem 3 now follows from Theorem 5, part (4). Part (1) is a special case of part (2), as ϕ is symmetric. □

4. Towards a Fourier decomposition

Throughout this section fix a symmetric type Φ, and a module F of type Φ.

4.1. **A counting principle.** Let X be a set and assume we are given a map $\varphi : X \to \mathcal{M}_F$. Define the following elements in $V(\mathcal{M}_F)$:

$$s_\varphi(y) = |\{x \in X | \varphi(x) = y\}|$$
$$t_\varphi(y) = |\{x \in X | \varphi(x) \supseteq y\}|$$

Clearly $t_\varphi = \zeta \cdot s_\varphi$. By multiplying both sides by μ, using proposition 2.5 applied to the map $\tau : \mathcal{M}_F \to \mathcal{T}$, we get $\hat{s}_\varphi = \hat{\mu} \cdot \hat{t}_\varphi$. Observing that $0 \in \mathcal{M}_F$ is the *unique* element above $0 \in \mathcal{T}$, we obtain an inclusion-exclusion type formula which will be useful in the sequel:

$$s_\varphi(0) = \hat{s}_\varphi(0) = \hat{\mu} \cdot \hat{t}_\varphi(0) = \sum_\alpha \chi(\alpha) \hat{t}_\varphi(\alpha) \tag{4}$$

Recall (Lemma 3.6) that $[\alpha \prec \dot{\kappa}]$ is the number of submodule of type κ which contain a given submodule of type α. Let x_κ and x_ω be disjoint submodules of F of types κ and ω (i.e $x_\kappa \wedge x_\omega = 0$). We introduce another convenient notation: $[\omega \prec \dot{\beta} \pitchfork \kappa]_\Phi$ (or simply $[\omega \prec \dot{\beta} \pitchfork \kappa]$) will denote the number of submodules of F of type β which contain x_ω, and are disjoint from x_κ.

Define $X = \{x \subset F \mid \tau(x) = \beta,\ x_\omega \subset x\}$, and $\varphi : X \to \mathcal{M}_F$ by $\varphi(x) = x \wedge x_\kappa$. Then, clearly,

$$s_\varphi(0) = [\omega \prec \dot{\beta} \pitchfork \kappa]$$

It is also easy to see that

$$\hat{t}_\varphi(\alpha) = [\omega \oplus \alpha \prec \dot{\beta}] \binom{\kappa}{\alpha}$$

Indeed,

$$\hat{t}_\varphi(\alpha) = \sum_{\{y \subset F \mid \tau(y) = \alpha\}} t_\varphi(y) = \left| \bigcup_{\{y \subset x_\kappa \mid \tau(y) = \alpha\}} \{x \subset F \mid \tau(x) = \beta,\ y, x_\omega \subset x\} \right| = \binom{\kappa}{\alpha} [\omega \oplus \alpha \prec \dot{\beta}]$$

Thus, by equation (4),

$$[\omega \prec \dot{\beta} \pitchfork \kappa] = \sum_\alpha \chi(\alpha) [\omega \oplus \alpha \prec \dot{\beta}] \binom{\kappa}{\alpha} \tag{5}$$

(Observe that the right hand side is independent of the choice of x_ω and x_κ, hence so is the left hand side, and the notation $[\omega \prec \dot{\beta} \pitchfork \kappa]$ is justified).

Example 4.1 (The case of a field)**.** *When R is a field and U, V are vector spaces, one has an embedding* $\mathrm{Hom}(V, U) \hookrightarrow V \oplus U$*, given by the* **graph** *of a transformation. Its image consists of those subspaces intersecting U trivially. If the field has q elements,*

$$[m \prec \dot{l} \pitchfork k]_n = [0 \prec (l \dot{-} m) \pitchfork k]_{n-m} = \binom{n-m-k}{l-m-k}_q \cdot q^{k(l-m-k)}$$

Thus, we get, using example 2.7,

$$\binom{n-m-k}{l-m-k}_q \cdot q^{k(l-m-k)} = \sum_i (-1)^i q^{\binom{i}{2}} \binom{n-m-i}{l-m-i}_q \binom{k}{i}_q$$

(compare with [2]*)*

4.2. **Computing some matrix coefficients.** We wish to compute the idempotents of $\mathcal{H}_\phi$ explicitly. Since the cellular structure must agree with the idempotent decomposition, we already know that there exist a lower triangular matrix $A_{\lambda\kappa}$ such that

$$\mathbf{c}_\lambda = \sum_{\kappa \le \lambda} A_{\lambda\kappa} \mathbf{e}_\kappa \tag{c-e}$$

where $\mathbf{e}_\lambda$ is the idempotent in $\mathcal{H}_\phi$ corresponding to its irreducible representation $\mathcal{K}_\lambda$. We have already seen that the transition matrix from the geometric basis to the cellular basis depends only on geometric invariants of the lattice of submodules (relation (**c-g**) above) in a very simple way. In some situations, we are able to give similar interpretation to various matrix coefficients of A. This is the main theme of this section.

Let $\kappa \le \omega \le \Phi$ be types and assume (Φ, ω) is a symmetric couple. Assume that κ satisfies the following duality axiom:

For a module x of type κ, and every type $\alpha \le \kappa$,

$$|\{y \mid y \le x,\ \tau(y) = \alpha\}| = |\{y \mid y \le x,\ \tau(x/y) = \alpha\}| \tag{duality}$$

Remark: By the principal divisor theorem, it is easy to see that every finite module over a principal ideal domain satisfies the duality axiom.

Recall that $[\kappa \prec \dot{\omega}]_\phi$ is the number of submodules of type ω which contain a given submodule of type κ and are contained in a given module of type ϕ (that is $[\kappa \prec \dot{\omega}]_\phi = [\kappa \prec \dot{\omega} \pitchfork 0]_\phi$).

Theorem 6. *Under the above assumptions,* $A_{\omega\kappa} = [\kappa \prec \dot{\omega}]_\phi [\omega \prec \dot{\phi} \pitchfork \kappa]_\Phi$.

Before proving Theorem 6 we state a simple Lemma.

Lemma 4.2. *Let* $\theta \underset{i}{\hookrightarrow} \kappa$ *be a map type. Let* w *and* x *be modules of types* ω *and* κ*, such that* $\vec{\tau}(x \wedge w \leq x) = i$*. The following diagram is cartesian.*

$$\begin{array}{ccc} w & \hookrightarrow & w \oplus \operatorname{coker}(i) \\ \uparrow & & \uparrow \\ w \wedge x & \underset{i}{\hookrightarrow} & x \end{array}$$

where

- $\operatorname{coker}(i)$ *is the type of* $x/(w \wedge x)$*.*
- *The map* $w \hookrightarrow w \oplus \operatorname{coker}(i)$ *is given by* $id \oplus 0$*.*
- *The map* $x \hookrightarrow w \oplus \operatorname{coker}(i)$ *is given* $a \oplus b$ *where* a *is an embedding, and* b *is the natural projection.*

Proof. By the fact ω is symmetric, the embedding $w \wedge x \hookrightarrow w$ can be taken to be $-a \circ i$. One easily sees that

$$\operatorname{Ker}\big(\mathrm{id} \oplus b : w \oplus x \to w \oplus \operatorname{coker}(i)\big) = \big((-a \circ i) \oplus i\big)(w \wedge x)$$

□

Proof of Theorem 6: Our strategy is to analyze the multiplication in the algebra with respect to the cellular basis. Let $B^\nu_{\omega\mu}$ be multiplication table with respect to the cellular basis:

$$\mathbf{c}_\omega \cdot \mathbf{c}_\kappa = \sum_{\nu \leq \omega \wedge \kappa} B^\nu_{\omega\kappa} \mathbf{c}_\nu \tag{6}$$

Observe that $B^\kappa_{\omega\kappa} = A_{\omega\kappa}$ for $\kappa \leq \omega$.
Substituting $\mathbf{c}_\eta = \mathbf{T}_{\phi \succ \eta} \mathbf{T}_{\eta \prec \phi}$ in equation (6) and using parts **3** and **3'** of Lemma 3.6 give (assume $\kappa \leq \omega$):

$$\begin{aligned} &\mathbf{T}_{\phi \succ \omega} \Big(\mathbf{T}_{\omega \prec \phi} \mathbf{T}_{\phi \succ \kappa} \Big) \mathbf{T}_{\kappa \prec \phi} \\ &\quad = \mathbf{T}_{\phi \succ \omega} \Big(\sum_{\nu \leq \kappa} \frac{B^\nu_{\omega\kappa}}{[\nu \prec \dot{\omega}]_\phi [\nu \prec \dot{\kappa}]_\phi} \mathbf{T}_{\omega \succ \nu} \mathbf{T}_{\nu \prec \kappa} \Big) \mathbf{T}_{\kappa \prec \phi} \\ &\quad \in \mathbf{T}_{\phi \succ \omega} \Big(\frac{A_{\omega\kappa}}{[\kappa \prec \dot{\omega}]_\phi} \mathbf{T}_{\omega \succ \kappa} + \mathcal{N}^{\kappa^-}_{\omega\kappa} \Big) \mathbf{T}_{\kappa \prec \phi} \end{aligned} \tag{7}$$

Recall that the sets of operators $\{\mathbf{G}_{\omega \succ \theta \underset{i}{\hookrightarrow} \kappa}\}_i$ form a basis of $\mathcal{N}_{\omega,\kappa}$. A direct calculation shows that (compare **c-g**, **C-G**)

$$\mathbf{T}_{\omega \prec \phi} \mathbf{T}_{\phi \succ \kappa} = \sum_{\theta \underset{i}{\hookrightarrow} \kappa} [\omega \oplus \operatorname{coker}(i) \prec \dot{\phi}]_\Phi \mathbf{G}_{\omega \succ \theta \underset{i}{\hookrightarrow} \kappa}$$

By the relations **C-G** between the geometric and the cellular bases, we have that the coefficient of $\mathbf{C}_{\omega\succ\kappa\underset{=}{\hookrightarrow}\kappa}$ when expanding $\mathbf{T}_{\omega\prec\phi}\mathbf{T}_{\phi\succ\kappa}$ with respect to the cellular basis is given by

$$\begin{aligned}
&\sum_{\theta\underset{i}{\hookrightarrow}\kappa}[\omega\oplus\operatorname{coker}(i)\prec\dot{\phi}]_\Phi\hat{\mu}_{\vec{\mathcal{M}}/\vec{\mathcal{T}}}(\theta\underset{i}{\hookrightarrow}\kappa,\kappa=\kappa)\\
&=\sum_{\theta\underset{i}{\hookrightarrow}\kappa}[\omega\oplus\operatorname{coker}(i)\prec\dot{\phi}]_\Phi\begin{pmatrix}\kappa=\kappa\\ \theta\underset{i}{\hookrightarrow}\kappa\end{pmatrix}\chi(\operatorname{coker}(i))\\
&=\sum_{\alpha}\sum_{i,\ \operatorname{coker}(i)=\alpha}[\omega\oplus\alpha\prec\dot{\phi}]_\Phi\begin{pmatrix}\kappa=\kappa\\ \theta\underset{i}{\hookrightarrow}\kappa\end{pmatrix}\chi(\alpha)\\
&=\sum_{\alpha}[\omega\oplus\alpha\prec\dot{\phi}]_\Phi\chi(\alpha)\sum_{i,\ \operatorname{coker}(i)=\alpha}\begin{pmatrix}\kappa=\kappa\\ \theta\underset{i}{\hookrightarrow}\kappa\end{pmatrix}\\
&=\sum_{\alpha}[\omega\oplus\alpha\prec\dot{\phi}]_\Phi\chi(\alpha)\begin{pmatrix}\kappa\\ \alpha\end{pmatrix} && \text{(the duality axiom)}\\
&=\sum_{\alpha}\chi(\alpha)\hat{w}(\alpha)=s(0)=[\omega\prec\dot{\phi}\pitchfork\kappa]_\Phi && \text{(equation (5))}
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathbf{T}_{\phi\succ\omega}\Big(\mathbf{T}_{\omega\prec\phi}\mathbf{T}_{\phi\succ\kappa}\Big)\mathbf{T}_{\kappa\prec\phi}\\
&\qquad\in\mathbf{T}_{\phi\succ\omega}\Big([\omega\prec\dot{\phi}\pitchfork\kappa]_\Phi\mathbf{T}_{\omega\succ\kappa}+\mathcal{N}^{\kappa^-}_{\omega\kappa}\Big)\mathbf{T}_{\kappa\prec\phi}
\end{aligned}\tag{8}$$

By Lemma 3.7, $\mathbf{T}_{\phi\succ\omega}\mathcal{N}^{\kappa^-}_{\omega\kappa}\mathbf{T}_{\kappa\prec\phi}<\mathcal{H}^{\kappa^-}_{\phi}$, thus comparing equations (7) and (8) we get the desired equation.

□

Example 4.3 (Fourier decomposition in the field case)**.** *Let $R=\mathbb{F}_q$ be the finite field with q elements. Fix two natural numbers m,n with $2m\le n$. Then (n,m) is a symmetric couple. The group is $\mathrm{GL}_n(\mathbb{F}_q)$, the representation is $\mathcal{F}_m$ and the Hecke algebra $\mathcal{H}_m$. $(g_k)_{k\le m}$ and $(e_k)_{k\le m}$ are two bases for $\mathcal{H}_m$. Example 4.1 gives for $k\le m$,*

$$\mathbf{g}_k=\sum_{i=k}^{m}\sum_{j=0}^{i}(-1)^{i-k}q^{\binom{i-k}{2}}\begin{pmatrix}i\\k\end{pmatrix}_q\begin{pmatrix}n-i-j\\m-i-j\end{pmatrix}_q\cdot q^{j(m-i-j)}\begin{pmatrix}l-j\\i-j\end{pmatrix}_q\mathbf{e}_j$$

(compare with [4]*).*

5. Generalization of the theory

The theory developed in sections 2 and 3 is valid (after minor changes of the terminology) for a large class of examples. In this section we explain the necessary terminology, re-phrase some of the theorems in a wider generality, and lastly, give a small list of examples which fit into this framework.

5.1. **The general setting.** We begin by replacing the category of modules over a ring by an arbitrary category. In order to speak about Grassmannians we need the notion of a "subobject" in this generality. Our reference for that is [6, V§7]. For their fundamental importance in our discussion we recall some of the definitions.

Definition 5.1. *A morphism $i:x\to y$ in the category $\mathcal{C}$ is called* **monic** *if for every object z of $\mathcal{C}$, the map*

$$i_*:\operatorname{Hom}(z,x)\to\operatorname{Hom}(z,y),\quad \phi\mapsto i\circ\phi$$

is injective.

Let $\mathcal{C}$ be a category. We abuse the notations and replace $\mathcal{C}$ with its subcategory which has the same objects, but its morphisms consist only the monics in $\mathcal{C}$. We fix the category $\mathcal{C}$ for the rest of the section.

Let $\mathcal{T} = \pi_0(\mathcal{C})$ denote the collection of types, that is isomorphism classes, in $\mathcal{C}$. We denote by $\tau : \mathrm{Ob}(\mathcal{C}) \to \mathcal{T}$ the type map. Let $* \longrightarrow *$ denote the category which consists of two objects and one nonidentity arrow. Define $\vec{\mathcal{C}}$ to be the category of functors (and natural equivalences) from $* \longrightarrow *$ to $\mathcal{C}$. Observe that all the morphisms in $\vec{\mathcal{C}}$ are monics (by the assumption on $\mathcal{C}$). We denote by $\vec{\mathcal{T}}$ the collection of types of $\vec{\mathcal{C}}$ and by $\vec{\tau}$ the type map.

Definition 5.2. *Let y be an object of $\mathcal{C}$. A* **subobject** *of y is an equivalence class of monics $x \xrightarrow{i} y$, under the relation*

$$x \xrightarrow{i} y \ \sim \ x' \xrightarrow{i'} y$$

if and only if there exist an isomorphism $x \xrightarrow{\phi} x'$ with $i' \circ \phi = i$.

We denote by $\mathcal{C}_y$ the collection of all subobjects of y. The subobject of y (represented by) $x \xrightarrow{i} y$ is said to be smaller than the subobject (represented by) $x' \xrightarrow{i'} y$ if there exist a (necessarily monic) morphism $x \xrightarrow{\phi} x'$ with $i' \circ \phi = i$. In our discussion we will assume that $\mathcal{C}_y$ is a **finite lattice** for every $y \in \mathrm{Ob}(\mathcal{C})$. Symbols $\binom{y}{\xi}$ and $\binom{y}{\iota}$ are, thus, readily understood. These are $\mathrm{Aut}(y)$-spaces, hence yield to representations of $\mathrm{Aut}(y)$ on the space of $\mathbb{Q}$-valued functions defined on them.

With the above terminology in hand, the reader is invited to observe that sections 2 and 3 are valid *mutatis-mutandis* in this generalized setting. In particular, we can define symmetric objects and symmetric couples (see definitions 2.1,2.2), and deduce (in analogy with Theorem 3):

Theorem 7. *$F \in \mathrm{Ob}(\mathcal{C})$ be of symmetric type Φ. For every $\lambda \leq \Phi$, denote by $\mathcal{F}_\lambda$ the vector space of $\mathbb{Q}$-valued functions on $\binom{F}{\lambda}$. Let ϕ be a type such that $\phi \leq \Phi$ is a symmetric couple, and let $G = \mathrm{Aut}_{\mathcal{C}}(F)$. There exists a collection of non-equivalent irreducible G-representations $\{\mathcal{U}_\lambda\}_{\lambda \leq \phi}$ such that:*

(1) $\mathcal{F}_\phi = \bigoplus_{\lambda \leq \phi} \mathcal{U}_\lambda$.
(2) *For every $\lambda, \nu \leq \phi$:*

$$\langle \mathcal{U}_\lambda, \mathcal{F}_\nu \rangle = |\{\lambda \hookrightarrow \nu\}|$$

I.e., the multiplicity of $\mathcal{U}_\lambda$ in $\mathcal{F}_\nu$ is the number of non-equivalent monics from an object of type λ to an object of type ν. In particular $\mathcal{U}_\lambda$ appears in $\mathcal{F}_\lambda$ with multiplicity one and does not appear in $\mathcal{F}_\nu$ unless $\lambda \leq \nu$.

5.2. **Examples.** We now give a list of examples of Categories. We will describe the symmetric objects and couples. We won't give proofs in all cases, as these can be regarded as easy exercises. We will try to describe the automorphism groups of symmetric objects, and their actions on the Grassmannians.

5.2.1. *Sets.* In the category of finite sets, **Sets**, the set of types $\mathcal{T}_{\mathbf{Sets}}$ can be identified with $\mathbb{N} \cup \{0\}$, the type map given by $\tau(A) = |A|$. Every object in **Sets** is symmetric, and the couple (n, m) is symmetric if and only if $n \geq 2m$. For an object A of type n, $\mathrm{Aut}(A)$ is identified with the permutation group S_n. The action of $\mathrm{Aut}(A)$ on $\binom{A}{m}$ is identified with the action of S_n on $\binom{[n]}{m}$.

5.2.2. *Vector spaces.* Denote by $\mathbf{Vec}_q$ (or simply **Vec** when q is given) the category of finite dimensional vector spaces over the finite field $\mathbb{F}_q$. $\mathcal{T}_{\mathbf{Vec}}$ is naturally identified with $\mathbb{N} \cup \{0\}$, by letting for every $V \in \mathrm{Ob}(\mathbf{Vec})$, $\tau(V) = \dim(V)$. Our notion of Grassmannian coincides with usual one. The cardinalities of the Grassmannians are given by the q-binomial functions:

$$\binom{n}{m}_{\mathbf{Vec}} = \binom{n}{m}_q = \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}},$$

where $(a;q)_n$ is the Pochammer symbol and equals to $\prod_{i=0}^{n-1}(1 - aq^i)$.

The automorphism groups $\mathrm{Aut}(V)$ is nothing but $\mathrm{GL}(V)$. It is easy to see that every type in $\mathcal{T}_{\mathbf{Vec}}$ is symmetric, and that the couple (n, m) is symmetric if and only if $n \geq 2m$ (this example is treated in [4]).

5.2.3. *Anti-symmetric bilinear forms.* Denote by $\mathbf{Symp}_q$ (or $\mathbf{Symp}$) the category which objects are couples (V, B), where V is an object of $\mathbf{Vec}_q$, and B is a (possibly degenerate) anti-symmetric bilinear form on V (we refer to such an object as a **symplectic space**). The morphisms in $\mathbf{Symp}$ are given by

$$\mathrm{Hom}_{\mathbf{Symp}}((V, B), (V', B')) = \{\phi \in \mathrm{Hom}_{\mathbf{Vec}}(V, V') \mid B = \phi^*(B')\}$$

$\mathbf{Vec}$ appears as a full subcategory of $\mathbf{Symp}$, by $V \mapsto (V, 0)$. The object of $\mathbf{Vec}$ are denoted **isotropic** when regarded as objects of $\mathbf{Symp}$. For the symplectic space (V, B), we will use $i(V, B)$ to denote the dimension of a maximal isotropic subspace, that is

$$i(V, B) = \max\{\dim(U) \mid (U, 0) < (V, B)\}$$

We denote $\mathrm{rad}(V, B) = \{v \in V \mid \forall u \in V,\ B(u, v) = 0\}$, and $n(V, B) = \frac{1}{2}\dim(V/\mathrm{rad}(V, B))$. Given $i = i(V, B)$ and $n = n(V, B)$, we have $\dim(V) = i + n$, and $\dim(\mathrm{rad}(V, B)) = i - n$, thus the set of types may be identified with

$$\mathcal{T}_{\mathbf{Symp}} = \{(i, n) \mid i, n \in \mathbb{N} \cup \{0\},\ i \geq n\}$$

The order structure on $\mathcal{T}_{\mathbf{Symp}}$ is given by $(i, n) \geq (i', n')$ if and only if both $i \geq i'$ and $n \geq n'$.

We already know that every isotropic object is symmetric. By Silvester theorem, if B is non-degenerate, then (V, B) is symmetric as well. On the other hand, it is easy to see that for $0 < n < i$, the type (i, n) is non-symmetric (compare radical and non-radical lines). Therefore, we conclude that The symmetric types are exactly the types of the form $(i, 0)$ or (n, n). One easily verifies that the couples $((n, n), (n, 0))$ are symmetric. The corresponding Grassmannians

$$\binom{(n, n)}{(n, 0)}$$

are known as the Lagrangian-Grassmannians. They are acted upon by $\mathrm{Aut}(n, n) = \mathrm{Symp}_n(\mathbb{F}_q)$.

5.2.4. *Bundles over sets.* Let X be a set. We consider the category whose objects are the points of X, and which has a unique morphism between every two points in X. We denote this category by X as well. Let $\mathcal{C}$ be a category. Let $\mathbf{Bun} = \mathbf{Bun}_X(\mathcal{C})$ be the category which objects are functors from X to $\mathcal{C}$, and a morphism between to such functors, E and F, is a couple (σ, ϕ), where σ is a permutation of X and ϕ is a natural transformation from E to $F \circ \sigma$. The types of $\mathbf{Bun}$ are given by the multiplicities of the various types in $\mathcal{C}$, that is

$$\mathcal{T}_{\mathbf{Bun}} = \{r : \mathcal{T}_{\mathcal{C}} \to \mathbb{N} \cup \{0\} \mid \sum_{\lambda \in \mathcal{T}_{\mathcal{C}}} r(\lambda) = |X|\}$$

in particular we denote by $\tilde{\lambda}$ the type in $\mathcal{T}_{\mathbf{Bun}}$ which satisfies $\tilde{\lambda}(\lambda) = |X|$. It is immediately seen that an object E in $\mathbf{Bun}$ is symmetric if and only if E_x is symmetric for every $x \in X$. Equivalently, the symmetric types of $\mathbf{Bun}$ are those supported on the set of symmetric types of $\mathcal{C}$. For a symmetric couple in $\mathcal{C}$, (Φ, ϕ), $(\tilde{\Phi}, \tilde{\phi})$ is a symmetric couple in $\mathbf{Bun}$. The group $\mathrm{Aut}(\tilde{\Phi})$ can be identified with the wreath product $S_{|X|} \propto \mathrm{Aut}(\Phi)$. It acts on the Grassmannian $\binom{\tilde{\Phi}}{\tilde{\phi}}$, which, in turn, can be identified with $\binom{\Phi}{\phi}^{|X|}$. Important special cases occur when $\mathcal{C}$ is $\mathbf{Sets}$, $\mathbf{Vec}$ or $\mathbf{Symp}$. For short we will consider in the sequel only $\mathcal{C} = \mathbf{Vec}$. The other two cases are similar.

5.2.5. *Vector bundles.* Let $\mathbf{VBun}_q = \mathbf{Bun}_X(\mathbf{Vec}_q)$ be the category of finite dimensional $\mathbb{F}_q$-vector bundles over the set X. The types in $\mathbf{VBun}$ are given by partitions of $k = |X|$, $r = (r_1, \ldots, r_m)$, where

$$r_i = |\{x \in X \ :\ \dim(E_x) = i\}|$$

The ordering is given by

$$r \geq r' \ \Leftrightarrow\ \forall j,\ \sum_{i \geq j} r_i \geq \sum_{i \geq j} r'_i$$

Another identification of the set of types is

$$\mathcal{T}_{\mathbf{VBun}} = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0\}$$

where the type map is given by

$$\tau(E) = \lambda(E) = (\dim(E_{x_1}), \dim(E_{x_2}), \dots, \dim(E_{x_k}))$$

Here we assume $X = \{x_1, \dots, x_k\}$ is an ordering of X, such that $\dim(E_{x_i}) \geq \dim(E_{x_{i+1}})$. The ordering is now given by the lexicographic order

$$\lambda \geq \lambda' \;\Leftrightarrow\; \forall i,\ \lambda_i \geq \lambda'_i$$

Using these coordinates, we rewrite $\tilde{m} = (m, m, \dots, m)$. Denote

$$\mathcal{T}_{\leq \tilde{m}} = \{\lambda \in \mathcal{T}_{\mathbf{VBun}} \mid \lambda \leq \tilde{m}\}$$

$$\Lambda_k^m = \{(\lambda_1, \dots, \lambda_k) \mid m \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0\}$$

Then $\mathcal{T}_{\leq \tilde{m}} = \Lambda_k^m$. Let m, n be given, and assume $n \geq 2m$. The couple $(\tilde{n}, \tilde{m})$ is a symmetric couple. The automorphism group of an object of type $\tilde{n}$ is isomorphic to the wreath product $S_k \propto \mathrm{GL}_n(\mathbb{F}_q)$. It acts on the corresponding Grassmannian of $\binom{n}{m}^k$ elements.

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