arXiv:0704.1689v2 [math.CV] 25 Feb 2008

SOME PROPERTIES OF AND OPEN PROBLEMS ON HESSIAN NILPOTENT POLYNOMIALS

WENHUA ZHAO

ABSTRACT. In the recent work [BE1], [M], [Z1] and [Z2], the well-known Jacobian conjecture ([BCW], [E]) has been reduced to a problem on HN (Hessian nilpotent) polynomials (the polynomials whose Hessian matrix are nilpotent) and their (deformed) inversion pairs. In this paper, we prove several results on HN polynomials, their (deformed) inversion pairs as well as on the associated symmetric polynomial or formal maps. We also propose some open problems for further study of these objects.

1. Introduction

In the recent work [BE1], [M], [Z1] and [Z2], the well-known Jacobian conjecture (see [BCW] and [E]) has been reduced to a problem on HN (Hessian nilpotent) polynomials, i.e. the polynomials whose Hessian matrix are nilpotent, and their (deformed) inversion pairs. In this paper, we prove some properties of HN polynomials, the (deformed) inversion pairs of (HN) polynomial, the associated symmetric polynomial or formal maps, the graphs assigned to homogeneous harmonic polynomials, etc. Another purpose of this paper is to draw the reader's attention to some open problems which we believe will be interesting and important for further study of these objects.

In this section we first discuss some backgrounds and motivations in Subsection 1.1 for the study of HN polynomials and their (deformed) inversion pairs. We also fix some terminology and notation in this subsection that will be used throughout this paper. Then in Subsection 1.2 we give an arrangement description of this paper.

Date: January 21, 2016.

2000 *Mathematics Subject Classification.* 14R15, 32H02, 32A50.

Key words and phrases. Hessian nilpotent polynomials, inversion pairs, harmonic polynomials, the Jacobian conjecture.

The author has been partially supported by NSA Grant R1-07-0053.

1.1. **Background and Motivation.** Let $z = (z_1, z_2, \dots, z_n)$ be n free commutative variables. We denote by $\mathbb{C}[z]$ (resp. $\mathbb{C}[[z]]$) the algebra of polynomials (resp. formal power series) of z over $\mathbb{C}$. A polynomial or formal power series $P(z)$ is said to be *HN* (*Hessian nilpotent*) if its Hessian matrix $\operatorname{Hes} P := (\frac{\partial^2 P}{\partial z_i \partial z_j})$ are nilpotent. The study of HN polynomials is mainly motivated by the recent progress achieved in [BE1], [M], [Z1] and [Z2] on the well-known JC (Jacobian conjecture), which we will briefly explain below.

Recall that the JC first proposed by Keller [Ke] in 1939 claims: *for any polynomial map F of $\mathbb{C}^n$ with the Jacobian $j(F) = 1$, its formal inverse map G must also be a polynomial map.* Despite intense study for more than half a century, the conjecture is still open even for the case $n = 2$. For more history and known results before 2000 on the Jacobian conjecture, see [BCW], [E] and references there. In 2003, M. de Bondt, A. van den Essen ([BE1]) and G. Meng ([M]) independently made the following breakthrough on the JC.

Let $D_i := \frac{\partial}{\partial z_i}$ $(1 \le i \le n)$ and $D = (D_1, \dots, D_n)$. For any $P(z) \in \mathbb{C}[[z]]$, denote by $\nabla P(z)$ the *gradient* of $P(z)$, i.e. $\nabla P(z) := (D_1 P(z), \dots, D_n P(z))$. We say a formal map $F(z) = z - H(z)$ is *symmetric* if $H(z) = \nabla P(z)$ for some $P(z) \in \mathbb{C}[[z]]$. Then, the *symmetric reduction* of the JC achieved in [BE1] and [M] is that, *to prove or disprove the JC, it will be enough to consider only symmetric polynomial maps.* Combining with the classical *homogeneous reduction* achieved in [BCW] and [Y], one may further assume that *the symmetric polynomial maps have the form $F(z) = z - \nabla P(z)$ with $P(z)$ homogeneous* (*of degree* 4). Note that, in this case the Jacobian condition $j(F) = 1$ is equivalent to the condition that $P(z)$ is HN. For some other recent results on symmetric polynomial or formal maps, see [BE1]–[BE5], [EW], [M], [Wr1], [Wr2], [Z1], [Z2] and [EZ].

Based on the homogeneous reduction and the symmetric reduction of the JC discussed above, the author further showed in [Z2] that the JC is actually equivalent to the following so-called *vanishing conjecture* of HN polynomials.

Conjecture 1.1. (**Vanishing Conjecture**) *Let $\Delta := \sum_{i=1}^n D_i^2$ be the Laplace operator of $\mathbb{C}[z]$. Then, for any HN polynomial $P(z)$* (*of homogeneous of degree $d = 4$*), *$\Delta^m P^{m+1}(z) = 0$ when $m >> 0$.*

Furthermore, the following criterion of Hessian nilpotency for formal power series was also proved in [Z2].

Proposition 1.2. *For any $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$, the following statements are equivalent.*

(1) *$P(z)$ is HN.*
(2) *$\Delta^m P^m = 0$ for any $m \geq 1$.*
(3) *$\Delta^m P^m = 0$ for any $1 \leq m \leq n$.*

One crucial idea of the proofs in [Z2] for the results above is to study a special formal deformation of symmetric formal maps. More precisely, let t be a central formal parameter. For any $P(z) \in \mathbb{C}[[z]]$, we call $F(z) = z - \nabla P(z)$ the *associated symmetric maps* of $P(z)$. Let $F_t(z) = z - t\nabla P(z)$. When the order $o(P(z))$ of $P(z)$ with respect to z is greater than or equal to 2, $F_t(z)$ is a formal map of $\mathbb{C}[[t]][[z]]$ with $F_{t=1}(z) = F(z)$. Therefore, we may view $F_t(z)$ as a formal deformation of the formal map $F(z)$. In this case, one can also show (see [M] or Lemma 3.14 in [Z1]) that the formal inverse map $G_t(z) := F_t^{-1}(z)$ of $F_t(z)$ does exist and is also symmetric, i.e. there exists a unique $Q_t(z) \in \mathbb{C}[[t]][[z]]$ with $o(Q_t(z)) \geq 2$ such that $G_t(z) = z + t\nabla Q_t(z)$. We call $Q_t(z)$ the *deformed inversion pair* of $P(z)$. Note that, whenever $Q_{t=1}(z)$ makes sense, the formal inverse map $G(z)$ of $F(z)$ is given by $G(z) = G_{t=1}(z) = z + \nabla Q_{t=1}(z)$, so in this case we call $Q(z) := Q_{t=1}(z)$ the *inversion pair* of $P(z)$.

Note that, under the condition $o(P(z)) \geq 2$, the deformed inversion pair $Q_t(z)$ of $P(z)$ might not be in $\mathbb{C}[t][[z]]$, so $Q_{t=1}(z)$ may not make sense. But, if we assume further that $J(F_t)(0) = 1$, or equivalently, $(\operatorname{Hes} P)(0)$ is nilpotent, then $F_t(z)$ is an automorphism of $\mathbb{C}[t][[z]]$, hence so is its inverse map $G_t(z)$. Therefore, in this case $Q_t(z)$ lies in $\mathbb{C}[t][[z]]$ and $Q_{t=1}(z)$ makes sense. Throughout this paper, whenever the inversion pair $Q(z)$ of a polynomial or formal power series $P(z) \in \mathbb{C}[[z]]$ (not necessarily HN) is under concern, our assumption on $P(z)$ will always be $o(P(z)) \geq 2$ and $(\operatorname{Hes} P)(0)$ is nilpotent. Note that, for any HN $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$, the condition that $(\operatorname{Hes} P)(0)$ is nilpotent holds automatically.

For later purpose, let us recall the following formula derived in [Z2] for the deformed inversion pairs of HN formal power series.

Theorem 1.3. *Suppose $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$ is HN. Then, we have*

$$Q_t(z) = \sum_{m=0}^{\infty} \frac{t^m}{2^m m!(m+1)!} \Delta^m P^{m+1}(z), \tag{1.1}$$

From the equivalence of the JC and the VC discussed above, we see that the study on the HN polynomials and their (deformed) inversion pairs becomes important and necessary, at least when the JC is concerned. Note that, due to the identity $\operatorname{Tr}\operatorname{Hes} P = \Delta P$, HN polynomials are just a special family of harmonic polynomials which are among the most classical objects in mathematics. Even though harmonic polynomials had been very well studied since the late of the eighteenth century, it seems that not much has been known on HN polynomials. We believe that these mysterious (HN) polynomials deserve much more attentions from mathematicians.

1.2. **Arrangement.** Considering the length of this paper, we here give a more detailed arrangement description of the paper.

In Section 2, we consider the following two questions. Let $P, S, T \in \mathbb{C}[[z]]$ with $P = S + T$ and Q, U, V their inversion pairs, respectively.

$\mathbf{Q_1}$: *Under what conditions,* P *is HN iff both* S *and* T *are HN?*

$\mathbf{Q_2}$: *Under what conditions, we have* $Q = U + V$*?*

We give some sufficient conditions in Theorems 2.1 and 2.7 for the two questions above. In Section 3, we employ a recursion formula of inversion pairs derived in [Z1] and Eq. (1.1) above to derive some estimates for the radius of convergence of inversion pairs of homogeneous (HN) polynomials (see Propositions 3.1 and 3.3).

For any $P(z) \in \mathbb{C}[[z]]$, we say it is *self-inverting* if its inversion pair $Q(z)$ is $P(z)$ itself. In Section 4, by using a general result on quasi-translations proved in [B], we derive some properties of HN *self-inverting* formal power series $P(z)$. Another purpose of this section is to draw the reader's attention to Open Problem 4.8 on classification of HN self-inverting polynomials or formal power series.

In Section 5, we show in Proposition 5.1, when the base field has characteristic $p > 0$, the VC, unlike the JC, actually holds for any polynomials $P(z)$ even without the HN condition on $P(z)$. It also holds in this case for any HN formal power series. One interesting question (see Open Problem 5.2) is to see if the VC like the JC fails over $\mathbb{C}$ when $P(z)$ is allowed to be any HN formal power series.

In Section 6, we prove a criterion of Hessian nilpotency for homogeneous polynomials over $\mathbb{C}$ (see Theorem 6.1). Considering the criterion in Proposition 1.2, this criterion is somewhat surprising but its proof turns out to be very simple.

Section 7 is mainly motivated by the following question raised by M. Kumar ([K]) and D. Wright ([Wr3]). Namely, for a symmetric formal map $F(z) = z - \nabla P(z)$, how to write $f(z) := \frac{1}{2}\sigma_2 - P(z)$ (where $\sigma_2 := \sum_{i=1}^n z_i^2$) and $P(z)$ itself as formal power series in $F(z)$? In this section, we derive some explicit formulas to answer the questions above and also for the same question for σ_2 (see Proposition 7.2). From these formulas, we also show in Theorem 7.4 that, the VC holds for a HN polynomial $P(z)$ iff one (hence, all) of σ_2, $P(z)$ and $f(z)$ can be written as a polynomial in F, where $F(z) = z - \nabla P(z)$ is the associated polynomial maps of $P(z)$.

Finally, in Section 8, we discuss a graph $\mathcal{G}(P)$ assigned to each homogeneous harmonic polynomials $P(z)$. The graph $\mathcal{G}(P)$ was first proposed by the author and later was further studied by Roel Willems in his master thesis [Wi] under direction of Professor Arno van den Essen. In Subsection 8.1 we give the definition of the graph $\mathcal{G}(P)$ for any homogeneous harmonic polynomial $P(z)$ and discuss the *connectedness reduction* (see Corollary 8.5) which says, to study the VC for homogeneous HN polynomials $P(z)$, it will be enough to consider the case when the graph $\mathcal{G}(P)$ is connected. In Subsection 8.2 we consider a connection of $\mathcal{G}(P)$ with the tree expansion formula derived in [M] and [Wr2] for the inversion pair $Q(z)$ of $P(z)$ (see also Proposition 8.9). As an application of the connection, we use it to give another proof for the connectedness reduction discussed in Corollary 8.5.

One final remark on the paper is as follows. Even though we could have focused only on (HN) polynomials, at least when only the JC is concerned, we will formulate and prove our results in the more general setting of (HN) formal power series whenever it is possible.

Acknowledgement: The author is very grateful to Professors Arno van den Essen, Mohan Kumar and David Wright for inspiring communications and constant encouragement. Section 7 was mainly motivated by some questions raised by Professors Mohan Kumar and David Wright. The author also would like to thank Roel Willems for sending the author his master thesis in which he has obtained some very interesting results on the graphs $\mathcal{G}(P)$ of homogeneous harmonic polynomials. At last but not the least, the author thanks the referee and the editor for many valuable suggestions.

2. Disjoint Formal Power Series and Their Deformed Inversion Pairs

Let $P, S, T \in \mathbb{C}[[z]]$ with $P = S + T$, and Q, U and V their inversion pairs, respectively. In this section, we consider the following two questions:

$\mathbf{Q_1}$: *Under what conditions, P is HN if and only if both S and T are HN?*

$\mathbf{Q_2}$: *Under what conditions, we have $Q = U + V$?*

We give some answers to the questions $\mathbf{Q_1}$ and $\mathbf{Q_2}$ in Theorems 2.1 and 2.7, respectively. The results proved here will also be needed in Section 8 when we consider a graph associated to homogeneous harmonic polynomials.

To question $\mathbf{Q_1}$ above, we have the following result.

Theorem 2.1. *Let $S, T \in \mathbb{C}[[z]]$ such that $\langle \nabla(D_i S), \nabla(D_j T)\rangle = 0$ for any $1 \le i, j \le n$, where $\langle \cdot, \cdot \rangle$ denotes the standard $\mathbb{C}$-bilinear form of $\mathbb{C}^n$. Let $P = S + T$. Then, we have*

(a) *$\text{Hes}\,(S)\,\text{Hes}\,(T) = \text{Hes}\,(T)\,\text{Hes}\,(S) = 0$.*

(b) *P is HN iff both S and T are HN.*

Note that statement (b) in the theorem above was first proved by R. Willems ([Wi]) in a special setting as in Lemma 2.6 below for homogeneous harmonic polynomials.

Proof: (a) For any $1 \le i, j \le n$, consider the $(i,j)^{th}$ entry of the product $\text{Hes}\,(S)\text{Hes}\,(T)$:

$$\sum_{k=1}^{n} \frac{\partial^2 S}{\partial z_i \partial z_k} \frac{\partial^2 T}{\partial z_k \partial z_j} = \langle \nabla(D_i S), \nabla(D_j T)\rangle = 0. \tag{2.1}$$

Hence $\text{Hes}\,(S)\,\text{Hes}\,(T) = 0$. Similarly, we have $\text{Hes}\,(T)\,\text{Hes}\,(S) = 0$.

(b) follows directly from (a) and the lemma below. $\square$

Lemma 2.2. *Let A, B and C be $n \times n$ matrices with entries in any commutative ring. Suppose that $A = B + C$ and $BC = CB = 0$. Then, A is nilpotent iff both B and C are nilpotent.*

Proof: The $(\Leftarrow)$ part is trivial because B and C in particular commute with each other.

To show $(\Rightarrow)$, note that $BC = CB = 0$. So for any $m \ge 1$, we have

$$A^m B = (B + C)^m B = (B^m + C^m)B = B^{m+1}.$$

Similarly, we have $C^{m+1} = A^m C$. Therefore, if $A^N = 0$ for some $N \geq 1$, we have $B^{N+1} = C^{N+1} = 0$. $\square$

Note that, for the ($\Leftarrow$) part of (b) in Theorem 2.1, we need only a weaker condition. Namely, for any $1 \leq i, j \leq n$,

$$\langle \nabla(D_i S), \nabla(D_j T)\rangle = \langle \nabla(D_j S), \nabla(D_i T)\rangle,$$

which will ensure that $\mathrm{Hes}\,(S)$ and $\mathrm{Hes}\,(T)$ commute.

To consider the second question $\mathbf{Q_2}$, let us first fix the following notation.

For any $P \in \mathbb{C}[[z]]$, let $\mathcal{A}(P)$ denote the subalgebra of $\mathbb{C}[[z]]$ generated by all partial derivatives of P (of any order). We also define a sequence $\{Q_{[m]}(z)\,|, m \geq 1\}$ by writing the deformed inversion pair $Q_t(z)$ of $P(z)$ as

$$Q_t(z) = \sum_{m\geq 1} t^{m-1} Q_{[m]}(z). \tag{2.2}$$

Lemma 2.3. *For any $P \in \mathbb{C}[[z]]$, we have*

(a) *$\mathcal{A}(P)$ is closed under the action of any differential operator of $\mathbb{C}[z]$ with constant coefficients.*

(b) *For any $m \geq 1$, we have $Q_{[m]}(z) \in \mathcal{A}(P)$.*

Proof: (a) Note that, by the definition of $\mathcal{A}(P)$, a formal power series $g(z) \in \mathbb{C}[[z]]$ lies in $\mathcal{A}(P)$ iff it can be written (not necessarily uniquely) as a polynomial in partial derivatives of $P(z)$. Then, by the Leibniz Rule, it is easy to see that, for any $g(z) \in \mathcal{A}(P)$, $D_i g(z) \in \mathcal{A}(P)$ $(1 \leq i \leq n)$. Repeating this argument, we see that any partial derivative of $g(z)$ is in $\mathcal{A}(P)$. Hence (a) follows.

(b) Recall that, by Proposition 3.7 in [Z1], we have the following recurrent formula for $Q_{[m]}(z)$ $(m \geq 1)$ in general:

$$Q_{[1]}(z) = P(z), \tag{2.3}$$

$$Q_{[m]}(z) = \frac{1}{2(m-1)} \sum_{\substack{k,l\geq 1\\ k+l=m}} \langle \nabla Q_{[k]}(z), \nabla Q_{[l]}(z)\rangle. \tag{2.4}$$

for any $m \geq 2$.

By using (a), the recurrent formulas above and induction on $m \geq 1$, it is easy to check that (b) holds too. $\square$

Definition 2.4. *For any $S, T \in \mathbb{C}[[z]]$, we say S and T are disjoint to each other if, for any $g_1 \in \mathcal{A}(S)$ and $g_2 \in \mathcal{A}(T)$, we have $\langle \nabla g_1, \nabla g_2\rangle = 0$.*

This terminology will be justified in Section 8 when we consider a graph $\mathcal{G}(P)$ associated to homogeneous harmonic polynomials P.

Lemma 2.5. *Let $S, T \in \mathbb{C}[[z]]$. Then S and T are disjoint to each other iff, for any $\alpha, \beta \in \mathbb{N}^n$, we have*

$$\langle \nabla(D^\alpha S), \nabla(D^\beta T)\rangle = 0. \tag{2.5}$$

Proof: The ($\Rightarrow$) part of the lemma is trivial. Conversely, for any $g_1 \in \mathcal{A}(S)$ and $g_2 \in \mathcal{A}(T)$ $(i = 1, 2)$, we need show

$$\langle \nabla g_1, \nabla g_2\rangle = 0.$$

But this can be easily checked by, first, reducing to the case that g_1 and g_2 are monomials of partial derivatives of S and T, respectively, and then applying the Leibniz rule and Eq. (2.5) above. $\square$

A family of examples of disjoint polynomials or formal power series are given as in the following lemma, which will also be needed later in Section 8.

Lemma 2.6. *Let I_1 and I_2 be two finite subsets of $\mathbb{C}^n$ such that, for any $\alpha_i \in I_i$ $(i = 1, 2)$, we have $\langle \alpha_1, \alpha_2\rangle = 0$. Denote by $\mathcal{A}_i$ $(i = 1, 2)$ the completion of the subalgebra of $\mathbb{C}[[z]]$ generated by $h_\alpha(z) := \langle \alpha, z\rangle$ $(\alpha \in I_i)$, i.e. $\mathcal{A}_i$ is the set of all formal power series in $h_\alpha(z)$ $(\alpha \in I_i)$ over $\mathbb{C}$. Then, for any $P_i \in \mathcal{A}_i$ $(i = 1, 2)$, P_1 and P_2 are disjoint.*

Proof: First, by a similar argument as the proof for Lemma 2.3, (a), it is easy to check that $\mathcal{A}_i$ $(i = 1, 2)$ are closed under action of any differential operator with constant coefficients. Secondly, since $\mathcal{A}_i$ $(i = 1, 2)$ are subalgebras of $\mathbb{C}[[z]]$, we have $\mathcal{A}(P_i) \subset \mathcal{A}_i$ $(i = 1, 2)$.

Therefore, to show P_1 and P_2 are disjoint to each other, it will be enough to show that, for any $g_i \in \mathcal{A}_i$ $(i = 1, 2)$, we have $\langle \nabla g_1, \nabla g_2\rangle = 0$. But this can be easily checked by first reducing to the case when g_i $(i = 1, 2)$ are monomials of $h_\alpha(z)$ $(\alpha \in I_i)$, and then applying the Leibniz rule and the following identity: for any $\alpha, \beta \in \mathbb{C}^n$,

$$\langle \nabla h_\alpha(z), \nabla h_\beta(z)\rangle = \langle \alpha, \beta\rangle.$$

$\square$

Now, for the second question $\mathbf{Q_2}$ on page 6, we have the following result.

Theorem 2.7. *Let $P, S, T \in \mathbb{C}[[z]]$ with order greater than or equal to 2, and Q_t, U_t, V_t their deformed inversion pairs, respectively. Assume that $P = S + T$ and S, T are disjoint to each other. Then*

(a) U_t and V_t are also disjoint to each other, i.e. for any $\alpha, \beta \in \mathbb{N}^n$, we have

$$\left\langle \nabla D^\alpha U_t(z), \nabla D^\beta V_t(z) \right\rangle = 0.$$

(b) We further have

$$Q_t = U_t + V_t. \tag{2.6}$$

Proof: (a) follows directly from Lemma 2.3, (b) and Lemma 2.5.

(b) Let $Q_{[m]}$, $U_{[m]}$ and $V_{[m]}$ $(m \geq 1)$ be defined as in Eq. (2.2). Hence it will be enough to show

$$Q_{[m]} = U_{[m]} + V_{[m]} \tag{2.7}$$

for any $m \geq 1$.

We use induction on $m \geq 1$. When $m = 1$, Eq. (2.7) follows from the condition $P = S + T$ and Eq. (2.3) . For any $m \geq 2$, by Eq. (2.4) and the induction assumption, we have

$$\begin{aligned} Q_{[m]} &= \frac{1}{2(m-1)} \sum_{\substack{k,l \geq 1 \\ k+l=m}} \langle \nabla Q_{[k]}, \nabla Q_{[l]} \rangle \\ &= \frac{1}{2(m-1)} \sum_{\substack{k,l \geq 1 \\ k+l=m}} \langle \nabla U_{[k]} + \nabla V_{[k]}, \nabla U_{[l]} + \nabla V_{[l]} \rangle \end{aligned}$$

Noting that, by Lemma 2.3, $U_{[j]} \in \mathcal{A}(S)$ and $V_{[j]} \in \mathcal{A}(T)$ $(1 \leq j \leq m)$:

$$= \frac{1}{2(m-1)} \sum_{\substack{k,l \geq 1 \\ k+l=m}} \langle \nabla U_{[k]}, \nabla U_{[l]} \rangle + \frac{1}{2(m-1)} \sum_{\substack{k,l \geq 1 \\ k+l=m}} \langle \nabla V_{[k]}, \nabla V_{[l]} \rangle$$

Applying the recursion formula Eq. (2.4) to both $U_{[m]}$ and $V_{[m]}$:

$$= U_{[m]} + V_{[m]}.$$

□

As later will be pointed out in Remark 8.11, one can also prove this theorem by using a tree expansion formula of inversion pairs, which was derived in [M] and [Wr2], in the setting as in Lemma 2.6.

From Theorems 2.1, 2.7 and Eqs. (1.1), (2.2), it is easy to see that we have the following corollary.

Corollary 2.8. *Let $P_i \in \mathbb{C}[[z]]$ $(1 \leq i \leq k)$ which are disjoint to each other. Set $P = \sum_{i=1}^{k} P_i$. Then, we have*

(a) P is HN iff each P_i is HN.

(b) Suppose that P is HN. Then, for any $m \geq 0$, we have

$$\Delta^m P^{m+1} = \sum_{i=1}^{k} \Delta^m P_i^{m+1}. \tag{2.8}$$

Consequently, if the VC holds for each P_i, then it also holds for P.

3. **Local Convergence of Deformed Inversion Pairs of Homogeneous (HN) Polynomials**

Let $P(z)$ be a formal power series which is convergent near $0 \in \mathbb{C}^n$. Then the associated symmetric map $F(z) = z - \nabla P$ is a well-defined analytic map from an open neighborhood of $0 \in \mathbb{C}^n$ to $\mathbb{C}^n$. If we further assume that $JF(0) = I_{n\times n}$, the formal inverse $G(z) = z + \nabla Q(z)$ of $F(z)$ is also locally well-defined analytic map. So the inversion pair $Q(z)$ of $P(z)$ is also locally convergent near $0 \in \mathbb{C}^n$. In this section, we use the formulas Eqs. (2.4), (1.1) and the Cauchy estimates to derive some estimates for the radius of convergence of inversion pairs $Q(z)$ of homogeneous (HN) polynomials $P(z)$ (see Propositions 3.1 and 3.3).

First let us fix the following notation.

For any $a \in \mathbb{C}^n$ and $r > 0$, we denote by $B(a, r)$ (resp. $S(a, r)$) the open ball (resp. the sphere) centered at $a \in \mathbb{C}$ with radius $r > 0$. The unit sphere $S(0, 1)$ will also be denoted by S^{2n-1}. Furthermore, we let $\Omega(a, r)$ be the polydisk centered at $a \in \mathbb{C}^n$ with radius $r > 0$, i.e. $\Omega(a, r) := \{z \in \mathbb{C}^n \,|\, |z_i - a_i| < r,\, 1 \leq i \leq n\}$. For any subset $A \subset \mathbb{C}^n$, we will use $\bar{A}$ to denote the closure of A in $\mathbb{C}^n$.

For any polynomial $P(z) \in \mathbb{C}[z]$ and a compact subset $D \subset \mathbb{C}^n$, we set $|P|_D$ to be the maximum value of $|P(z)|$ over D. In particular, when D is the unit sphere S^{2n-1}, we also write $|P| = |P|_D$, i.e.

$$|P| := \max\{|P(z)| \,|\, z \in S^{2n-1}\}. \tag{3.1}$$

Note that, for any $r \geq 0$ and $a \in B(0,r)$, we have $\Omega(a,r) \subset B(a,r) \subset B(0,2r)$. Combining with the well-known Maximum Principle of holomorphic functions, we get

$$|P|_{\overline{\Omega(a,r)}} \leq |P|_{\overline{B(a,r)}} \leq |P|_{\overline{B(0,2r)}} = |P|_{S(0,2r)}. \tag{3.2}$$

For the inversion pairs Q of homogeneous polynomials P without HN condition, we have the following estimate for the radius of convergence at $0 \in \mathbb{C}^n$.

Proposition 3.1. *Let $P(z)$ be a non-zero homogeneous polynomial (not necessarily HN) of degree $d \geq 3$ and $r_0 = (n2^{d-1}|P|)^{\frac{1}{2-d}}$. Then the inversion pair $Q(z)$ converges over the open ball $B(0,r_0)$.*

To prove the proposition, we need the following lemma.

Lemma 3.2. *Let $P(z)$ be any polynomial and $r > 0$. Then, for any $a \in B(0,r)$ and $m \geq 1$, we have*

$$\left|Q_{[m]}(a)\right| \leq \frac{n^{m-1}|P|^m_{S(0,2r)}}{2^{m-1}r^{2m-2}}. \tag{3.3}$$

Proof: We use induction on $m \geq 1$. First, when $m = 1$, by Eq. (2.3) we have $Q_{[1]} = P$. Then Eq. (3.3) follows from the fact $B(a,r) \subset B(0,2r)$ and the maximum principle of holomorphic functions.

Assume Eq. (3.3) holds for any $1 \leq k \leq m-1$. Then, by the Cauchy estimates of holomorphic functions (e.g. see Theorem 1.6 in [R]), we have

$$\left|(D_iQ_{[k]})(a)\right| \leq \frac{1}{r}\left|Q_{[k]}\right|_{\overline{\Omega(0,r)}} \leq \frac{n^{k-1}|P|^k_{B(0,2r)}}{2^{k-1}r^{2k-1}}. \tag{3.4}$$

By Eqs. (2.4) and (3.4), we have

$$\begin{aligned} |Q_{[m]}(a)| &\leq \frac{1}{2(m-1)} \sum_{\substack{k,l\geq 1\\ k+l=m}} \left|\langle \nabla Q_{[k]}, \nabla Q_{[l]}\rangle\right| \\ &\leq \frac{1}{2(m-1)} \sum_{\substack{k,l\geq 1\\ k+l=m}} n\, \frac{n^{k-1}|P|^k_{S(0,2r)}}{2^{k-1}r^{2k-1}} \frac{n^{\ell-1}|P|^\ell_{S(0,2r)}}{2^{\ell-1}r^{2\ell-1}} \\ &= \frac{n^{m-1}|P|^m_{S(0,2r)}}{2^{m-1}r^{2m-2}}. \end{aligned}$$

$\square$

Proof of Proposition 3.1: By Eq. (2.2) , we know that,

$$Q(z) = \sum_{m\geq 1} Q_{[m]}(z). \tag{3.5}$$

To show the proposition, it will be enough to show the infinite series above converges absolutely over $B(0, r)$ for any $r < r_0$.

First, for any $m \geq 1$, let A_m be the RHS of the inequality Eq. (3.3). Note that, since P is homogeneous of degree $d \geq 3$, we further have

$$|P|^m_{B(0,2r)} = \left((2r)^d |P|_{S^{2n-1}}\right)^m = (2r)^{dm}|P|^m. \tag{3.6}$$

Therefore, for any $m \geq 1$, we have

$$A_m = 2^{(d-1)m+1} n^{m-1} r^{(d-2)m+2} |P|^m, \tag{3.7}$$

and by Lemma 3.2,

$$|Q_{[m]}(a)| \leq A_m \tag{3.8}$$

for any $a \in B(0, r)$.

Since $0 < r < r_0 = (n2^{d-1}|P|)^{2-d}$, it is easy to see that

$$\lim_{m\to+\infty} \frac{A_{m+1}}{A_m} = n2^{d-1} r^{d-2} |P| < 1.$$

Therefore, by the comparison test, the infinite series in Eq. (3.5) converges absolutely and uniformly over the open ball $B(0, r)$. $\square$

Note that the estimate given in Proposition 3.1 depends on the number n of variables. Next we show that, with the HN condition on P, an estimate independent of n can be obtained as follows.

Proposition 3.3. *Let $P(z)$ be a homogeneous HN polynomial of degree $d \geq 4$ and set $r_0 := (2^{d+1}|P|)^{\frac{1}{2-d}}$. Then, the inversion pair $Q(z)$ of $P(z)$ converges over the open ball $B(0, r_0)$.*

Note that, when $d = 2$ or 3, by Wang's Theorem ([Wa]), the JC holds in general. Hence it also holds for the associated symmetric map $F(z) = z - \nabla P$ when $P(z)$ is HN. Therefore $Q(z)$ in this case is also a polynomial of z and converges over the whole space $\mathbb{C}^n$.

To prove the proposition above, we first need the following two lemmas.

Lemma 3.4. *Let $P(z)$ be a homogeneous polynomial of degree $d \geq 1$ and $r > 0$. For any $a \in B(0,r)$, $m \geq 0$ and $\alpha \in \mathbb{N}^n$, we have*

$$|(D^\alpha P^{m+1})(a)| \leq \frac{\alpha!}{r^{|\alpha|}}(2r)^{d(m+1)}|P|^{m+1}. \tag{3.9}$$

Proof: First, by the Cauchy estimates and Eq. (3.2), we have

$$|(D^\alpha P^{m+1})(a)| \leq \frac{\alpha!}{r^{|\alpha|}}|P^{m+1}|_{\overline{\Omega(a,r)}} \leq \frac{\alpha!}{r^{|\alpha|}}|P^{m+1}|_{\overline{B(0,2r)}}. \tag{3.10}$$

On the other hand, by the maximum principle and the condition that P is homogeneous of degree $d \geq 3$, we have

$$\begin{aligned} |P^{m+1}|_{\overline{B(0,2r)}} &= |P|^{m+1}_{\overline{B(0,2r)}} = |P|^{m+1}_{\overline{S(0,2r)}} = ((2r)^d|P|)^{m+1} \\ &= (2r)^{d(m+1)}|P|^{m+1}. \end{aligned} \tag{3.11}$$

Then, combining Eqs. (3.10) and (3.11), we get Eq. (3.9). □

Lemma 3.5. *For any $m \geq 1$, we have*

$$\sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=m}} \alpha! \leq m!\binom{m+n-1}{m} = \frac{(m+n-1)!}{(n-1)!}. \tag{3.12}$$

Proof: First, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$, we have $\alpha! \leq m!$ since the binomial $\binom{m}{\alpha} = \frac{m!}{\alpha!}$ is always a positive integer. Therefore, we have

$$\sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=m}} \alpha! \leq m! \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=m}} 1.$$

Secondly, note that $\sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=m}} 1$ is just the number of distinct $\alpha \in \mathbb{N}^n$ with $|\alpha| = m$, which is the same as the number of distinct monomials in n free commutative variables of degree m. Since the latter is well-known to be the binomial $\binom{m+n-1}{m}$, we have

$$\sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=m}} \alpha! \leq m!\binom{m+n-1}{m} = \frac{(m+n-1)!}{(n-1)!}.$$

□

Proof of Proposition 3.3: By Eq. (1.1) , we know that,

$$Q(z) = \sum_{m\geq 1} \frac{\Delta^m P^{m+1}}{2^m m!(m+1)!}. \tag{3.13}$$

To show the proposition, it will be enough to show the infinite series above converges absolutely over $B(0,r)$ for any $r < r_0$.

We first give an upper bound for the general terms in the series Eq. (3.13) over $B(0,r)$.

Consider

$$\Delta^m P^{m+1} = (\sum_{i=1}^n D_i^2)^m P^{m+1} = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=m}} \frac{m!}{\alpha!} D^{2\alpha} P^{m+1}. \tag{3.14}$$

Therefore, we have

$$|\Delta^m P^{m+1}(a)| \leq \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=m}} \frac{m!}{\alpha!} |D^{2\alpha} P^{m+1}(a)|$$

Applying Lemma 3.4 with α replaced by 2α:

$$\leq \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=m}} \frac{m!}{\alpha!} \frac{(2\alpha)!}{r^{2m}} (2r)^{d(m+1)} |P|^{m+1}$$

Noting that $(2\alpha)! \leq [(2\alpha)!!]^2 = 2^{2m}(\alpha!)^2$:

$$\begin{aligned} &\leq \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=m}} \frac{m!}{\alpha!} \frac{2^{2m}(\alpha!)^2}{r^{2m}} (2r)^{d(m+1)} |P|^{m+1} \\ &= m! 2^{2m+d(m+1)} r^{d(m+1)-2m} |P|^{m+1} \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=m}} \alpha! \end{aligned}$$

Applying Lemma 3.5:

$$= \frac{m!(m+n-1)! 2^{2m+d(m+1)} r^{d(m+1)-2m} |P|^{m+1}}{(n-1)!}.$$

Therefore, for any $m \geq 1$, we have

$$\left| \frac{\Delta^m P^{m+1}}{2^m m!(m+1)!} \right| \leq \frac{2^{m+d(m+1)} r^{d(m+1)-2m} |P|^{m+1} (m+n-1)!}{(m+1)!(n-1)!}. \tag{3.15}$$

For any $m \geq 1$, let A_m be the right hand side of Eq. (3.15) above. Then, by a straightforward calculation, we see that the ratio

$$\frac{A_{m+1}}{A_m} = \frac{m+n}{m+2}\, 2^{d+1} r^{d-2} |P|. \tag{3.16}$$

Since $r < r_0 = (2^{d+1}|P|)^{\frac{1}{2-d}}$, it is easy to see that

$$\lim_{m\to+\infty} \frac{A_{m+1}}{A_m} = 2^{d+1} r^{d-2} |P| < 1.$$

Therefore, by the comparison test, the infinite series in Eq. (3.13) converges absolutely and uniformly over the open ball $B(0, r)$. $\square$

4. Self-Inverting Formal Power Series

Note that, by the definition of inversion pairs (see page 3), $Q \in \mathbb{C}[[z]]$ is the inversion pair of $P \in \mathbb{C}[[z]]$ iff P is the inversion pair of Q. In other words, the relation that Q and P are inversion pair of each other in some sense is a duality relation. Naturally, one may ask, for which $P(z)$, it is self-dual or self-inverting? In this section, we discuss this special family of polynomials or formal power series.

Another purpose of this section is to draw the reader's attention to the problem of classification of (HN) self-inverting polynomials (see Open Problem 4.8). Even though the classification of HN polynomials seems to be out of reach at the current time, we believe that the classification of (HN) self-inverting polynomials is much more approachable.

Definition 4.1. *A formal power series $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$ and $(\mathrm{Hes}\, P)(0)$ nilpotent is said to be self-inverting if its inversion pair $Q(z) = P(z)$.*

Following the terminology introduced in [B], we say a formal map $F(z) = z - H(z)$ with $H(z) \in \mathbb{C}[[z]]^{\times n}$ and $o(H(z)) \geq 1$ is a *quasi-translation* if $j(F)(0) \neq 0$ and its formal inverse map is given by $G(z) = z + H(z)$.

Therefore, for any $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$ and $(\mathrm{Hes}\, P)(0)$ nilpotent, it is self-inverting iff the associated symmetric formal map $F(z) = z - \nabla P(z)$ is a quasi-translation.

For quasi-translations, the following general result has been proved in Proposition 1.1 of [B] for polynomial quasi-translations.

Proposition 4.2. *A formal map $F(z) = z - H(z)$ with $o(H) \geq 1$ and $JH(0)$ nilpotent is a quasi-translation if and only if $JH \cdot H = 0$.*

Even though the proposition above was proved in [B] only in the setting of polynomial maps, the proof given there works equally well for formal quasi-translations under the condition that $JH(0)$ is nilpotent. Since it has also been shown in Proposition 1.1 in [B] that, for any polynomial quasi-translations $F(z) = z - H(z)$, $JH(z)$ is always nilpotent, so the condition that $JH(0)$ is nilpotent in the proposition above does not put any extra restriction for the case of polynomial quasi-translations.

From Proposition 4.2 above, we immediately have the following criterion for self-inverting formal power series.

Proposition 4.3. *For any $P(z) \in \mathbb{C}[[z]]$ with $o(P) \geq 2$ and $(Hes\, P)(0)$ nilpotent, it is self-inverting if and only if $\langle \nabla P, \nabla P \rangle = 0$.*

Proof: Since $o(P) \geq 2$ and $(\mathrm{Hes}\, P)(0)$ is nilpotent, by Proposition 4.2, we see that, $P(z) \in \mathbb{C}[[z]]$ is self-inverting iff $J(\nabla P) \cdot \nabla P = (\mathrm{Hes}\, P) \cdot \nabla P = 0$. But, on the other hand, it is easy to check that, for any $P(z) \in \mathbb{C}[[z]]$, we have the following identity:

$$(\mathrm{Hes}\, P) \cdot \nabla P = \frac{1}{2} \nabla \langle \nabla P, \nabla P \rangle.$$

Therefore, $(\mathrm{Hes}\, P) \cdot \nabla P = 0$ iff $\nabla \langle \nabla P, \nabla P \rangle = 0$, and iff $\langle \nabla P, \nabla P \rangle = 0$ because $o(\langle \nabla P, \nabla P \rangle) \geq 2$. $\square$

Corollary 4.4. *For any $P(z) \in \mathbb{C}[[z]]$ with $o(P) \geq 2$ and $(Hes\, P)(0)$ nilpotent, if it is self-inverting, then so is $P^m(z)$ for any $m \geq 1$.*

Proof: Note that, for any $m \geq 2$, we have $o(P^m(z)) \geq 2m > 2$ and $(\mathrm{Hes}\, P)(0) = 0$. Then, the corollary follows immediately from Proposition 4.3 and the following general identity:

$$\langle \nabla P^m, \nabla P^m \rangle = m^2 P^{2m-2} \langle \nabla P, \nabla P \rangle. \tag{4.1}$$

$\square$

Corollary 4.5. *For any harmonic formal power series $P(z) \in \mathbb{C}[[z]]$ with $o(P) \geq 2$ and $(Hes\, P)(0)$ nilpotent, it is self-inverting iff $\Delta P^2 = 0$.*

Proof: This follows immediately from Proposition 4.3 and the following general identity:

$$\Delta P^2 = 2(\Delta P)P + 2\langle \nabla P, \nabla P \rangle. \tag{4.2}$$

$\square$

Proposition 4.6. *Let $P(z)$ be a harmonic self-inverting formal power series. Then, for any $m \geq 1$, P^m is HN.*

Proof: First, we use the mathematical induction on $m \geq 1$ to show that $\Delta P^m = 0$ for any $m \geq 1$.

The case of $m = 1$ is given. For any $m \geq 2$, consider

$$\begin{aligned} \Delta P^m &= \Delta(P \cdot P^{m-1}) \\ &= (\Delta P)P^{m-1} + P(\Delta P^{m-1}) + 2\langle \nabla P, \nabla P^{m-1} \rangle \\ &= (\Delta P)P^{m-1} + P(\Delta P^{m-1}) + 2(m-1)P^{m-2}\langle \nabla P, \nabla P \rangle. \end{aligned}$$

Then, by the mathematical induction assumption and Proposition 4.3, we get $\Delta P^m = 0$.

Secondly, for any fixed $m \geq 1$ and $d \geq 1$, we have

$$\Delta^d[(P^m)^d] = \Delta^{d-1}(\Delta P^{dm}) = 0.$$

Then, by the criterion in Proposition 1.2, P^m is HN. $\square$

Example 4.7. *Note that, in Section* 5.2 *of* [Z2]*, a family of self-inverting HN formal power series has been constructed as follows.*

Let Ξ be any non-empty subset of $\mathbb{C}^n$ such that, for any $\alpha, \beta \in \Xi$, $\langle \alpha, \beta \rangle = 0$. Let $\mathcal{A}$ be the completion of the subalgebra of $\mathbb{C}[[z]]$ generated by $h_\alpha(z) := \langle \alpha, z \rangle$ $(\alpha \in \Xi)$, i.e. $\mathcal{A}$ is the set of all formal power series in $h_\alpha(z)$ $(\alpha \in \Xi)$ over $\mathbb{C}$. Then it is straightforward to check (*or see Section* 5.2 *of* [Z2] *for details*) *that any element $P(z) \in \mathcal{A}$ is HN and self-inverting.*

It is unknown if all HN self-inverting polynomials or formal power series can be obtained by the construction above. More generally, we believe the following open problem is worth investigating.

Open Problem 4.8. (*a*) *Decide whether or not all self-inverting polynomials or formal power series are HN.*

(*b*) *Classify all* (*HN*) *self-inverting polynomials and formal power series.*

Finally, let us point out that, for any self-inverting $P(z) \in \mathbb{C}[[z]]$, the deformed inversion pair $Q_t(z)$ (not just $Q(z) = Q_{t=1}(z)$) is also same as $P(z)$.

Proposition 4.9. *Let $P(z) \in \mathbb{C}[[z]]$ with $o(P) \geq 2$ and $(Hes\, P)(0)$ nilpotent. Then $P(z)$ is self-inverting if and only if $Q_t(z) = P(z)$.*

Proof: First, let us point out the following observations.

Let t be a formal central parameter and $F_t(z) = z - t\nabla P(z)$ as before. Since $o(P) \geq 2$ and $(\operatorname{Hes} P)(0)$ is nilpotent, we have $j(F_t)(0) = 1$. Therefore, $F_t(z)$ is an automorphism of the algebra $\mathbb{C}[t][[z]]$ of formal power series of z over $\mathbb{C}[t]$. Since the inverse map of $F_t(z)$ is given by $G_t(z) = z + t\nabla Q_t(z)$, we see that $Q_t(z) \in \mathbb{C}[t][[z]]$. Therefore, for any $t_0 \in \mathbb{C}$, $Q_{t=t_0}(z)$ makes sense and lies in $\mathbb{C}[[z]]$. Furthermore, by the uniqueness of inverse maps, it is easy to see that the inverse map of $F_{t_0} = z - t_0\nabla P$ of $\mathbb{C}[t][[z]]$ is given by $G_{t_0}(z) = z + t_0\nabla Q_{t=t_0}$. Therefore the inversion pair of $t_0P(z)$ is given by $t_0Q_{t=t_0}(z)$.

With the notation and observations above, by choosing $t_0 = 1$, we have $Q_{t=1}(z) = Q(z)$ and the $(\Leftarrow)$ part of the proposition follows immediately. Conversely, for any $t_0 \in \mathbb{C}$, we have $\langle \nabla(t_0P), \nabla(t_0P)\rangle = t_0^2\langle \nabla P, \nabla P\rangle$. Then, by Proposition 4.3, $t_0P(z)$ is self-inverting and its inversion pair $t_0Q_{t=t_0}(z)$ is same as $t_0P(z)$, i.e. $t_0Q_{t=t_0}(z) = t_0P(z)$. Therefore, we have $Q_{t=t_0}(z) = P(z)$ for any $t_0 \in \mathbb{C}^\times$. But on the other hand, we have $Q_t(z) \in \mathbb{C}[t][[z]]$ as pointed above, i.e. the coefficients of all monomials of z in $Q_t(z)$ are polynomials of t, hence we must have $Q_t(z) = P(z)$ which is the $(\Rightarrow)$ part of the proposition. $\square$

5. The Vanishing Conjecture over Fields of Positive Characteristic

It is well-known that the JC may fail when $F(z)$ is not a polynomial map (e.g. $F_1(z_1, z_2) = e^{-z_1}$; $F_2(z_1, z_2) = z_2e^{z_1}$). It also fails badly over fields of positive characteristic even in one variable case (e.g. $F(x) = x - x^p$ over a field of characteristic $p > 0$). However, the situation for the VC over fields of positive characteristic is dramatically different from the JC even through these two conjectures are equivalent to each other over fields of characteristic zero. Actually, as we will show in the proposition below, the VC over fields of positive characteristic holds for any polynomials (not even necessarily HN) and also for any HN formal power series.

Proposition 5.1. *Let k be a field of characteristic $p > 0$. Then*

(a) For any polynomial $P(z) \in k[z]$ (not necessarily homogeneous nor HN) of degree $d \geq 1$, $\Delta^m P^{m+1} = 0$ for any $m \geq \frac{d(p-1)}{2}$.

(*b*) *For any HN formal power series $P(z) \in k[[z]]$, i.e. $\Delta^m P^m = 0$ for any $m \geq 1$, we have, $\Delta^m P^{m+1} = 0$ for any $m \geq p-1$.*

In other words, over the fields of positive characteristic, the VC holds even for HN formal power series $P(z) \in k[[z]]$; while for polynomials, it holds even without the HN condition nor any other conditions.

Proof: The main reason that the proposition above holds is because of the following simple fact due to the Leibniz rule and positiveness of the characteristics of the base field k, namely, for $m \geq 1$, $u(z), v(z) \in k[[z]]$ and any differential operator Λ of $k[z]$, we have

$$\Lambda(u^{mp}v) = u^{mp}\Lambda v. \tag{5.1}$$

Now let $P(z)$ be any polynomial or formal series as in the proposition. For any $m \geq 1$, write $m+1 = q_m p + r_m$ with $q_m, r_m \in \mathbb{Z}$ and $0 \leq r_m \leq p-1$. Then by Eq. (5.1) , we have

$$\Delta^m P^{m+1} = \Delta^m(P^{q_m p}P^{r_m}) = P^{q_m p}\Delta^m P^{r_m}. \tag{5.2}$$

If $P(z)$ is a polynomial of degree $d \geq 1$, we have $\Delta^m P^{r_m} = 0$ when $m \geq \frac{d(p-1)}{2}$, since in this case $2m > \deg(P^{r_m})$. If $P(z)$ is a HN formal power series, we have $\Delta^m P^{r_m} = 0$ when $m \geq p-1 \geq r_m$. Therefore, (a) and (b) in the proposition follow from Eq. (5.2) and the observations above. $\square$

One interesting question is whether or not the VC fails (as the JC does) for any HN formal power series $P(z) \in \mathbb{C}[[z]]$ but $P(z) \notin \mathbb{C}[z]$? To our best knowledge, no such counterexample has been known yet. We here put it as an open problem.

Open Problem 5.2. *Find a HN formal power series $P(z) \in \mathbb{C}[[z]]$ but $P(z) \notin \mathbb{C}[z]$, if there are any, such that the VC fails for $P(z)$.*

One final remark about Proposition 5.1 is as follows. Note that the crucial fact used in the proof is that any differential operator Λ of $k[z]$ commutes with the multiplication operator by the p^{th} power of any element of $k[[z]]$. Then, by a parallel argument as in the proof of Proposition 5.1, it is easy to see that the following more general result also holds.

Proposition 5.3. *Let k be a field of characteristics $p > 0$ and Λ a differential operator of $k[z]$. Let $f \in k[[z]]$. Assume that, for any $1 \leq m \leq p-1$, there exists $N_m > 0$ such that $\Lambda^{N_m} f^m = 0$. Then, we have $\Lambda^m f^{m+1} = 0$ when $m >> 0$.*

In particular, if Λ strictly decreases the degree of polynomials. Then, for any polynomial $f \in k[z]$, we have $\Lambda^m f^{m+1} = 0$ when $m >> 0$.

6. A Criterion of Hessian Nilpotency for Homogeneous Polynomials

Recall that $\langle \cdot, \cdot \rangle$ denotes the standard $\mathbb{C}$ bilinear form of $\mathbb{C}^n$. For any $\beta \in \mathbb{C}^n$, we set $h_\beta(z) := \langle \beta, z \rangle$ and $\beta_D := \langle \beta, D \rangle$.

The main result of this section is the following criterion of Hessian nilpotency for homogeneous polynomials. Considering the criterion given in Proposition 1.2, it is somewhat surprising but the proof turns out to be very simple.

Theorem 6.1. *For any $\beta \in \mathbb{C}^n$ and homogeneous polynomial $P(z)$ of degree $d \geq 2$, set $P_\beta(z) := \beta_D^{d-2} P(z)$. Then, we have*

$$Hes\, P_\beta = (d-2)!\, (Hes\, P)(\beta). \tag{6.1}$$

In particular, $P(z)$ is HN iff, for any $\beta \in \mathbb{C}^n$, $P_\beta(z)$ is HN.

To prove the theorem, we need first the following lemma.

Lemma 6.2. *Let $\beta \in \mathbb{C}^n$ and $P(z) \in \mathbb{C}[z]$ homogeneous of degree $N \geq 1$. Then*

$$\beta_D^N P(z) = N!\, P(\beta). \tag{6.2}$$

Proof: Since both sides of Eq. (6.2) are linear on $P(z)$, we may assume $P(z)$ is a monomial, say $P(z) = z^{\mathbf{a}}$ for some $\mathbf{a} \in \mathbb{N}^n$ with $|\mathbf{a}| = N$.

Consider

$$\begin{aligned} \beta_D^N P(z) = (\sum_{i=1}^n \beta_i D_i)^N z^{\mathbf{a}} &= \sum_{\substack{\mathbf{k} \in \mathbb{N}^n \\ |\mathbf{k}|=N}}^n \frac{N!}{\mathbf{k}!} \beta^{\mathbf{k}} D^{\mathbf{k}} z^{\mathbf{a}} \\ &= \frac{N!}{\mathbf{a}!} \beta^{\mathbf{a}} D^{\mathbf{a}} z^{\mathbf{a}} = N! \beta^{\mathbf{a}} = N! P(\beta). \end{aligned}$$

$\square$

Proof of Theorem 6.1: We consider

$$\text{Hes}\, P_\beta(z) = \left(\frac{\partial^2 (\beta_D^{d-2} P)}{\partial z_i \partial z_j}(z) \right)_{n\times n} = \left(\beta_D^{d-2} \frac{\partial^2 P}{\partial z_i \partial z_j}(z) \right)_{n \times n}$$

Applying Lemma 6.2 to $\frac{\partial^2 P}{\partial z_i \partial z_j}(z)$:

$$= (d-2)! \left(\frac{\partial^2 P}{\partial z_i \partial z_j}(\beta) \right)_{n \times n} = (d-2)! \, (\operatorname{Hes} P)(\beta).$$

□

Let $\{e_i \,|\, 1 \leq i \leq n\}$ be the standard basis of $\mathbb{C}^n$. Applying the theorem above to $\beta = e_i$ $(1 \leq i \leq n)$, we have the following corollary, which was first proved by M. Kumar [K].

Corollary 6.3. *For any homogeneous HN polynomial $P(z) \in \mathbb{C}[z]$ of degree $d \geq 2$, $D_i^{d-2} P(z)$ $(1 \leq i \leq n)$ are also HN.*

The reason that we think the criteria given in Theorem 6.1 and Corollary 6.3 interesting is that, $P_\beta(z) = \beta_D^{d-2} P(z)$ is homogeneous of degree 2, and it is much easier to decide whether a homogeneous polynomial of degree 2 is HN or not. More precisely, for any homogeneous polynomial $U(z)$ of degree 2, there exists a unique symmetric $n \times n$ matrix A such that $U(z) = z^\tau A z$. Then it is easy to check that $\operatorname{Hes} U(z) = 2A$. Therefore, $U(z)$ is HN iff the symmetric matrix A is nilpotent.

Finally we end this section with the following open question on the criterion given in Proposition 1.2.

Recall that Proposition 1.2 was proved in [Z2]. We now sketch the argument.

For any $m \geq 1$, we set

$$u_m(P) = \operatorname{Tr} \operatorname{Hes}^m(P), \tag{6.3}$$

$$v_m(P) = \Delta^m P^m. \tag{6.4}$$

For any $k \geq 1$, we define $\mathcal{U}_k(P)$ (resp. $\mathcal{V}_k(P)$) to be the ideal in $\mathbb{C}[[z]]$ generated by $\{u_m(P) | 1 \leq m \leq k\}$ (resp. $\{v_m(P) | 1 \leq m \leq k\}$) and all their partial derivatives of any order. Then it has been shown (in a more general setting) in Section 4 in [Z2] that $\mathcal{U}_k(P) = \mathcal{V}_k(P)$ for any $k \geq 1$.

It is well-known in linear algebra that, if $u_m(P(z)) = 0$ when $m >> 0$, then $\operatorname{Hes} P$ is nilpotent and $u_m(P) = 0$ for any $m \geq 1$. One natural question is whether or not this is also the case for the sequence $\{v_m(P) \,|\, m \geq 1\}$. More precisely, we believe the following conjecture which was proposed in [Z2] is worth investigating.

Conjecture 6.4. *Let $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$. If $\Delta^m P^m(z) = 0$ for $m >> 0$, then $P(z)$ is HN.*

7. Some Results on Symmetric Polynomial Maps

Let $P(z)$ be any formal power series with $o(P(z)) \geq 2$ and $(\operatorname{Hes} P)(0)$ nilpotent, and $F(z)$ and $G(z)$ as before. Set

$$\sigma_2 := \sum_{i=1}^{n} z_i^2, \tag{7.1}$$

$$f(z) := \frac{1}{2}\sigma_2 - P(z). \tag{7.2}$$

Professors Mohan Kumar [K] and David Wright [Wr3] once asked how to write $P(z)$ and $f(z)$ in terms of $F(z)$? More precisely, find $U(z), V(z) \in \mathbb{C}[[z]]$ such that

$$U(F(z)) = P(z), \tag{7.3}$$

$$V(F(z)) = f(z). \tag{7.4}$$

In this section, we first derive in Proposition 7.2 some explicit formulas for $U(z)$ and $V(z)$, and also for $W(z) \in \mathbb{C}[[z]]$ such that

$$W(F(z)) = \sigma_2(z). \tag{7.5}$$

We then show in Theorem 7.4 that, when $P(z)$ is a HN polynomial, the VC holds for P or equivalently, the JC holds for the associated symmetric polynomial map $F(z) = z - \nabla P$, iff one of U, V and W is polynomial.

Let t be a central parameter and $F_t(z) = z - t\nabla P$. Let $G_t(z) = z + t\nabla Q_t$ be the formal inverse of $F_t(z)$ as before. We set

$$f_t(z) := \frac{1}{2}\sigma_2 - tP(z), \tag{7.6}$$

$$U_t(z) := P(G_t(z)), \tag{7.7}$$

$$V_t(z) := f_t(G_t(z)), \tag{7.8}$$

$$W_t(z) := \sigma_2(G_t(z)). \tag{7.9}$$

Note first that, under the conditions that $o(P(z)) \geq 2$ and $(\operatorname{Hes} P)(0)$ is nilpotent, we have $G_t(z) \in \mathbb{C}[t][[z]]^{\times n}$ as mentioned in the proof of Proposition 4.9. Therefore, we have $U_t(z), V_t(z), W_t(z) \in \mathbb{C}[t][[z]]$, and $U_{t=1}(z)$,

$V_{t=1}(z)$ and $W_{t=1}(z)$ all make sense. Secondly, from the definitions above, we have

$$W_t(z) = 2V_t(z) + 2tU_t(z), \tag{7.10}$$

$$F_t(z) = \nabla f_t(z), \tag{7.11}$$

$$f_{t=1}(z) = f(z). \tag{7.12}$$

Lemma 7.1. *With the notations above, we have*

$$P(z) = U_{t=1}(F(z)), \tag{7.13}$$

$$f(z) = V_{t=1}(F(z)), \tag{7.14}$$

$$\sigma_2(z) = W_{t=1}(F(z)). \tag{7.15}$$

In particular, $f(z)$*,* $P(z)$ *and* $\sigma_2(z)$ *lie in* $\mathbb{C}[F]$ *iff* $U_{t=1}(z)$*,* $V_{t=1}(z)$ *and* $W_{t=1}(z)$ *lie in* $\mathbb{C}[z]$*.*

In other words, by setting $t = 1$, U_t, V_t and W_t will give us U, V and W in Eqs. (7.3)–(7.5), respectively.

Proof: From the definitions of $U_t(z)$, $V_t(z)$ and $W_t(z)$ (see Eqs. (7.7)–(7.9), we have

$$\begin{aligned} P(z) &= U_t(F_t(z)), \\ f_t(z) &= V_t(F_t(z)), \\ \sigma_2(z) &= W_t(F_t(z)). \end{aligned}$$

By setting $t = 1$ in the equations above and noticing that $F_{t=1}(z) = F(z)$, we get Eqs. (7.13)–(7.15). $\square$

For $U_t(z)$, $V_t(z)$ and $W_t(z)$, we have the following explicit formulas in terms of the deformed inversion pair Q_t of P.

Proposition 7.2. *For any formal power series* $P(z) \in \mathbb{C}[[z]]$ (*not necessarily HN*) *with* $o(P(z)) \geq 2$ *and* $(Hes\, P)(0)$ *nilpotent, we have*

$$U_t(z) = Q_t + t\,\frac{\partial Q_t}{\partial t}, \tag{7.16}$$

$$V_t(z) = \frac{1}{2}\sigma_2 + t(z\frac{\partial Q_t}{\partial z} - Q_t), \tag{7.17}$$

$$W_t(z) = \sigma_2 + 2tz\frac{\partial Q_t}{\partial z} + 2t^2\frac{\partial Q_t}{\partial t}. \tag{7.18}$$

Proof: Note first that, Eq. (7.18) follows directly from Eqs. (7.16), (7.17) and (7.10).

To show Eq. (7.16), by Eqs. (3.4) and (3.6) in [Z1], we have

$$U_t(z) = P(G_t) = Q_t + \frac{t}{2}\langle \nabla Q_t, \nabla Q_t \rangle = Q_t + t\,\frac{\partial Q_t}{\partial t}. \tag{7.19}$$

To show Eq. (7.17), we consider

$$\begin{aligned} V_t(z) &= f_t(G_t) \\ &= \frac{1}{2}\langle z + t\nabla Q_t(z), z + t\nabla Q_t(z)\rangle - tP(G_t) \\ &= \frac{1}{2}\sigma_2 + t\langle z, \nabla Q_t(z)\rangle + \frac{t^2}{2}\langle \nabla Q_t, \nabla Q_t\rangle - tP(G_t) \end{aligned}$$

By Eq. (7.19), substituting $Q_t + \frac{t}{2}\langle \nabla Q_t, \nabla Q_t\rangle$ for $P(G_t)$:

$$\begin{aligned} &= \frac{1}{2}\sigma_2 + t\langle z, \nabla Q_t(z)\rangle - tQ_t(z) \\ &= \frac{1}{2}\sigma_2 + t(z\frac{\partial Q_t}{\partial z} - Q_t). \end{aligned}$$

□

When $P(z)$ is homogeneous and HN, we have the following more explicit formulas which in particular give solutions to the questions raised by Professors Mohan Kumar and David Wright.

Corollary 7.3. *For any homogeneous HN polynomial $P(z)$ of degree $d \geq 2$, we have*

$$U_t(z) = \sum_{m=0}^{\infty} \frac{t^m}{2^m (m!)^2} \Delta^m P^{m+1}(z) \tag{7.20}$$

$$V_t(z) = \frac{1}{2}\sigma_2 + \sum_{m=0}^{\infty} \frac{(d_m - 1)t^{m+1}}{2^m m!(m+1)!}\,\Delta^m P^{m+1}(z)\,, \tag{7.21}$$

$$W_t(z) = \sigma_2 + \sum_{m=0}^{\infty} \frac{(d_m + m)t^{m+1}}{2^{m-1} m!(m+1)!}\,\Delta^m P^{m+1}(z)\,, \tag{7.22}$$

where $d_m = \deg(\Delta^m P^{m+1}) = d(m+1) - 2m$ $(m \geq 0)$.

Proof: We give a proof for Eq. (7.20). Eqs. (7.21) can be proved similarly. (7.22) follows directly from Eqs. Eq. (7.20), (7.21) and (7.10).

By combining Eqs. (7.16) and (1.1), we have

$$U_t(z) = \sum_{m=0}^{\infty} \frac{t^m \Delta^m P^{m+1}(z)}{2^m m!(m+1)!} + \sum_{m=1}^{\infty} \frac{m t^m \Delta^m P^{m+1}(z)}{2^m m!(m+1)!}$$

$$= P(z) + \sum_{m=1}^{\infty} \frac{t^m}{2^m (m!)^2} \Delta^m P^{m+1}(z)$$
$$= \sum_{m=0}^{\infty} \frac{t^m}{2^m (m!)^2} \Delta^m P^{m+1}(z).$$

Hence, we get Eq. (7.20). $\Box$

One consequence of the proposition above is the following result on symmetric polynomials maps.

Theorem 7.4. *For any HN polynomial $P(z)$ (not necessarily homogeneous) with $o(P) \geq 2$, the following statements are equivalent:*

(1) *The VC holds for $P(z)$.*
(2) *$P(z) \in \mathbb{C}[F]$.*
(3) *$f(z) \in \mathbb{C}[F]$.*
(4) *$\sigma_2(z) \in \mathbb{C}[F]$.*

Note that, the equivalence of the statements (1) and (3) was first proved by Mohan Kumar ([K]) by a different method.

Proof: Note first that, by Lemma 7.1, it will be enough to show that, $\Delta^m P^{m+1} = 0$ when $m >> 0$ iff one of $U_t(z)$, $V_t(z)$ and $W_t(z)$ is a polynomial in t with coefficients in $\mathbb{C}[z]$. Secondly, when $P(z)$ is homogeneous, the statement above follows directly from Eqs. (7.20)–(7.22).

To show the general case, for any $m \geq 0$ and $M_t(z) \in \mathbb{C}[t][[z]]$, we denote by $[t^m](M_t(z))$ the coefficient of t^m when we write $M_t(z)$ as a formal power series of t with coefficients in $\mathbb{C}[[z]]$. Then, from Eqs. (7.16)–(7.18) and Eq. (1.1), it is straightforward to check that the coefficients of t^m ($m \geq 1$) in $U_t(z)$, $V_t(z)$ and $W_t(z)$ are given as follows.

$$[t^m](U_t(z)) = \frac{\Delta^m P^{m+1}}{2^m (m!)^2}, \tag{7.23}$$

$$[t^m](V_t(z)) = \frac{1}{2^{m-1}(m-1)!m!} \left(z\frac{\partial}{\partial z}(\Delta^{m-1} P^m) - \Delta^{m-1} P^m \right), \tag{7.24}$$

(7.25)
$$[t^m](W_t(z)) = \frac{1}{2^{m-2}(m-1)!m!} \left(z\frac{\partial}{\partial z}(\Delta^{m-1} P^m) + (m-1)\Delta^{m-1} P^m \right).$$

From Eq. (7.23), we immediately have (1) $\Leftrightarrow$ (2). To show the equivalences (1) $\Leftrightarrow$ (3) and (1) $\Leftrightarrow$ (4), note first that $o(P) \geq 2$, so $o(\Delta^{m-1} P^m) \geq 2$ for any $m \geq 1$. While, on the other hand, for any polynomial $h(z) \in \mathbb{C}[z]$ with $o(h(z)) \geq 2$, we have, $h(z) = 0$ iff $(z\frac{\partial}{\partial z} - 1)h(z) = 0$, and iff $(z\frac{\partial}{\partial z} + (m-1))h(z) = 0$ for some $m \geq 1$. This is simply because that, for any monomial z^α ($\alpha \in \mathbb{N}^n$), we have $(z\frac{\partial}{\partial z} - 1)z^\alpha = (|\alpha| - 1)z^\alpha$ and $(z\frac{\partial}{\partial z} + (m-1))z^\alpha = (|\alpha| + (m-1))z^\alpha$. From this general fact, we see that (1) $\Leftrightarrow$ (3) follows from Eq. (7.24) and (1) $\Leftrightarrow$ (4) from Eq. (7.25). $\square$

8. A Graph Associated with Homogeneous HN Polynomials

In this section, we would like to draw the reader's attention to a graph $\mathcal{G}(P)$ assigned to each homogeneous harmonic polynomials $P(z)$. The graph $\mathcal{G}(P)$ was first proposed by the author and later was further studied by R. Willems in his master thesis [Wi] under direction of Professor A. van den Essen. The introduction of the graph $\mathcal{G}(P)$ is mainly motivated by a criterion of Hessian nilpotency given in [Z2] (see also Theorem 8.2 below), via which one hopes more necessary or sufficient conditions for a homogeneous harmonic polynomial $P(z)$ to be HN can be obtained or described in terms of the graph structure of $\mathcal{G}(P)$.

We first give in Subsection 8.1 the definition of the graph $\mathcal{G}(P)$ for any homogeneous harmonic polynomial $P(z)$ and discuss the connectedness reduction (see Corollary 8.5), i.e. a reduction of the VC to the homogeneous HN polynomials P such that $\mathcal{G}(P)$ is connected. We then consider in Subsection 8.2 a connection of $\mathcal{G}(P)$ with the tree expansion formula derived in [M] and [Wr2] for the inversion pair $Q(z)$ of $P(z)$ (see Proposition 8.9). As an application of the connection, we give another proof for the connectedness reduction given in Corollary 8.5.

8.1. Definition and the Connectedness Reduction.

For any $\beta \in \mathbb{C}^n$, set $h_\beta(z) := \langle \beta, z \rangle$ and $\beta_D := \langle \beta, D \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard $\mathbb{C}$-bilinear form of $\mathbb{C}^n$. Let $X(\mathbb{C})$ denote the set of all isotropic elements of $\mathbb{C}^n$, i.e. the set of all elements $\alpha \in \mathbb{C}^n$ such that $\langle \alpha, \alpha \rangle = 0$.

Recall that we have the following fundamental theorem on homogeneous harmonic polynomials.

Theorem 8.1. *For any homogeneous harmonic polynomial $P(z)$ of degree $d \geq 2$, we have*

$$P(z) = \sum_{i=1}^{k} c_i h_{\alpha_i}^d(z) \tag{8.1}$$

for some $c_i \in \mathbb{C}^\times$ and $\alpha_i \in X(\mathbb{C}^n)$ $(1 \leq i \leq k)$.

Note that, replacing α_i in Eq. (8.1) by $c_i^{-\frac{1}{d}}\alpha_i$, we may also write $P(z)$ as

$$P(z) = \sum_{i=1}^{k} h_{\alpha_i}^d(z) \tag{8.2}$$

with $\alpha_i \in X(\mathbb{C}^n)$ $(1 \leq i \leq k)$.

For the proof of Theorem 8.1, see, for example, [I] and [Wi].

We fix a homogeneous harmonic polynomial $P(z) \in \mathbb{C}[z]$ of degree $d \geq 2$, and assume that $P(z)$ is given by Eq. (8.2) for some $\alpha_i \in X(\mathbb{C}^n)$ $(1 \leq i \leq k)$. We may and will always assume $\{h_{\alpha_i}^d(z) | 1 \leq i \leq k\}$ are linearly independent in $\mathbb{C}[z]$.

Recall the following matrices had been introduced in [Z2]:

$$A_P = (\langle \alpha_i, \alpha_j \rangle)_{k \times k}, \tag{8.3}$$

$$\Psi_P = (\langle \alpha_i, \alpha_j \rangle h_{\alpha_j}^{d-2}(z))_{k \times k}. \tag{8.4}$$

Then we have the following criterion of Hessian nilpotency for homogeneous harmonic polynomials. For its proof, see Theorem 4.3 in [Z2].

Theorem 8.2. *Let $P(z)$ be as above. Then, for any $m \geq 1$, we have*

$$Tr\, Hes^m(P) = (d(d-1))^m\, Tr\, \Psi_P^m. \tag{8.5}$$

In particular, $P(z)$ is HN if and only if the matrix Ψ_P is nilpotent.

One simple remark on the criterion above is as follows.

Let B be the $k \times k$ diagonal matrix with the i^{th} $(1 \leq i \leq k)$ diagonal entry being $h_{\alpha_i}(z)$. For any $1 \leq j \leq k$, set

$$\Psi_{P;j} := B^j A_P B^{d-2-j} = (h_{\alpha_i}^j \langle \alpha_i, \alpha_j \rangle h_{\alpha_j}^{d-2-j}). \tag{8.6}$$

Then, by repeatedly applying the fact that, for any two $k \times k$ matrices C and D, CD is nilpotent iff so is DC, it is easy to see that Theorem 8.2 can also be re-stated as follows.

Corollary 8.3. *Let $P(z)$ be given by Eq. (8.2) with $d \geq 2$. Then, for any $1 \leq j \leq d-2$ and $m \geq 1$, we have*

$$\text{(8.7)} \qquad Tr\, Hes^m(P) = (d(d-1))^m \, Tr\, \Psi^m_{P;j}.$$

In particular, $P(z)$ is HN if and only if the matrix $\Psi_{P;j}$ is nilpotent.

Note that, when d is even, we may choose $j = (d-2)/2$. So P is HN iff the symmetric matrix

$$\text{(8.8)} \qquad \Psi_{P;(d-2)/2}(z) = (\, h_{\alpha_i}^{(d-2)/2}(z) \, \langle \alpha_i, \alpha_j \rangle \, h_{\alpha_j}^{(d-2)/2}(z) \,)$$

is nilpotent.

Motivated by the criterion above, we assign a graph $\mathcal{G}(P)$ to any homogeneous harmonic polynomial $P(z)$ as follows.

We fix an expression as in Eq. (8.2) for $P(z)$. The set of vertices of $\mathcal{G}(P)$ will be the set of positive integers $[\mathbf{k}] := \{1, 2, \dots, k\}$. The vertices i and j of $\mathcal{G}(P)$ are connected by an edge iff $\langle \alpha_i, \alpha_j \rangle \neq 0$. In this case, we get a finite graph.

Furthermore, we may also label edges of $\mathcal{G}(P)$ by assigning $\langle \alpha_i, \alpha_j \rangle$ or $(h_{\alpha_i}^{(d-2)/2} \langle \alpha_i, \alpha_j \rangle h_{\alpha_i}^{(d-2)/2})$, when d is even, for the edge connecting vertices $i, j \in [\mathbf{k}]$. We then get a labeled graph whose adjacency matrix is exactly A_P or $\Psi_{P,(d-2)/2}$ (depending on the labels we choose for the edges of $\mathcal{G}(P)$).

Naturally, one may also ask the following (open) questions.

Open Problem 8.4. (*a*) *Find some necessary or sufficient conditions on the* (*labeled*) *graph $\mathcal{G}(P)$ such that the homogeneous harmonic polynomial $P(z)$ is HN.*

(*b*) *Find some necessary or sufficient conditions on the* (*labeled*) *graph $\mathcal{G}(P)$ such that the VC holds for the homogeneous HN polynomial $P(z)$.*

First, let us point out that, to approach the open problems above, it will be enough to focus on homogeneous harmonic polynomials P such that the graph $\mathcal{G}(P)$ is connected.

Suppose that the graph $\mathcal{G}(P)$ is a disconnected graph with $r \geq 2$ connected components. Let $[\mathbf{k}] = \sqcup_{i=1}^r I_i$ be the corresponding partition of the set $[\mathbf{k}]$ of vertices of $\mathcal{G}(P)$. For each $1 \leq i \leq r$, we set $P_i(z) := \sum_{\alpha \in I_i} h_\alpha^d(z)$.

Note that, by Lemma 2.6, P_i $(1 \leq i \leq r)$ are disjoint to each other, so Corollary 2.8 applies to the sum $P = \sum_{i=1}^r P_i$. In particular, we have,

(*a*) *P is HN iff each P_i is HN.*

(*b*) *if the VC holds for each P_i, then it also holds for P.*

Therefore, we have the following *connectedness reduction.*

Corollary 8.5. *To study homogeneous HN polynomials P or the VC for homogeneous HN polynomials P, it will be enough to consider the case when $\mathcal{G}(P)$ is connected.*

Note that, the property (a) above was first proved by R. Willems ([Wi]) by using the criterion in Theorem 8.2. (b) was first proved by the author by a different argument, and with the author's permission, it had also been included in [Wi].

Finally, let us point out that R. Willems ([Wi]) has proved the following very interesting results on Open Problem 8.4.

Theorem 8.6. ([Wi]) *Let P be a homogeneous HN polynomial as in Eq.(8.2) with $d \geq 4$. Let $l(P)$ be the dimension of the vector subspace of $\mathbb{C}^n$ spanned by $\{\alpha_i \,|\, 1 \leq i \leq k\}$. Then*

(1) *If $l(P) = 1, 2, k-1$ or k, the graph $\mathbb{G}(P)$ is totally disconnected (i.e. $\mathcal{G}(P)$ is the graph with no edges).*
(2) *If $l(P) = k-2$ and $\mathcal{G}(P)$ is connected, then $\mathcal{G}(P)$ is the complete bi-graph $K(4, k-4)$.*
(3) *In the case of (a) and (b) above, the VC holds.*

Furthermore, it has also been shown in [Wi] that, for any homogeneous HN polynomials P, the graph $\mathcal{G}(P)$ can not be any path nor cycles of any positive length. For more details, see [Wi].

8.2. **Connection with the Tree Expansion Formula of Inversion Pairs.** First let us recall the tree expansion formula derived in [M], [Wr2] for the inversion pair $Q(z)$.

Let $\mathbb{T}$ denote the set of all trees, i.e. the set of all connected and simply connected finite simple graphs. For each tree $T \in \mathbb{T}$, denote by $V(T)$ and $E(T)$ the sets of all vertices and edges of T, respectively. Then we have the following tree expansion formula for inversion pairs.

Theorem 8.7. ([M], [Wr2]) *Let $P \in \mathbb{C}[[z]]$ with $o(P) \geq 2$ and Q its inversion pair. For any $T \in \mathbb{T}$, set*

$$Q_{T,P} = \sum_{\ell: E(T) \to [\mathbf{n}]} \prod_{v \in V(T)} D_{adj(v),\ell} P, \tag{8.9}$$

where $adj(v)$ is the set $\{e_1, e_2, \ldots, e_s\}$ of edges of T adjacent to v, and $D_{adj(v),\ell} = D_{\ell(e_1)} D_{\ell(e_2)} \cdots D_{\ell(e_s)}$.

Then the inversion pair Q of P is given by

$$Q = \sum_{T\in\mathbb{T}} \frac{1}{|Aut(T)|} Q_{T,P}. \tag{8.10}$$

Now we assume $P(z)$ is a homogeneous harmonic polynomial $d \geq 2$ and has expression in Eq. (8.2). Under this assumption, it is easy to see that $Q_{T,P}$ $(T \in \mathbb{T})$ becomes

$$Q_{T,P} = \sum_{f:V(T)\to[\mathbf{k}]} \sum_{\ell:E(T)\to[\mathbf{n}]} \prod_{v\in V(T)} D_{adj(v),\ell} h^d_{\alpha_{f(v)}}(z). \tag{8.11}$$

The role played by the graph $\mathcal{G}(P)$ of P is to restrict the maps $f : V(T) \to V(\mathcal{G}(P))(= [\mathbf{k}])$ in Eq. (8.11) to a special family of maps. To be more precise, let $\Omega(T, \mathcal{G}(P))$ be the set of maps $f : V(T) \to [\mathbf{k}]$ such that, *for any distinct adjoint vertices $u, v \in V(T)$, $f(u)$ and $f(v)$ are distinct and adjoint in $\mathcal{G}(P)$.* Then we have the following lemma.

Lemma 8.8. *For any $f : V(T) \to [\mathbf{k}]$ with $f \notin \Omega(T, \mathcal{G}(P))$, we have*

$$\sum_{\ell:E(T)\to[\mathbf{n}]} \prod_{v\in V(T)} D_{adj(v),\ell} h^d_{\alpha_{f(v)}}(z) = 0. \tag{8.12}$$

Proof: Let $f : V(T) \to [\mathbf{k}]$ as in the lemma. Since $f \notin \Omega(T, \mathcal{G}(P))$, there exist distinct adjoint $v_1, v_2 \in V(T)$ such that, either $f(v_1) = f(v_2)$ or $f(v_1)$ and $f(v_2)$ are not adjoint in the graph $\mathcal{G}(P)$. In any case, we have $\langle \alpha_{f(v_1)}, \alpha_{f(v_2)} \rangle = 0$.

Next we consider contributions to the RHS of Eq. (8.11) from the vertices v_1 and v_2. Denote by e the edge of T connecting v_1 and v_2, and $\{e_1, \dots e_r\}$ (resp. $\{\tilde{e}_1, \dots \tilde{e}_s\}$) the set of edges connected with v_1 (resp. v_2) besides the edge e. Then, for any $\ell : E(T) \to [\mathbf{n}]$, the factor in the RHS of Eq. (8.11) from the vertices v_1 and v_2 is the product

$$\left(D_{\ell(e)} D_{\ell(e_1)} \cdots D_{\ell(e_r)} h^d_{\alpha_{f(v_1)}}(z)\right) \left(D_{\ell(e)} D_{\ell(\tilde{e}_1)} \cdots D_{\ell(\tilde{e}_s)} h^d_{\alpha_{f(v_2)}}(z)\right). \tag{8.13}$$

Define an equivalent relation for maps $\ell : E(T) \to [\mathbf{n}]$ by setting $\ell_1 \sim \ell_2$ iff ℓ_1, ℓ_2 have same image at each edge of T except e. Then, by taking sum of the terms in Eq. (8.13) over each equivalent class, we get the factor

$$\left\langle \nabla D_{\ell(e_1)} \cdots D_{\ell(e_r)} h^d_{\alpha_{f(v_1)}}(z),\, \nabla D_{\ell(\tilde{e}_1)} \cdots D_{\ell(\tilde{e}_s)} h^d_{\alpha_{f(v_2)}}(z) \right\rangle. \tag{8.14}$$

Note that $D_{\ell(e_1)} \cdots D_{\ell(e_r)} h^d_{\alpha_{f(v_1)}}(z)$ and $D_{\ell(\tilde{e}_1)} \cdots D_{\ell(\tilde{e}_s)} h^d_{\alpha_{f(v_2)}}(z)$ are constant multiples of some integral powers of $h_{\alpha_{f(v_1)}}(z)$ and $h_{\alpha_{f(v_2)}}(z)$, respectively. Therefore, $\langle \alpha_{f(v_1)}, \alpha_{f(v_2)} \rangle (= 0)$ appears as a multiplicative constant factor in the term in Eq. (8.14), which makes the term zero. Hence the lemma follows. $\square$

One immediate consequence of the lemma above is the following proposition.

Proposition 8.9. *With the setting and notation as above, we have*

$$Q_{T,P} = \sum_{f\in\Omega(T,\mathcal{G}(P))} \sum_{\ell:E(T)\to[\mathbf{n}]} \prod_{v\in V(T)} D_{adj(v),\ell} h^d_{\alpha_{f(v)}}(z). \tag{8.15}$$

Remark 8.10. (*a*) *For any $f \in \Omega(T,\mathcal{G}(P))$, $\{f^{-1}(j)\,|\,j\in Im(f)\}$ gives a partition of $V(T)$ since no two distinct vertices in $f^{-1}(j)$ $(j\in Im(f))$ can be adjoint. In other words, f is nothing but a proper coloring for the tree T, which is also subject to certain more conditions from the graph structure of $\mathcal{G}(P)$. It is interesting to see that the coloring problem of graphs also plays a role in the inversion problem of symmetric formal maps.*

(*b*) *It will be interesting to see if more results can be derived from the graph $\mathcal{G}(P)$ via the formulas in Eqs.* (*8.10*) *and* (*8.15*).

Remark 8.11. *By similar arguments as those in proofs of Lemma 8.8, one may get another proof for Theorem 2.7 in the setting as in Lemma 2.6.*

Finally, as an application of Proposition 8.9 above, we give another proof for the connectedness reduction given in Corollary 8.5.

Let P as given in Eq. (8.2) with the inversion pair Q. Suppose that there exists a partition $[\mathbf{k}] = I_1 \sqcup I_2$ with $I_i \neq \emptyset$. Let $P_i = \sum_{\alpha\in I_i} h^d_\alpha(z)$ $(i=1,2)$ and Q_i the inversion pair of P_i. Then we have $P = P_1 + P_2$ and $\mathcal{G}(P_1)\sqcup\mathcal{G}(P_2) = \mathcal{G}(P)$. Therefore, to show the connectedness reduction discussed in the previous subsection, it will be enough to show $Q = Q_1 + Q_2$. But this will follow immediately from Eqs. (8.10), (8.15) and the following lemma.

Lemma 8.12. *Let P, P_1 and P_2 as above, then, for any tree $T\in\mathbb{T}$, we have*

$$\Omega(T,\mathcal{G}(P)) = \Omega(T,\mathcal{G}(P_1)) \sqcup \Omega(T,\mathcal{G}(P_2)).$$

Proof: For any $f\in\Omega(T,\mathcal{G}(P))$, f preserves the adjacency of vertices of $\mathcal{G}(P)$. Since T as a graph is connected, $\mathrm{Im}(f)\subset V(\mathcal{G}(P))$ as a (full) subgraph of $\mathcal{G}(P)$ must also be connected. Therefore, $\mathrm{Im}(f)\subset V(\mathcal{G}(P_1))$

or $\mathrm{Im}(f) \subset V(\mathcal{G}(P_2))$. Hence $\Omega(T, \mathcal{G}(P)) \subset \Omega(T, \mathcal{G}(P_1)) \sqcup \Omega(T, \mathcal{G}(P_2))$. The other way of containess is obvious. $\square$

References

[BCW] H. Bass, E. Connell, D. Wright, *The Jacobian Conjecture, Reduction of Degree and Formal Expansion of the Inverse*. Bull. Amer. Math. Soc. **7**, (1982), 287–330. [MR 83k:14028]. [Zbl.539.13012].

[B] M. de Bondt, *Quasi-translations and Counterexamples to the Homogeneous Dependence Problem.* Proc. Amer. Math. Soc. 134 (2006), no. 10, 2849–2856 (electronic). [MR2231607].

[BE1] M. de Bondt and A. van den Essen, *A reduction of the Jacobian Conjecture to the Symmetric Case.* Proc. Amer. Math. Soc. 133 (2005), no. 8, 2201–2205 (electronic). [MR2138860].

[BE2] M. de Bondt and A. van den Essen, *Nilpotent Symmetric Jacobian Matrices and the Jacobian Conjecture*, J. Pure Appl. Algebra 193 (2004), no. 1-3, 61–70. [MR2076378].

[BE3] M. de Bondt and A. van den Essen, *Singular Hessians*, J. Algebra 282 (2004), no. 1, 195–204. [MR2095579].

[BE4] M. de Bondt and A. van den Essen, *Nilpotent Symmetric Jacobian Matrices and the Jacobian Conjecture II*, J. Pure Appl. Algebra 196 (2005), no. 2-3, 135–148. [MR2110519].

[BE5] M. de Bondt and A. van den Essen, *Hesse and the Jacobian Conjecture*, Affine algebraic geometry, 63–76, Contemp. Math., 369, Amer. Math. Soc., Providence, RI, 2005. [MR2126654].

[E] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture.* Progress in Mathematics, 190. Birkhuser Verlag, Basel, 2000. [MR1790619].

[EW] A. van den Essen and S. Washburn, *The Jacobian Conjecture for Symmetric Jacobian Matrices*, J. Pure Appl. Algebra, **189** (2004), no. 1-3, 123–133. [MR2038568]

[EZ] A. van den Essen and W. Zhao, *Two Results on Hessian Nilpotent Polynomials.* To appear in *J. Pure Appl. Algebra.* See also arXiv:0704.1690v1 [math.AG].

[I] H. Iwaniec, *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics, 17. American Mathematical Society, Providence, RI, 1997. [MR1474964]

[Ke] O. H. Keller, *Ganze Gremona-Transformation*, Monats. Math. Physik **47** (1939), no. 1, 299-306. [MR1550818].

[K] M. Kumar, *Personal commucations.*

[M] G. Meng, *Legendre Transform, Hessian Conjecture and Tree Formula*, Appl. Math. Lett. 19 (2006), no. 6, 503–510. [MR2221506]. See also math-ph/0308035.

[R] R. M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Graduate Texts in Mathematics, 108. Springer-Verlag New York Inc., 1986. [MR0847923].

[Wa] S. Wang, *A Jacobian Criterion for Separability*, J. Algebra **65** (1980), 453-494. [MR83e:14010].

[Wi] R. Willems, *Graphs and the Jacobian Conjecture*, Master Thesis, July 2005. Radboud University Nijmegen, The Netherlands.

[Wr1] D. Wright, *The Jacobian Conjecture: Ideal Membership Questions and Recent advances*, Affine algebraic geometry, 261–276, Contemp. Math., 369, Amer. Math. Soc., Providence, RI, 2005. [MR2126667].

[Wr2] D. Wright, *The Jacobian Conjecture as a Combinatorial Problem.* Affine algebraic geometry, 483–503, Osaka Univ. Press, Osaka, 2007. See also math.CO/0511214. [MR2330486].

[Wr3] D. Wright, *Personal communications.*

[Y] A. V. Jagžev, *On a problem of O.-H. Keller.* (Russian) Sibirsk. Mat. Zh. 21 (1980), no. 5, 141–150, 191. [MR0592226].

[Z1] W. Zhao, *Inversion Problem, Legendre Transform and Inviscid Burgers' Equation*, J. Pure Appl. Algebra, **199** (2005), no. 1-3, 299–317. [MR2134306]. See also math. CV/0403020.

[Z2] W. Zhao, *Hessian Nilpotent Polynomials and the Jacobian Conjecture*, Trans. Amer. Math. Soc. 359 (2007), no. 1, 249–274 (electronic). [MR2247890]. See also math.CV/0409534.

Department of Mathematics, Illinois State University, Normal, IL 61790-4520.

E-mail: wzhao@ilstu.edu.