

TWO RESULTS ON HOMOGENEOUS HESSIAN NILPOTENT POLYNOMIALS

ARNO VAN DEN ESSEN* AND WENHUA ZHAO**

ABSTRACT. Let $z = (z_1, \dots, z_n)$ and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2}$ the Laplace operator. A formal power series $P(z)$ is said to be *Hessian Nilpotent*(HN) if its Hessian matrix $\text{Hes } P(z) = (\frac{\partial^2 P}{\partial z_i \partial z_j})$ is nilpotent. In recent developments in [BE1], [M] and [Z], the Jacobian conjecture has been reduced to the following so-called *vanishing conjecture*(VC) of HN polynomials: *for any homogeneous HN polynomial $P(z)$ (of degree $d = 4$), we have $\Delta^m P^{m+1}(z) = 0$ for any $m \gg 0$.* In this paper, we first show that, the VC holds for any homogeneous HN polynomial $P(z)$ provided that the projective subvarieties \mathcal{Z}_P and \mathcal{Z}_{σ_2} of $\mathbb{C}P^{n-1}$ determined by the principal ideals generated by $P(z)$ and $\sigma_2(z) := \sum_{i=1}^n z_i^2$, respectively, intersect only at regular points of \mathcal{Z}_P . Consequently, the Jacobian conjecture holds for the symmetric polynomial maps $F = z - \nabla P$ with $P(z)$ HN if F has no non-zero fixed point $w \in \mathbb{C}^n$ with $\sum_{i=1}^n w_i^2 = 0$. Secondly, we show that the VC holds for a HN formal power series $P(z)$ if and only if, for any polynomial $f(z)$, $\Delta^m(f(z)P(z)^m) = 0$ when $m \gg 0$.

1. Introduction and Main Results

Let $z = (z_1, z_2, \dots, z_n)$ be commutative free variables. Recall that the well-known Jacobian conjecture claims that: *any polynomial map $F(z) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with the Jacobian $j(F)(z) \equiv 1$ is an automorphism of \mathbb{C}^n and its inverse map must also be a polynomial map.* Despite intense study from mathematicians in more than sixty years, the conjecture is still open even for the case $n = 2$. In 1998, S. Smale [S] included the Jacobian conjecture in his list of 18 important mathematical problems for 21st century. For more history and known results on the Jacobian conjecture, see [BCW], [E] and references there.

Recently, M. de Bondt and the first author [BE1] and G. Meng [M] independently made the following remarkable breakthrough on the Jacobian conjecture. Namely, they reduced the Jacobian conjecture to

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the so-called *symmetric* polynomial maps, i.e the polynomial maps of the form $F = z - \nabla P$, where $\nabla P := (\frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2}, \dots, \frac{\partial P}{\partial z_n})$, i.e. $\nabla P(z)$ is the *gradient* of $P(z) \in \mathbb{C}[z]$.

For more recent developments on the Jacobian conjecture for symmetric polynomial maps, see [BE1]–[BE4].

Based on the symmetric reduction above and also the classical homogeneous reduction in [BCW] and [Y], the second author in [Z] further reduced the Jacobian conjecture to the following so-called vanishing conjecture.

Let $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2}$ the Laplace operator and call a formal power series $P(z)$ *Hessian nilpotent*(HN) if its Hessian matrix $\text{Hes } P(z) := (\frac{\partial^2 P}{\partial z_i \partial z_j})$ is nilpotent. It has been shown in [Z] that the Jacobian conjecture is equivalent to

Conjecture 1.1. (Vanishing Conjecture of HN Polynomials)

For any homogeneous HN polynomial $P(z)$ (of degree $d = 4$), we have $\Delta^m P^{m+1} = 0$ when $m \gg 0$.

Note that, it has also been shown in [Z] that $P(z)$ is HN if and only if $\Delta^m P^m = 0$ for $m \geq 1$.

In this paper, we will prove the following two results on HN polynomials.

Let $P(z)$ be a homogeneous HN polynomial of degree $d \geq 3$ and $\sigma_2(z) := \sum_{i=1}^n z_i^2$. We denote by \mathcal{Z}_P and \mathcal{Z}_{σ_2} the projective subvarieties of $\mathbb{C}P^{n-1}$ determined by the principal ideals generated by $P(z)$ and $\sigma_2(z)$, respectively. The first main result of this paper is the following theorem.

Theorem 1.2. *Let $P(z)$ be a homogeneous HN polynomial of degree $d \geq 4$. Assume that \mathcal{Z}_P intersects with \mathcal{Z}_{σ_2} only at regular points of \mathcal{Z}_P , then the vanishing conjecture holds for $P(z)$. In particular, the vanishing conjecture holds if the projective variety \mathcal{Z}_P is regular.*

Remark 1.3. *Note that, when $\deg P(z) = d = 2$ or 3 , the Jacobian conjecture holds for the symmetric polynomial map $F = z - \nabla P$. This is because, when $d = 2$, F is a linear map with $j(F) \equiv 1$. Hence F is an automorphism of \mathbb{C}^n ; while when $d = 3$, we have $\deg F = 2$. By Wang's theorem [W], the Jacobian conjecture holds for F again. Then, by the equivalence of the vanishing conjecture for the homogeneous HN polynomial $P(z)$ and the Jacobian conjecture for the symmetric map $F = z - \nabla P$ established in [Z], we see that, when $\deg P(z) = d = 2$ or 3 , Theorem 1.2 actually also holds even without the condition on the projective variety \mathcal{Z}_P .*

For any non-zero $z \in \mathbb{C}^n$, denote by $[z]$ its image in the projective space $\mathbb{C}P^{n-1}$. Set

$$(1.1) \quad \tilde{\mathcal{Z}}_{\sigma_2} := \{z \in \mathbb{C}^n \mid z \neq 0; [z] \in \mathcal{Z}_{\sigma_2}\}.$$

In other words, $\tilde{\mathcal{Z}}_{\sigma_2}$ is the set of non-zero $z \in \mathbb{C}^n$ such that $\sum_{i=1}^n z_i^2 = 0$.

Note that, for any homogeneous polynomial $P(z)$ of degree d , it follows from the Euler's formula $dP = \sum_{i=1}^n z_i \frac{dP}{dz_i}$, that any non-zero $w \in \mathbb{C}^n$, $[w] \in \mathbb{C}P^{n-1}$ is a singular point of \mathcal{Z}_P if and only if w is a fixed point of the symmetric map $F = z - \nabla P$. Furthermore, it is also well-known that, $j(F) \equiv 1$ if and only if $P(z)$ is HN.

By the observations above and Theorem 1.2, it is easy to see that we have the following corollary on symmetric polynomial maps.

Corollary 1.4. *Let $F = z - \nabla P$ with P homogeneous and $j(F) \equiv 1$ (or equivalently, P is HN). Assume that F does not fix any $w \in \tilde{\mathcal{Z}}_{\sigma_2}$. Then the Jacobian holds for $F(z)$. In particular, if F has no non-zero fixed point, the Jacobian conjecture holds for F .*

Our second main result is following theorem which says that the vanishing conjecture is actually equivalent to a formally much stronger statement.

Theorem 1.5. *For any HN polynomial $P(z)$, the vanishing conjecture holds for $P(z)$ if and only if, for any polynomial $f(z) \in \mathbb{C}[z]$, $\Delta^m(f(z)P(z)^m) = 0$ when $m \gg 0$.*

2. Proof of the Main Results

Let us first fix the following notation. Let $z = (z_1, z_2, \dots, z_n)$ be free complex variables and $\mathbb{C}[z]$ (resp. $\mathbb{C}[[z]]$) the algebra of polynomials (resp. formal power series) in z . For any $d \geq 0$, we denote by V_d the vector space of homogeneous polynomials in z of degree d .

For any $1 \leq i \leq n$, we set $D_i = \frac{\partial}{\partial z_i}$ and $D = (D_1, D_2, \dots, D_n)$. We define a \mathbb{C} -bilinear map $\{\cdot, \cdot\} : \mathbb{C}[z] \times \mathbb{C}[z] \rightarrow \mathbb{C}[z]$ by setting

$$\{f, g\} := f(D)g(z)$$

for any $f(z), g(z) \in \mathbb{C}[z]$.

Note that, for any $m \geq 0$, the restriction of $\{\cdot, \cdot\}$ on $V_m \times V_m$ gives a \mathbb{C} -bilinear form of the vector subspace V_m , which we will denote by $B_m(\cdot, \cdot)$. It is easy to check that, for any $m \geq 1$, $B_m(\cdot, \cdot)$ is symmetric and non-singular.

The following lemma will play a crucial role in our proof of the first main result.

Lemma 2.1. *For any homogeneous polynomials $g_i(z)$ ($1 \leq i \leq k$) of degree $d_i \geq 1$, let S be the vector space of polynomial solutions of the following system of PDEs:*

$$(2.2) \quad \begin{cases} g_1(D) u(z) = 0, \\ g_2(D) u(z) = 0, \\ \dots \\ g_k(D) u(z) = 0. \end{cases}$$

Then, $\dim S < +\infty$ if and only if $g_i(z)$ ($1 \leq i \leq k$) have no non-zero common zeroes.

Proof: Let I the homogeneous ideal of $\mathbb{C}[z]$ generated by $\{g_i(z) | 1 \leq i \leq k\}$. Since all $g_i(z)$'s are homogeneous, S is a homogeneous vector subspace S of $\mathbb{C}[z]$.

Write

$$(2.3) \quad S = \bigoplus_{m=0}^{\infty} S_m,$$

$$(2.4) \quad I = \bigoplus_{m=0}^{\infty} I_m.$$

where $I_m := I \cap V_m$ and $S_m := S \cap V_m$ for any $m \geq 0$.

Claim: *For any $m \geq 1$ and $u(z) \in V_m$, $u(z) \in S_m$ if and only if $\{u, I_m\} = 0$, or in other words, $S_m = I_m^\perp$ with respect to the \mathbb{C} -bilinear form $B_m(\cdot, \cdot)$ of V_m .*

Proof of the Claim: First, by the definitions of I and S , we have $\{I_m, S_m\} = 0$ for any $m \geq 1$, hence $S_m \subseteq I_m^\perp$. Therefore, we need only show that, for any $u(z) \in I_m^\perp \subset V_m$, $g_i(D)u(z) = 0$ for any $1 \leq i \leq n$.

We first fix any $1 \leq i \leq n$. If $m < d_i$, there is nothing to prove. If $m = d_i$, then $g_i(z) \in I_m$, hence $\{g_i, u\} = g_i(D)u = 0$. Now suppose $m > d_i$. Note that, for any $v(z) \in V_{m-d_i}$, $v(z)g_i(z) \in I_m$. Hence we have

$$\begin{aligned} 0 &= \{v(z)g_i(z), u(z)\} \\ &= v(D)g_i(D)u(z) \\ &= v(D)(g_i(D)u)(z) \\ &= \{v(z), (g_i(D)u)(z)\}. \end{aligned}$$

Therefore, we have

$$B_{m-d_i}((g_i(D)u)(z), V_{m-d_i}) = 0.$$

Since $B_{m-d_i}(\cdot, \cdot)$ is a non-singular \mathbb{C} -bilinear form of V_{m-d_i} , we have $g_i(D)u = 0$. Hence, the Claim holds. \square

By a well-known fact in Algebraic Geometry (see Exercise 2.2 in [H], for example), we know that the homogeneous polynomials $g_i(z)$ ($1 \leq i \leq k$) have no non-zero common zeroes if and only if $I_m = V_m$ when $m \gg 0$. While, by the Claim above, we know that, $I_m = V_m$ when $m \gg 0$ if and only if $S_m = 0$ when $m \gg 0$, and if and only if the solution space S of the system (2.2) is finite dimensional. Hence, the lemma follows. \square

Now we are ready to prove our first main result, Theorem 1.2.

Proof of Theorem 1.2: Let $P(z)$ be a homogeneous HN polynomial of degree $d \geq 4$ and S the vector space of polynomial solutions of the following system of PDEs:

$$(2.5) \quad \begin{cases} \frac{\partial P}{\partial z_1}(D) u(z) = 0, \\ \frac{\partial P}{\partial z_2}(D) u(z) = 0, \\ \dots \\ \frac{\partial P}{\partial z_n}(D) u(z) = 0, \\ \Delta u(z) = 0. \end{cases}$$

First, note that the projective subvariety \mathcal{Z}_P intersects with \mathcal{Z}_{σ_2} only at regular points of \mathcal{Z}_P if and only if $\frac{\partial P}{\partial z_i}(z)$ ($1 \leq i \leq n$) and $\sigma_2 = \sum_{i=1}^n z_i^2$ have no non-zero common zeros (again use Euler's formula). Then, by Lemma 2.1, we have $\dim S < +\infty$.

On the other hand, by Theorem 6.3 in [Z], we know that $\Delta^m P^{m+1} \in S$ for any $m \geq 0$. Note that $\deg \Delta^m P^{m+1} = (d-2)m + d$ for any $m \geq 0$. So $\deg \Delta^m P^{m+1} > \deg \Delta^k P^{k+1}$ for any $m > k$. Since $\dim S < +\infty$ (from above), we have $\Delta^m P^{m+1} = 0$ when $m \gg 0$, i.e. the vanishing conjecture holds for $P(z)$. \square

Next, we give a proof for our second main result, Theorem 1.5.

Proof of Theorem 1.5: The (\Leftarrow) part follows directly by choosing $f(z)$ to be $P(z)$ itself.

To show (\Rightarrow) part, let $d = \deg f(z)$. If $d = 0$, f is a constant. Then, $\Delta^m(f(z)P(z)^m) = f(z)\Delta^m P^m = 0$ for any $m \geq 1$.

So we assume $d \geq 1$. By Theorem 6.2 in [Z], we know that, if the vanishing conjecture holds for $P(z)$, then, for any fixed $a \geq 1$, $\Delta^m P^{m+a} = 0$ when $m \gg 0$. Therefore there exists $N > 0$ such that, for any $0 \leq b \leq d$ and any $m > N$, we have $\Delta^m P^{m+b} = 0$.

By Lemma 6.5 in [Z], for any $m \geq 1$, we have

$$(2.6) \quad \Delta^m(f(z)P(z)^m) = \sum_{\substack{k_1+k_2+k_3=m \\ k_1, k_2, k_3 \geq 0}} 2^{k_2} \binom{m}{k_1, k_2, k_3} \sum_{\substack{\mathbf{s} \in \mathbb{N}^n \\ |\mathbf{s}|=k_2}} \binom{k_2}{\mathbf{s}} \frac{\partial^{k_2} \Delta^{k_1} f(z)}{\partial z^{\mathbf{s}}} \frac{\partial^{k_2} \Delta^{k_3} P^m(z)}{\partial z^{\mathbf{s}}},$$

where $\binom{m}{k_1, k_2, k_3}$ and $\binom{k_2}{\mathbf{s}}$ denote the usual binomials.

Note first that, the general term in the sum above is non-zero only if $2k_1 + k_2 \leq d$. But on the other hand, since

$$(2.7) \quad 0 \leq k_1 + k_2 \leq 2k_1 + k_2 \leq d,$$

by the choice of $N \geq 1$, we have $\Delta^{k_3} P^m(z) = \Delta^{k_3} P^{k_3+(k_1+k_2)}(z)$ is non-zero only if

$$(2.8) \quad k_3 \leq N.$$

From the observations above and Eqs. (2.6), (2.7), (2.8) it is easy to see that, $\Delta^m(f(z)P(z)^m) \neq 0$ only if $m = k_1 + k_2 + k_3 \leq d + N$. In other words, $\Delta^m(f(z)P(z)^m) = 0$ for any $m > d + N$. Hence Theorem 1.5 holds. \square

Note that all results used in the proof above for the (\Leftarrow) part of the theorem also hold for all HN formal power series. Therefore we have the following corollary.

Corollary 2.2. *Let $P(z)$ be a HN formal power series such that the vanishing conjecture holds for $P(z)$. Then, for any polynomial $f(z)$, we have $\Delta^m(f(z)P(z)^m) = 0$ when $m \gg 0$.*

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* DEPARTMENT OF MATHEMATICS, RADBOUD UNIVERSITY NIJMEGEN,
POSTBUS 9010, 6500 GL NIJMEGEN, THE NETHERLANDS.

E-mail: essen@math.ru.nl

** DEPARTMENT OF MATHEMATICS, ILLINOIS STATE UNIVERSITY, NOR-
MAL, IL 61790-4520.

E-mail: wzhao@ilstu.edu.