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# Cusps on cosmic superstrings with junctions

**Anne-Christine Davis**[1], **William Nelson**[2],
**Senthooran Rajamanoharan**[1], **Mairi Sakellariadou**[2]

[1]DAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, U.K.

[2]Department of Physics, King's College, University of London, Strand WC2R 2LS, London, U.K.

E-mail: a.c.davis@damtp.cam.ac.uk, william.nelson@kcl.ac.uk, s.rajamanoharan@damtp.cam.ac.uk , mairi.sakellariadou@kcl.ac.uk

**Abstract.** The existence of cusps on non-periodic strings ending on D-branes is demonstrated and the conditions, for which such cusps are generic, are derived. The dynamics of F-, D-string and FD-string junctions are investigated. It is shown that pairs of FD-string junctions, such as would form after intercommutations of F- and D-strings, generically contain cusps. This new feature of cosmic superstrings opens up the possibility of extra channels of energy loss from a string network. The phenomenology of cusps on such cosmic superstring networks is compared to that of cusps formed on networks of their field theory analogues, the standard cosmic strings.

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## 1. Introduction

Fundamental (F) strings and Dirichlet branes with one non-compact spatial dimension (D-strings) are generically formed [1, 2, 3, 4] at the end of brane inflation [5] within the context of string inspired cosmological models. Such strings, known as cosmic superstrings, are of cosmological size and could play the rôle of cosmic strings [6, 7], false vacuum remnants formed generically at the end of hybrid inflation within Grand Unified Theories [8, 9]. Cosmic superstrings have gained a lot of interest, particularly since it is believed that they may be observed in the sky, providing both a means of testing string theory and a hint for a physically motivated inflationary model (for a recent reviews, see e.g. Ref. [10, 11, 12]).

The most significant difference between cosmic superstrings and the field theory cosmic strings we are more familiar with, is the existence of three string junctions, the presence of which could strongly effect the dynamics of the string network. Understanding these new dynamical effects is critical if there is to be any hope of differentiating cosmic superstrings from their solitonic analogues. A number of analytical [13, 14, 15] and numerical [16, 17, 18, 19, 20, 21, 22] studies have addressed

cosmic superstring dynamics. We note that, in principle, cosmic superstring dynamics ought to be studied using the Dirac-Born-Infeld action, the low-energy effective action for many varieties of strings arising in the context of string theory.

In what follows, we investigate (generic) string solutions ending on parallel Dirichlet branes as a pedagogical example of the effects possible at a three string junction. We then look at the dynamics of a junction made up of an F-string, a D-string, and an FD-string, which are expected to form through intercommuting of initial configurations composed from F- and D-string networks. We show that cusps are generic features for such strings, opening up a new energy loss mechanism for the network, in addition to the formation and subsequent decay of closed loops and the formation of bound states [22]. Studies of the phenomenological implications of cosmic superstrings, particularly the gravitational [23, 24, 25, 21, 26] and Ramond-Ramond [16, 27] radiation emitted from cosmic superstrings — predominantly from cusps and to some extent from kinks — are then justified. Some of the phenomenological consequences of cosmic superstring dynamics are discussed here.

## 2. DBI string ending on parallel D-branes

The world-history of a string can be represented by its world-sheet

$$x^\mu = x^\mu(\tau, \sigma);$$

a two-dimensional surface in the four-dimensional space-time. The world-sheet's coordinates $\tau, \sigma$ are arbitrary time-like and space-like parameters, respectively. A metric for the two-dimensional world-sheet is induced by pulling-back the space-time metric

$$\gamma_{\alpha\beta} = g_{\mu\nu} x^\mu_{,\alpha} x^\nu_{,\beta} \ ,$$

where $g_{\mu\nu}$ denotes the four-dimensional metric.

Consider a Dirac-Born-Infeld (DBI) string in a Minkowski background, with end-points that are constrained to lie on two stationary, parallel and flat D$n$-branes, where $n$ is the spatial dimensionality of the branes. Without loss of generality, we choose Cartesian space-time coordinates in which the separation vector between the two branes lies in the $z$-direction.

The action for the DBI string is

$$S = -\mu \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-|\gamma_{\alpha\beta} + \lambda F_{\alpha\beta}|} \ , \tag{1}$$

where $\mu$ is the string tension parameter and $\lambda = 2\pi\alpha'$. For a $(p, q)$-string (a bound state of $p$ coincident F-strings and $q$ coincident D-strings) the string tension is $\mu = |q|/(g_s\lambda)$, where $g_s$ is the perturbative string coupling. The electromagnetic field strength, $F_{\alpha\beta}$, associated with the $U(1)$ gauge field $A_\alpha$ on the world-sheet, reads

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \ .$$

The DBI action describes the low-energy dynamics of certain classical string-like objects found in string theory.

Due to the re-parametrisation invariance of the string world-sheet $x^\mu(\tau,\sigma)$, we are free to impose the conformal gauge condition:

$$\dot{x}^2 + x'^2 = 0 \quad \text{and} \quad \dot{x}\cdot x' = 0 \ , \tag{2}$$

where $\dot{} \equiv \partial/\partial\tau$ and $' \equiv \partial/\partial\sigma$. In this gauge, the DBI equation of motion for the string world-sheet reads [15]

$$\ddot{x} - x'' = 0 \ . \tag{3}$$

The electric flux along the string,

$$p \equiv \frac{\partial\mathcal{L}}{\partial F_{\tau\sigma}} = \frac{\lambda^2\mu F_{\tau\sigma}}{\sqrt{-x'^2\dot{x}^2 - \lambda^2 {F_{\tau\sigma}}^2}} \ , \tag{4}$$

is a conserved quantity; in the $(p,q)$-string picture it corresponds to the number of coincident F-strings that make up the bound state.

The conformal gauge condition allows for a residual gauge symmetry, which can be fixed in a manner that is consistent with the equations of motion by imposing the temporal gauge condition $x^0 \equiv t = \tau$. Once we have totally fixed the world-sheet parametrisation in this way, the equation of motion reads

$$\ddot{\mathbf{x}} - \mathbf{x}'' = 0 \ , \tag{5}$$

with gauge constraints

$$\dot{\mathbf{x}}^2 + \mathbf{x}'^2 = 1 \quad \text{and} \quad \dot{\mathbf{x}}\cdot\mathbf{x}' = 0 \ , \tag{6}$$

where $x^\mu(t,\sigma) = (t, \mathbf{x}(t,\sigma))$.

The general solution of the equation of motion, Eq. (5), is

$$\mathbf{x} = \frac{1}{2}\left[\mathbf{a}(t-\sigma) + \mathbf{b}(t+\sigma)\right] \ , \tag{7}$$

where $\mathbf{a}(t-\sigma)$ and $\mathbf{b}(t+\sigma)$ are arbitrary vector-valued functions. However, the gauge constraints, Eq. (6), impose the additional (necessary and sufficient) restrictions

$$|\mathbf{a}'|^2 = |\mathbf{b}'|^2 = 1 \ , \tag{8}$$

where primes ($'$) here refer to total derivative of these single-variable functions. Therefore, prior to taking boundary conditions into account, a DBI string solution may be completely specified by the arbitrary choice of two parametrised curves on the unit sphere.

Let us consider what additional conditions are imposed by the boundary conditions for the current problem – that of a DBI string ending on two stationary and parallel D$n$-branes. Without loss of generality, we choose the parametrisation of the world-sheet so that at time $t$ the end-point of the string attached to the upper brane (towards the $+z$-direction) is at $\sigma = 0$, whilst the end-point of the string attached to the lower brane is at $\sigma = L(t)$. Note that $L(t)$ denotes the *parameter length* of the string (as opposed to its physical length). When the string has time-independent boundary conditions (as is the

case considered here), the parameter length, $L(t)$, is assumed to be time-independent. However, for more general configurations (which we consider later on), it is important to look for solutions where $L(t)$ is not necessarily constant. Thus, we do not make here the (usual) assumption that $\dot{L}(t) = 0$. Certainly, when we analyse a time-independent brane configuration, we recover the expected result, namely that the parameter length of the string remains constant.

It turns out to be convenient to decompose all spatial three-vectors (e.g., $\mathbf{v}$) into a part parallel to the D-branes and one perpendicular to them:

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \ . \tag{9}$$

This decomposition (which is clearly unique), allows us to distinguish between the boundary conditions for Neumann directions (parallel to the branes) and the boundary conditions for Dirichlet directions (perpendicular to the branes).

On the one hand, the fact that the end-points of the string are constrained to the two D-branes implies, in Dirichlet directions, that

$$\dot{\mathbf{x}}_{\perp}(t,0) = 0 \quad \text{and} \quad \dot{\mathbf{x}}_{\perp}(t,L(t)) + \dot{L}(t)\mathbf{x}'_{\perp}(t,L(t)) = 0 \ , \tag{10}$$

whilst in Neumann directions, that

$$\mathbf{x}'_{\parallel}(t,0) = 0 \quad \text{and} \quad \mathbf{x}'_{\parallel}(t,L(t)) = 0 \ . \tag{11}$$

If we consider what these boundary conditions mean in terms of $\mathbf{a}$ and $\mathbf{b}$, we find that the boundary conditions at the $\sigma = 0$ end of the string lead to

$$\mathbf{a}'_{\parallel}(t) \ = \mathbf{b}'_{\parallel}(t) \ , \tag{12}$$

$$\mathbf{a}'_{\perp}(t) = \ -\mathbf{b}'_{\perp}(t) \ . \tag{13}$$

Geometrically speaking, Eqs. (12), (13) imply that the curves $\mathbf{a}'(t)$ and $\mathbf{b}'(t)$ are related by inversion through a surface of identical dimension and orientation to the D-branes, that passes through the centre of the unit sphere. On the other hand, the boundary conditions at the $\sigma = L(t)$ end of the string imply that

$$\mathbf{a}'_{\parallel}(t - L(t)) \qquad\qquad = \mathbf{b}'_{\parallel}(t + L(t)) \ , \tag{14}$$

$$\left[\dot{L}(t) - 1\right] \mathbf{a}'_{\perp}(t - L(t)) = \ -\left[\dot{L}(t) + 1\right] \mathbf{b}'_{\perp}(t + L(t)) \ . \tag{15}$$

Any curves $\mathbf{a}'$ and $\mathbf{b}'$ that satisfy Eqs. (12), (13), (14), (15) must also necessarily satisfy the conditions that we obtain by squaring them and imposing the gauge constraints Eq. (8), written in the form

$$|\mathbf{a}'_{\parallel}|^2 + |\mathbf{a}'_{\perp}|^2 = |\mathbf{b}'_{\parallel}|^2 + |\mathbf{b}'_{\perp}|^2 = 1 \ . \tag{16}$$

It is easily checked that the necessary conditions that we have obtained in this manner can only be satisfied when $\dot{L} = 0$. We have thus confirmed explicitly that the string solutions for this stationary brane configuration must have constant parameter length, $L(t) = L$.

Using $\dot{L} = 0$, as well as the boundary conditions Eqs. (12), (13), it is straightforward to re-write Eqs. (14), (15) in the form

$$\mathbf{a}'(t - L) = \mathbf{a}'(t + L) \ , \tag{17}$$

and similarly for $\mathbf{b}'$. Geometrically speaking, $\mathbf{a}'(t)$ and $\mathbf{b}'(t)$ describe closed curves of parameter length $2L$ on the unit sphere.

Equations (14) and (15) are necessary but not sufficient to satisfy the Dirichlet boundary conditions; they ensure that the string end-points do not travel in any direction perpendicular to the D-branes, but they do not ensure that the end-points are actually on the D-branes. Satisfying this extra requirement has two consequences: firstly, it determines the initial values $\mathbf{a}_\perp(0)$ and $\mathbf{b}_\perp(0)$, and secondly — and more importantly for our purposes — it implies that the curve $\mathbf{a}'(\xi)$ must satisfy the condition

$$\frac{1}{2}\int_{-L}^{L} \mathbf{a}'_\perp(\xi)\mathrm{d}\xi = \mathbf{\Delta} \ , \tag{18}$$

where $\mathbf{\Delta}$ is a normal vector stretching from the lower brane at $\sigma = L$ to the upper brane at $\sigma = 0$. In other words, $\mathbf{\Delta}$ points in the $+z$-direction and has magnitude equal to the inter-brane separation.

Finally, we have to fix the remaining symmetry, namely the space-time symmetry corresponding to a free choice of space-time coordinates $x^\mu$. In other words, we need to choose a unique inertial frame. One obvious choice would be the zero momentum frame. This choice is indeed possible, as one can check that the total momentum of the string

$$P^\mu \sim \int_0^L \dot{x}^\mu(t,\sigma)\mathrm{d}\sigma \tag{19}$$

is a time-like vector, and therefore there exists a Lorentz transformation that would set the spatial part of this vector equal to zero. However, we need to avoid Lorentz transformations that involve the Dirichlet directions, as these would change the configuration of the D-branes. Therefore, the correct solution is to choose the frame in which only the Neumann component of the momentum, $\mathbf{P}_\parallel$, is zero. In other words, we choose the unique frame in which the momentum in directions parallel to the branes vanishes. Obviously, such a choice of frame will not affect the configuration of the D-branes. It follows that in this frame,

$$\int_0^L \dot{\mathbf{x}}_\parallel(t,\sigma)\mathrm{d}\sigma = 0 \ . \tag{20}$$

Combining Eqs. (18) and (20) we get the condition

$$\langle \mathbf{a}' \rangle = \frac{\mathbf{\Delta}}{L} \ , \tag{21}$$

with the definition

$$\langle \mathbf{v} \rangle \equiv \frac{1}{2L}\int_{-L}^{L} \mathbf{v}(\xi)\mathrm{d}\xi \ , \tag{22}$$

for any closed curve $\mathbf{v}(\xi)$ on the unit sphere, having parameter length equal to $2L$. (This quantity can be geometrically visualised as the average or centre-of-mass of such a curve.) Notice that this condition has a corollary that we should have expected on physical grounds: there is a lower bound on our choice of string parameter length, $L \geq |\mathbf{\Delta}|$.

We have thus arrived at a geometric method for classifying solutions for a system of a DBI string constrained by two parallel and stationary D-branes. Working with a conformal and temporal gauge parametrisation of the world-sheet, and also working in unique zero Neumann-momentum space-time coordinates, we have found that any string solution has a unique description in terms of one constant scalar, $L$, two constant vectors, $\mathbf{a}_{\parallel}(0)$ and $\mathbf{b}_{\parallel}(0)$, as well as a single closed, parametrised curve on the unit sphere, $\mathbf{a}'(\xi)$ (or, equivalently, $\mathbf{b}'(\xi)$), which satisfies the averaging condition Eq. (21) (or an equivalent averaging condition for $\mathbf{b}'$).

*Cusps*

To investigate cusps formation in DBI strings ending on two stationary and parallel D$n$-branes, we will follow the same approach as for Nambu-Goto string loops, we are more familiar with. Actually, the geometric visualisation of the DBI string solutions described here is similar to that of closed Nambu-Goto string loops [6], but with two important differences. Firstly, whilst with closed string loops we are free to choose both curves $\mathbf{a}'$ and $\mathbf{b}'$ independently, for a DBI string ending on two stationary and parallel D$n$-branes we can only choose one of these curves freely; the other one is automatically determined by the inversion described by Eqs. (12), (13). Secondly, with closed string loops we can always choose the zero momentum frame, so that the average of the curves $\mathbf{a}'$ and $\mathbf{b}'$ can always be constrained to the centre of the unit sphere. Here we can only constrain the average of $\mathbf{a}'$ to a unique point within the unit sphere determined by the separation and orientation of the D-branes, $\mathbf{\Delta}$. When we investigate the formation of cusps in this system, we will see the consequences of the above mentioned differences.

Cusps are found wherever the instantaneous velocity of a part of the string reaches the speed of light. As with closed strings, it is straightforward to show that cusps are present in DBI strings ending in two stationary and parallel branes if, and only if, the curves $\mathbf{a}'$ and $\mathbf{b}'$ intersect on the unit sphere. Therefore, the prevalence of cusps in a string configuration is directly related to the prevalence of intersections amongst pairs of closed curves on the unit sphere, that satisfy a number of properties listed above.

For smooth closed string loops, where $\mathbf{a}'$ and $\mathbf{b}'$ are independent smooth curves whose centres-of-mass are at the centre of the unit sphere, it is very difficult for $\mathbf{a}'$ and $\mathbf{b}'$ curves to avoid intersection. Therefore, as it is usually said, closed string loops have *generically* cusps [28]. For the DBI string under consideration here, the curves are no longer independent. The exact nature of their relationship to each other depends on the dimensionality of the D$n$-branes. The important case for cosmological implications is $n = 1$ (D1-branes), which we discuss in detail in the following sections. For $n = 2$, the two curves are reflections of each other across a plane passing through the centre of the unit sphere (see, Fig.1); for $n = 1$ the two curves are inversions of one another through a line passing through the unit sphere; and for $n = 0$ the two curves are inversions of one another through the centre of the unit sphere. Furthermore, the centres-of-mass of the two curves are no longer necessarily located at the centre of the unit sphere – their

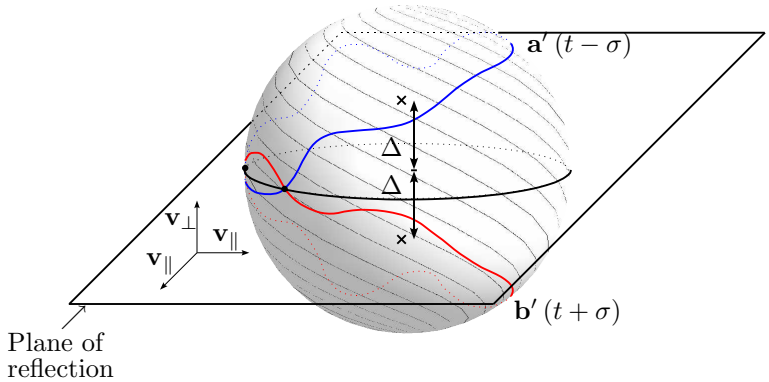


**Figure 1.** The vectors $\mathbf{a}'(t-\sigma)$ and $\mathbf{b}'(t+\sigma)$ trace out closed curves on unit spheres that are separated by the inter-brane distance.

locations are now governed by the inter-brane separation.

Even though we have managed to reduce the problem of cusps to a rather simple-looking game involving intersections of curves satisfying certain properties, it turns out to be very difficult to make any quantitative predictions about this geometric puzzle. Nevertheless, there are a few observations that are worth pointing out.

Firstly, for any $n$, the likelihood of cusps falls to zero as $L$ approaches the inter-brane separation distance $|\mathbf{\Delta}|$ (i.e., the centres-of-mass of the curves approach the surface of the unit sphere). In this limit, the two curves, related by inversion, are confined to shrinking antipodal regions of the sphere and the likelihood of intersection becomes vanishingly small.

Secondly, for coincident D2-branes (with the DBI string having its end-points on these coincident branes) there are always cusps. This is because the two curves have centres-of-mass at the centre of the unit sphere, and are related by reflections through a plane passing through the centre of the unit sphere. The first condition forces each curve to cross the plane of reflection at least twice, whilst the second condition implies that the two curves intersect whenever they cross the reflection plane. Assuming continuity, it is therefore reasonable to expect that even for the case of non-coincident D2-branes, the likelihood of cusps will grow in the limit $L \gg |\mathbf{\Delta}|$.

Thirdly, the situation is different for coincident D1- and D0-branes, as although the averaging condition still forces the two curves to pass through the same plane as before, whereas the inversion property no longer implies that the two curves have to cross this plane at the same point. Therefore, intersections are not guaranteed in the same way as they are for coincident D2-branes.

For D1-branes however we can see that the two curves will generically intersect whenever the line though which they are inverted is enclosed by the closed curves. To see this consider the two vectors perpendicular to the inversion line that end on the closed curve. Typically these will not be anti-parallel, however if they are, then these points will be unaffected by the inversion and the $\mathbf{a}'$ and $\mathbf{b}'$ curves will intersect. If the inversion line is enclosed by the closed curve, then at one extreme the two vectors

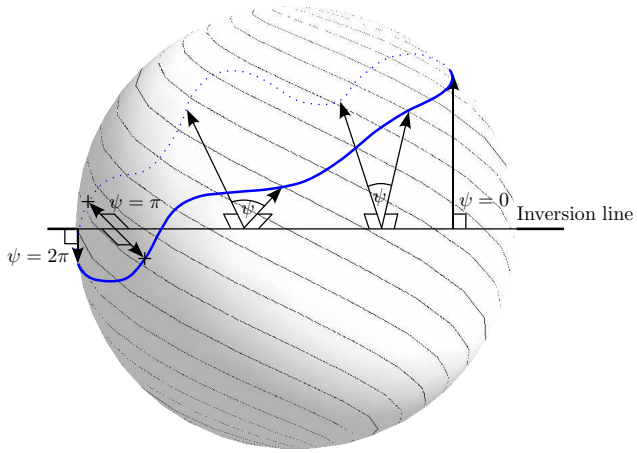


**Figure 2.** When the line through which the two closed curves are inverted pierces the closed curve, then there will generically be and intersection of the two curves. This can be seen by considering the two vectors perpendicular to the inversion line, ending on the curve. When the angle between these vectors, $\psi$, is $\pi$ the inverted curve will intersect the original curve (marked with crosses).

will have zero angle between them and at the other extreme this angle will tend to $2\pi$. By continuity there must be a point at which the angle between the two vectors is $\pi$ and hence we will get the intersection between the $\mathbf{a}'$ and $\mathbf{b}'$ curves, necessary for cusps formation (see, Fig. 2).

We can extend the above, by considering closed curves that do not encircle the inversion line, but for which there exists a line, $AB$, that intersects the inversion line at ninety degrees and is (topologically) on the opposite side of the closed curve (see, Fig. 3). To see that this scenario also has points on the closed curve that are invariant under the inversion, consider the plane containing the inversion line and the line $AB$. The closed curve will have to intersect this plane at least four times (with the possibility that two or more of these intersection points are coincident in the extreme cases). Then as before we can construct the vectors perpendicular to the inversion line that end on the closed curve. The angle between these two vectors will be zero at both extremes and because the line $AB$ in on the opposite side of the closed curve to the inversion line, there will be at least one point at which the angle between the two vectors is greater than $\pi$. By continuity, there exists a pair of points on the closed curve at which the angle between the vectors is $\pi$ and hence a pair of points on the closed curve that are invariant under the inversion, i.e. there will be cusps. We can then classify all possible closed curves by considering a general line intersecting the inversion line at right angles and describing the closed curve by whether it goes over or under each leg of the resulting cross (see, Fig. 4). If we label the legs of the cross on the inversion line by $1, 3$ and the remaining legs of the cross by $2, 4$ then the following curves have cusps,

$$\begin{aligned} &(1_{\mathbf{o}}, 2_{\mathbf{u}}, 3_{\mathbf{u}}, 4_{\mathbf{u}}) \,, \quad (1_{\mathbf{o}}, 2_{\mathbf{o}}, 3_{\mathbf{u}}, 4_{\mathbf{u}}) \,, \quad (1_{\mathbf{o}}, 2_{\mathbf{u}}, 3_{\mathbf{o}}, 4_{\mathbf{u}}) \,, \\ &(1_{\mathbf{u}}, 2_{\mathbf{u}}, 3_{\mathbf{o}}, 4_{\mathbf{u}}) \,, \quad (1_{\mathbf{o}}, 2_{\mathbf{u}}, 3_{\mathbf{u}}, 4_{\mathbf{o}}) \,, \end{aligned} \tag{23}$$

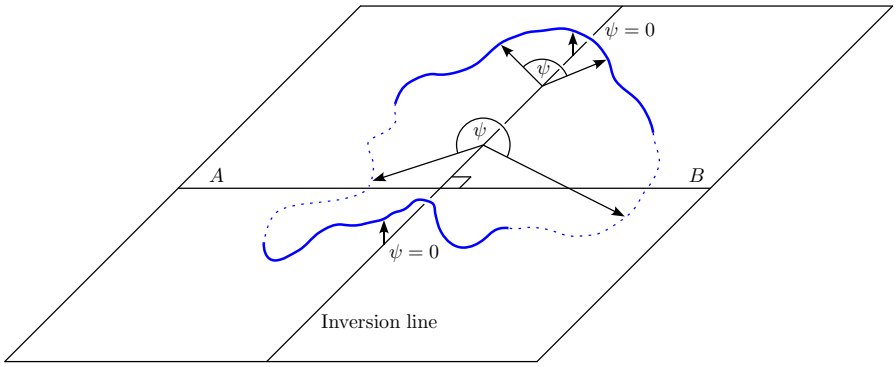


**Figure 3.** If there exists a line that intersects the inversion line at right angles, $AB$, that is topologically on the opposite side of the closed curve, then there will be a pair of points that are invariant under the inversion, i.e. there will be cusps. The sphere has not been drawn for clarity.

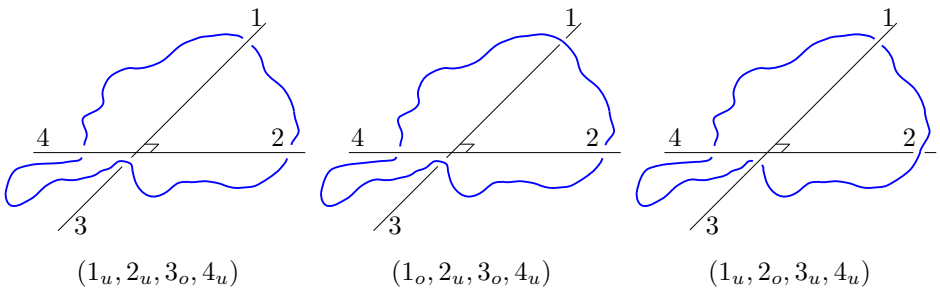


**Figure 4.** The legs of the inversion line are labelled $(1, 3)$ and the legs of the line intersecting it at right angles are labelled $(2, 4)$. We can then classify possible curves by whether they go over or under each leg. Three examples are give above, the first two will generically contain cusps, whilst the last will not.

as well as their reflections $\mathbf{o} \leftrightarrow \mathbf{u}$, where the subscripts indicate whether the curve went over $X_{\mathbf{o}}$, or under $X_{\mathbf{u}}$ the leg $X$. The first and second groups in Eq. (23) have cusps because the inversion line is encircled by the closed curve, whilst the right-hand most curve has cusps by the generalisation given above. In total there are another three curves (and their reflections) that will not produce cusps, $(1_{\mathbf{o}}, 2_{\mathbf{o}}, 3_{\mathbf{o}}, 4_{\mathbf{o}})$, $(1_{\mathbf{u}}, 2_{\mathbf{o}}, 3_{\mathbf{u}}, 4_{\mathbf{u}})$ and $(1_{\mathbf{u}}, 2_{\mathbf{u}}, 3_{\mathbf{u}}, 4_{\mathbf{o}})$. If each of these possibilities are equally likely, which is a reasonable assumption if the centre of the mass of the closed curve and the centre of the cross are at the origin, then we would expect to have cusps in more than half of curves. As the centre of mass of the closed curve is moved away from the origin, the probability of each of these curves would no longer be equal. In particular as the closed curves become restricted to shrinking antipodal region we can see that the $(1_{\mathbf{o}}, 2_{\mathbf{o}}, 3_{\mathbf{o}}, 4_{\mathbf{o}})$ curve (and its reflection) would become increasingly more likely, thus reducing the probability of having cusps, in line with our earlier expectations.

Thus, we have shown that whilst cusps may not be a generic feature of a string stretched between two D1-branes, when $|\Delta| \ll L$ we would expect to find cusps in a

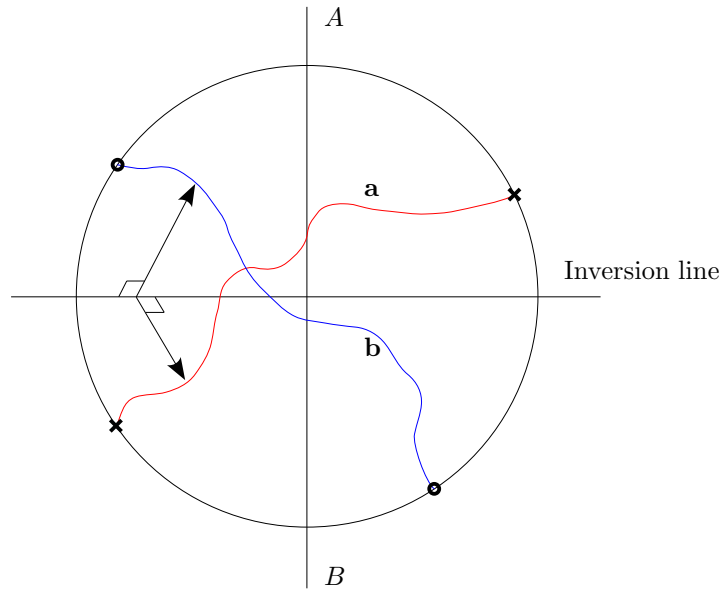


**Figure 5.** If a closed curve **a** intersects the plane made up of the inversion line and a line perpendicular to it as shown then it will encircle both lines. The inversion of **a** through the inversion line results in a curve that intersects the plane through two opposite quadrants. For any deformation of this inverted closed curve, **b**, that respects the positions of these intersection, there will be a point at which the two curves **a** and **b** intersect. In the diagram a top down view of the plane and unit sphere is given, with the curves **a** and **b** extending out of the page.

significant fraction of cases. In the following we will need a slight generalisation of the above proofs.

For the cases of the closed curve encircling the inversion line, we have seen that the closed curve will intersect its inversion, here we extend this to deformations of this inverted curve. In particular, consider a closed curve **a** that encircles both the inversion line and a line $AB$ intersecting the inversion line at right angles (the two curves in the second column of Eq. (23)). The inversion of **a** will yield a closed curve **b** that also encircles both these lines, but with the opposite orientation. Any subsequent deformation of **b** (or indeed **a**) that preserves the fact that the curve encircles both lines, will result in intersections between the two closed curves. To see this consider the plane consisting of the inversion line and $AB$. We arbitrarily define one side of this plane to be positive, then we have for the closed curves **a** and **b** to correctly encircle the inversion line and the line $AB$, that they must intersect the plane twice each, in opposite quadrants as in Fig. 5. The angle between the two vectors perpendicular to the inversion line (or $AB$) and ending on **a** and **b** must at some point reach zero, which implies the closed curves intersect. This holds equally for the negative side of the plane. Essentially what this says is that the result for closed curves and their inversions through a line is a topological one and holds equally well for subsequent deformations of either closed curve that preserves the topological relations between the closed curves and the inversion line. The requirement that the relation between the line perpendicular to the

inversion line and the closed curves is there to ensure that the subsequent deformation does not 'undo' the inversion.

## 3. Three string junctions

Let us consider the simplest DBI string system that contains a junction: that of three DBI strings joined at a single junction. For the sake of simplicity, we attach the free ends of the strings (the ends that are not connected to the junction) to three flat, stationary and parallel D$n$-branes, which (without loss of generality) are spatially separated in the $z$-direction. In what follows, only the boundary conditions for the three DBI strings associated with these D$n$-branes are important and we need not be concerned with the precise nature (dimensionality) of the branes themselves‡.

We label the three strings with an index $i = 1, 2, 3$. As in Section 2, we parametrise each string's world-sheet, $x_i^\mu(\tau, \sigma)$, so that $\sigma = 0$ corresponds to the end of the string attached to a D$n$-brane, whilst $\sigma = L_i(\tau)$ corresponds to the three-string junction. Thus, $L_i(\tau)$ is the parameter length of the $i^{\text{th}}$-string at world-sheet time $\tau$. We denote the world-line of the junction by $\bar{x}^\mu(\tau)$ and the (single-component) gauge field present on the junction by $\bar{A}(\tau)$.

We exploit the re-parametrisation invariance of each string world-sheet to work in conformal gauge. In this gauge, the action for this configuration of three DBI strings reads [15]

$$\begin{aligned} S = & -\sum_i \mu_i \int \mathrm{d}\tau \int_0^{L_i(\tau)} \mathrm{d}\sigma \sqrt{-x_i'^2 \dot{x}_i^2 - \lambda (F^i_{\tau\sigma})^2} \\ & + \sum_i \int \mathrm{d}\tau \, \{ \mathbf{f}_i(\tau) \cdot [x_i(\tau, L_i(\tau)) - \bar{x}(\tau)] \\ & \qquad + g_i(\tau) \left[ A^i_\tau(\tau, L_i(\tau)) + \dot{L}_i A^i_\sigma(\tau, L_i(\tau)) - \bar{A}(\tau) \right] \Big\} \ , \end{aligned} \tag{24}$$

where $\mu_i$ is the tension of the $i^{\text{th}}$-string, which for a $(p,q)$-string is given by $\mu_i = |q_i|/(g_s \lambda)$, and $A^i_\alpha$ and $F^i_{\alpha\beta}$ are the gauge field and gauge field strengths respectively on the $i^{\text{th}}$-string. The Lagrange multipliers $\mathbf{f}_i(\tau)$ constrain the strings to meet at the junction $\bar{x}(\tau)$, whilst the Lagrange multipliers $g_i(\tau)$ impose to the component of each string's gauge field that is tangential to the junction world-line to coincide with the junction gauge field $\bar{A}(\tau)$.

We then derive the equations of motion for this action§. By varying the action with respect to ${x_i}^\mu(\tau, \sigma)$ and $A^i_\alpha$, we find that the equations of motion for the string world-sheets are:

$$\ddot{x}_i^\mu - x_i^{\mu\prime\prime} = 0 \ , \tag{25}$$

‡ This will be important later on, when we discuss (as an example) the case of a three string junction composed by an F-,D-string, and their FD-string bound state.

§ This was first done in Ref. [15]; however we shall include the results here for the sake of clarity and completeness.

and the equations of motion for the gauge fields once again imply the conservation of electric flux along each string, i.e., $\partial_\tau p_i = \partial_\sigma p_i = 0$, where

$$p_i = \frac{\lambda^2 \mu_i F^i_{\tau\sigma}}{\sqrt{-x_i'^2 \dot{x}_i^2 - \lambda \left(F^i_{\tau\sigma}\right)^2}} \ . \tag{26}$$

The above variations of the action also lead to certain boundary conditions at the junction which, when combined with the equations obtained by varying the action with respect to $\bar{x}(\tau)$ and $\bar{A}(\tau)$, give the following conservation laws:

$$\sum_i \bar{\mu}_i \left( x_i^{\mu\prime} + \dot{L}_i \dot{x}_i^\mu \right) = 0 \tag{27}$$

and

$$\sum_i p_i = 0 \ , \tag{28}$$

where the effective string tension $\bar{\mu}_i$ is

$$\bar{\mu}_i = \sqrt{\lambda^2 \mu_i^2 + p_i^2} = \sqrt{\frac{q_i^2}{g_s^2} + p_i^2} \ . \tag{29}$$

Finally, variation of the action with respect to the Lagrange multipliers, $f_i(\tau)$ and $g_i(\tau)$, leads to the required constraints

$$x_i \left(\tau, L_i(\tau)\right) = \bar{x} \left(\tau\right) \tag{30}$$

and

$$A^i_\tau \left(\tau, L_i(\tau)\right) + \dot{L}_i A^i_\sigma \left(\tau, L_i(\tau)\right) = \bar{A}(\tau) \ . \tag{31}$$

The action given in Eq. (24) can only represent a network of $(p,q)$-strings when none of the $q_i$'s are zero. In the presence of a $(p,0)$-string, this action has to be modified by replacing the corresponding DBI kinetic term by a Nambo-Goto (gauge-free) kinetic term with string tension $\mu_i = p$. Nevertheless, it is straightforward to show that the equations of motion, boundary conditions and conservation equations, derived from this modified action, agree with those given above for the original action.

Furthermore, when we are dealing with a system of $(p,q)$-strings (rather than generic DBI strings, which can have arbitrary tensions), we can show that the strings must satisfy the additional conservation condition [15]

$$\sum_i q_i = 0 \ , \tag{32}$$

which prevents any of the D-strings contained in the three $(p,q)$ bound states from ending at the junction. [Equation (28) has much the same consequence for the F-strings.]

Finally, as mentioned in the previous section, the conformal gauge condition admits a residual re-parametrisation invariance, which can be fixed by providing a supplementary gauge condition. Once again, one can show that temporal gauge,

$$\tau = t, \quad x_i^\mu(t,\sigma) = (t, \mathbf{x}(t,\sigma)) \quad \text{and} \quad \bar{x}^\mu(t) = (t, \bar{\mathbf{x}}(t)) \ , \tag{33}$$

is both consistent with the equations of motion, and also completely fixes the world-sheet parametrisation.

We note that the above equations of motion, the boundary conditions and the conservation laws combine to give us a system that exhibits self-duality under the S-duality transformation

$$p \leftrightarrow q \quad \text{and} \quad g_s \to \frac{1}{g_s} \ . \tag{34}$$

This shall become important later, when we come to discuss the properties of this system in the weak coupling limit $g_s \ll 1$; any conclusion we make about the behaviour of the system in this regime also applies in the strong coupling limit $g_s \gg 1$, provided that we interchange the roles of the F- and D-strings.

We shall now try to classify the string solutions of this system in a geometric manner, by attempting to follow steps similar to those taken in Section 2.

In conformal and temporal gauge, we have the usual wave equation for each string world-sheet

$$\ddot{\mathbf{x}}_i - \mathbf{x}_i'' = 0 \ . \tag{35}$$

This admits the general solution

$$\mathbf{x}_i = \frac{1}{2}\left[\mathbf{a}_i\left(t-\sigma\right) + \mathbf{b}_i\left(t+\sigma\right)\right] \ , \tag{36}$$

for arbitrary single-variable functions $\mathbf{a}_i$ and $\mathbf{b}_i$, subject to the conformal gauge constraints

$$|\mathbf{a}_i'|^2 = |\mathbf{b}_i'|^2 = 1 \ . \tag{37}$$

As with the previous single-string system, we must investigate how the boundary conditions on the three strings affect our *hitherto* free choice in selecting the two curves on the unit sphere that $\mathbf{a}_i'$ and $\mathbf{b}_i'$ represent.

The D$n$-brane boundary conditions at the $\sigma = 0$ end of each string are the most straightforward; they provide the following conditions:

$$\mathbf{a}_{i\parallel}'(t) = \mathbf{b}_{i\parallel}'(t) \ , \tag{38}$$

$$\mathbf{a}_{i\perp}'(t) = -\mathbf{b}_{i\perp}'(t) \ , \tag{39}$$

which are analogous to Eqs. (12) and (13).

We next consider the implications of the boundary conditions at the junction, given by Eqs. (27) and (30), which in temporal gauge are more conveniently expressed as:

$$\sum_i \bar{\mu}_i \dot{L}_i = 0 \ , \tag{40}$$

$$\sum_i \bar{\mu}_i \left(\mathbf{x}_i' + \dot{L}_i \dot{\mathbf{x}}_i\right) = 0 \ , \tag{41}$$

$$\mathbf{x}_i^\mu\left(t, L_i(t)\right) = \mathbf{x}_j^\mu\left(t, L_j(t)\right) \quad , \quad \forall\ i,j \ . \tag{42}$$

By substituting the general solution, Eq. (36), into Eq. (41) and the $t$-derivative of Eq. (42), we obtain three independent conditions, given by

$$\begin{aligned}[\bar{\mu}_1+\bar{\mu}_2+\bar{\mu}_3]\left[1+\dot{L}_1\right]\mathbf{b}_1'\left(t+L_1(t)\right)= \\ [\bar{\mu}_1-\bar{\mu}_2-\bar{\mu}_3]\left[1-\dot{L}_1\right]\mathbf{a}_1'\left(t-L_1(t)\right) \\ +2\bar{\mu}_2\left[1-\dot{L}_2\right]\mathbf{a}_2'\left(t-L_2(t)\right)+2\bar{\mu}_3\left[1-\dot{L}_3\right]\mathbf{a}_3'\left(t-L_3(t)\right)\ ,\end{aligned} \tag{43}$$

and its two counterparts under cyclic permutation of the string label indices.

Following the same methodology as in Section 2, we look for necessary conditions that need to be satisfied by curves that are subject to the above constraints, as well as to those we obtain by squaring the above equations and imposing the gauge constraints, Eq. (37). By doing so, we arrive at a system of three simultaneous equations which are polynomial in the three $\dot{L}_i$. We can "solve" this system of simultaneous equations, in the sense that we can re-arrange them to give a set of expressions for $\dot{L}_i$. However, since the coefficients in these simultaneous polynomial equations involve scalar products between the various $\mathbf{a}_i'(t-L_i(t))$, which themselves depend on $\dot{L}_i$, these expressions for $\dot{L}_i$ will in general be ordinary differential equations in $L_i(t)$. Therefore, by inverting this system of equations, we find seven independent solutions; six of those are

$$\dot{L}_1=\pm 1\ ,\quad \dot{L}_2=\pm 1\ ,\quad \dot{L}_3=\mp\frac{\bar{\mu}_1+\bar{\mu}_2}{\bar{\mu}_3}\ , \tag{44}$$

and cyclic permutations. Notice that these solutions happen to be independent of $L_i$, and therefore can be integrated directly. The seventh solution however does depend on $L_i$, via the scalar products

$$c_{ij}=\mathbf{a}_i'(t-L_i(t))\cdot\mathbf{a}_j'(t-L_j(t))\ , \tag{45}$$

and is given by

$$\frac{\bar{\mu}_1}{\bar{\mu}_1+\bar{\mu}_2+\bar{\mu}_3}\left(1-\dot{L}_i\right)=\frac{M_1\left(1-c_{23}\right)}{M_1\left(1-c_{23}\right)+M_2\left(1-c_{13}\right)+M_3\left(1-c_{12}\right)}\ , \tag{46}$$

with cyclic permutations giving expressions for $\dot{L}_2$ and $\dot{L}_3$, where

$$M_1=\bar{\mu}_1^2-\left(\bar{\mu}_2-\bar{\mu}_3\right)^2\ , \tag{47}$$

with cyclic permutations giving $M_2$ and $M_3$, respectively.

Let us briefly examine the first six solutions. By substituting these solutions back into Eq. (43), we find that three of them require that

$$\mathbf{a}_1'(t-L_1)=\mathbf{a}_2'(t-L_2)=\mathbf{a}_3'(t-L_3)\ , \tag{48}$$

whilst the other three require that

$$\mathbf{b}_1'(t+L_1)=\mathbf{b}_2'(t+L_2)=\mathbf{b}_3'(t+L_3)\ . \tag{49}$$

The seventh solution, Eq. (46), gives us greater freedom in choosing $\mathbf{a}_i'(t-L_i)$ and $\mathbf{b}_i'(t+L_i)$ – that is, for *any* given set of $\mathbf{a}_i'(t-L_i)$ (as long as they are not all equal), we can calculate $\dot{L}_i$ using Eq. (46) and $\mathbf{b}_i'(t+L_i)$ using Eq. (43).

Furthermore, consider calculating $\dot{L}_i$ using Eq. (46) for initially unequal $\mathbf{a}'_i(t-L_i)$, and subsequently taking the limit where $\mathbf{a}'_i(t-L_i)$ become equal. The values of $\dot{L}_i$ in this limit are, in general, different from the values of $\dot{L}_i$ suggested by Eq. (44) for exactly equal $\mathbf{a}'_i(t-L_i)$. In other words, there is no way of continuously deforming a string solution that is described by Eq. (46) into one of the special solutions described by Eq. (44)‖.

Combining the two results above, we conclude that the special string solutions characterised by Eq. (44) form a disconnected and effectively lower dimensional part of the space of solutions. Thus, when we discuss *generic* properties of this system, we may safely neglect these special solutions as, although they are perfectly valid as dynamical solutions, they take up an effectively zero-volume portion of the total space of solutions.

Finally, notice that the conditions, Eq. (38), (39), (40) and Eq.(43) and its two counterparts under cyclic permutation, are not quite enough to fully enforce the Dirichlet and junction boundary conditions. Equation (39) forces the non-junction ends of the strings to move tangentially to the D$n$-branes, but in order to ensure that these ends actually meet the branes, we need to impose the extra conditions

$$\mathbf{x}_{i\perp}(0,0) = \mathbf{D}_i \ , \tag{50}$$

where $\mathbf{D}_i$ is the position vector of the point where the D$n$-brane, attached to the $i^{\text{th}}$-string, intersects the $z$-axis. Similarly, Eq. (43) and its cyclic permutations force the junction ends of the string to move in tandem with one another, but in order to ensure that these ends actually meet at the junction, we need to impose the extra conditions

$$\mathbf{x}_i(0, L_i(0)) = \mathbf{x}_j(0, L_j(0)) \qquad , \quad \forall\ i,j \ . \tag{51}$$

It is straightforward to check that the effect of enforcing these conditions (apart from fixing the constants $\mathbf{a}_{i\perp}(0)$ and $\mathbf{b}_{i\perp}(0)$, which are inconsequential for our purposes) is to introduce the following averaging constraints on the curves $\mathbf{a}_i$:

$$\int_{-L_i(0)}^{L_i(0)} \mathbf{a}'_{i\perp}(\xi)\mathrm{d}\xi - \int_{-L_j(0)}^{L_j(0)} \mathbf{a}'_{j\perp}(\xi)\mathrm{d}\xi = \mathbf{\Delta}_{ij} \qquad , \quad \forall\ i,j \ , \tag{52}$$

where

$$\mathbf{\Delta}_{ij} = \mathbf{D}_i - \mathbf{D}_j \ , \tag{53}$$

is the normal separation vector from the $i^{\text{th}}$ to the $j^{\text{th}}$ D$n$-brane.

Finally, as discussed in Section 2, we fix the remaining space-time symmetry in this system by choosing the unique inertial frame in which the Neumann component total momentum of the system, $P_\parallel$, is zero. This leads to the averaging condition

$$\sum_i \int_{-L_i(0)}^{L_i(0)} \mathbf{a}'_{i\parallel}(\xi)\mathrm{d}\xi = 0 \ . \tag{54}$$

At the end of this long process, we have succeeded in classifying dynamical solutions of the DBI string system containing a junction in terms of the curves on the unit sphere

‖ We note that in Ref. [14] only Eq. (46), and its cyclic permutations, have been discussed as string solutions.

$\mathbf{a}_i'(t-\sigma)$ and $\mathbf{b}_i'(t+\sigma)$, the scalar functions $L_i(t)$ and the constants $\mathbf{a}_i(0)$ and $\mathbf{b}_i(0)$. In particular, we have found that any choice of these quantities, subject to the constraints given by Eqs. (38), (39), (43), (46), (52) and (54), corresponds to a unique string solution (and vice versa).

Although most of these constraints have clear geometrical interpretations (as it was the case in Section 2), the constraints represented by Eqs. (43) and (46) are not at all easy to visualise for general string tensions $\bar{\mu}_i$. Moreover, as cusps on these strings are associated with intersections of the curves $\mathbf{a}_i'$ and $\mathbf{b}_i'$, we need a full geometric picture of these constraints in order to consider the likelihood of cusps in this system. Fortunately however, we shall find that progress can be made in the case of $(p,q)$-string junctions for certain limits of the coupling constants.

*Example: An F-, D-, FD-string junction*

Suppose that the string labelled by "1" is a (1,0)-string (F-string); the string labelled by "2" is a (0,1)-string (D-string) and the string labelled by "3" is a (1,1)-string (FD-string). Using Eq. (29) we can expand the effective tensions of these strings as a series in the perturbative string coupling, $g_{\rm s}$:

$$\bar{\mu}_1 = 1 \ , \quad \bar{\mu}_2 = \frac{1}{g_{\rm s}} \ , \quad \bar{\mu}_3 = \sqrt{1+\frac{1}{g_{\rm s}^2}} = \frac{1}{g_{\rm s}} + \frac{g_{\rm s}}{2} + \mathcal{O}\left(g_{\rm s}^3\right) \ . \tag{55}$$

Therefore, for $g_{\rm s} \ll 1$, we can write down a perturbative expansion for $\mathbf{a}_i'$:

$$\mathbf{a}_i' = \mathbf{a}_i'^{(0)}(\xi) + g_s \mathbf{a}_i'^{(1)}(\xi) + g_s^2 \mathbf{a}_i'^{(2)}(\xi) + \cdots \tag{56}$$

and similarly for $\mathbf{b}_i'$ and $\dot{L}_i$, and substitute these expansions into the boundary conditions given in Eqs. (43) and (46), again fixing the boundary conditions at the $\sigma_i = 0$ ends of the three strings by attaching them to parallel D branes. By doing so, and matching coefficients order-by-order, we find the following boundary conditions for the leading order terms:

$$\left(S_{23}^{(0)} - 2S_{13}^{(0)} - 2S_{12}^{(0)}\right)\mathbf{b}_1'^{\,(0)} = S_{23}^{(0)}\mathbf{a}_1'^{\,(0)} - 2S_{13}^{(0)}\mathbf{a}_2'^{\,(0)} - 2S_{12}^{(0)}\mathbf{a}_3'^{\,(0)} \ , \tag{57}$$

$$\mathbf{b}_2'^{\,(0)} = \mathbf{a}_3'^{\,(0)} \ , \tag{58}$$

$$\mathbf{b}_3'^{\,(0)} = \mathbf{a}_2'^{\,(0)} \ , \tag{59}$$

and

$$\dot{L}_1^{(0)} = 1 - \frac{S_{23}^{(0)}}{S_{12}^{(0)} + S_{13}^{(0)}} \ , \quad \dot{L}_2^{(0)} = \frac{S_{12}^{(0)} - S_{13}^{(0)}}{S_{12}^{(0)} + S_{13}^{(0)}} \ , \quad \dot{L}_3^{(0)} = \frac{S_{13}^{(0)} - S_{12}^{(0)}}{S_{12}^{(0)} + S_{13}^{(0)}} \ , \tag{60}$$

where $S_{ij} = \frac{1}{2}(1-c_{ij})$ and, for the sake of clarity, we have used the short-hand

$$\mathbf{a}_i' = \mathbf{a}_i'(t - L_i(t)) \quad \text{and} \quad \mathbf{b}_i' = \mathbf{b}_i'(t + L_i(t)) \ . \tag{61}$$

Being interested only to the leading-order behaviour of the strings, we shall henceforth drop the superscripts $^{(0)}$ in the discussion on the perturbative dynamics of the system.

The above equations have a highly intuitive physical interpretation, which is best seen by rewriting Eqs. (58) and (59) in the form

$$\dot{\mathbf{x}}_2(t, L_2(t)) = \dot{\mathbf{x}}_3(t, L_3(t)) \quad \text{and} \quad \mathbf{x}_2'(t, L_2(t)) = -\mathbf{x}_3'(t, L_3(t)) \ . \tag{62}$$

In other words, as $g_s \to 0$, the D-string and FD-string effectively become one *continuous* string of constant (overall) length, $L_2(t) + L_3(t)$, obeying the usual equation of motion, which is unaffected by the dynamics of the much lighter F-string.

The dynamics of the F-string are then determined by the boundary conditions, Eqs. (57) and (60), imposed by the combined string (composed by the D- and the FD-string) on the $\sigma = L_1(t)$ end of the F-string. Therefore, as far as the F-string is concerned, the combined string is effectively a D1-brane (with its own prescribed motion) on which the F-string produces no back-reaction. These boundary conditions may be better understood when expressed in terms of the orientation, $\mathbf{x}_2'(t, L_2(t))$, and velocity, $\dot{\mathbf{x}}_2(t, L_2(t))$, of the effective D1-brane, as:

$$\left[|\mathbf{x}_2'|^2 - 2(1 - \mathbf{a}_1' \cdot \dot{\mathbf{x}}_2)\right] \mathbf{b}_1' = -|\mathbf{x}_2'|^2 \mathcal{R}\mathbf{a}_1' - 2(1 - \mathbf{a}_1' \cdot \dot{\mathbf{x}}_2)\dot{\mathbf{x}}_2 \ , \tag{63}$$

and

$$\dot{L}_1 = \frac{|\dot{\mathbf{x}}_2|^2 - \mathbf{a}_1' \cdot \dot{\mathbf{x}}_2}{1 - \mathbf{a}_1' \cdot \dot{\mathbf{x}}_2} \ , \tag{64}$$

where

$$\mathcal{R}\mathbf{a}_1' = -\mathbf{a}_1' + \frac{2(\mathbf{a}_1' \cdot \mathbf{x}_2')\mathbf{x}_2'}{|\mathbf{x}_2'|^2} \ , \tag{65}$$

is a linear transformation that inverts $\mathbf{a}_1'$ through the line $\{\lambda \mathbf{x}'; \lambda \in \mathbb{R}\}$, which is parallel to the combined string and passes through the origin of the unit sphere.

After a little thought, it is apparent that Eq. (63) causes $\mathbf{b}_1'$ to lie on the semi-circle that is obtained by projecting the line $\{\mathcal{R}\mathbf{a}_1' + \lambda \dot{\mathbf{x}}_2; \lambda \in \mathbb{R}\}$ onto the unit sphere ($\dot{\mathbf{x}}_2$ is perpendicular to $\mathbf{x}_2'$ by the conformal gauge conditions). However, the exact position of $\mathbf{b}_1'$ on this semi-circle depends on both the magnitude, $|\dot{\mathbf{x}}_2|$, and the angle that $\dot{\mathbf{x}}_2$ makes with $\mathbf{a}_1'$, in a rather complicated manner. Nevertheless, it is possible to make further progress in two interesting limits: firstly, when the combined string is moving slowly at the junction ($|\dot{\mathbf{x}}_2| \ll 1$); and secondly, when the combined string is moving highly relativistically at the junction, ($|\mathbf{x}_2'| \ll 1$). In both limits, we follow a perturbative procedure as before by expanding Eqs. (63) and (64) in the relevant small parameter and extracting the leading order behaviour.

When the combined string is moving slowly ($|\dot{\mathbf{x}}_2| \ll 1$), we obtain the following leading order boundary conditions:

$$\mathbf{b}_1' = \mathcal{R}\mathbf{a}_1' \quad \text{and} \quad \dot{L}_1 = 0 \ , \tag{66}$$

which are exactly the Dirichlet boundary conditions that we derived in Section 2 for a single DBI string attached to a non-dynamical D1-brane at its $\sigma = L$ end.

As we saw in Section 2, a string ending on D1-branes will have cusps a significant fraction of the time. Here, we have found that precisely the boundary conditions necessary for these cusps to form, occur for an F-string at a three string junction,

in the limit $|\dot{\mathbf{x}}_2| \ll 1$. We have taken the $\sigma = 0$ end of the F-string to have boundary conditions associated with a D-brane, which can now be replaced with a second three string junction. Thus, we see that cusps would be expected to form on F-strings ending on two three string junctions, in the limit that the D/FD-string is moving slowly. This can be extended by noting that for a general $\dot{\mathbf{x}}_2$ we have a deformation of a pure inversion. In particular, a point on the curve $\mathbf{a}_1'$ is inverted through $\mathbf{x}_2'$ and then *pulled* along the semi-circle made up of the inverted point and $\pm\widehat{\dot{\mathbf{x}}_2} \equiv \dot{\mathbf{x}}_2/|\dot{\mathbf{x}}_|$. In Section 2, we showed that, if we consider the plane containing the lines $\mathbf{x}_2'$ and $\dot{\mathbf{x}}_2$, then a curve $\mathbf{a}_1'$ that encircles both lines, i.e. pierces the plane in opposite quadrants, will intersect the curve $\mathbf{b}_1'$ provided that $\mathbf{b}_1'$ also pierces the plane in opposite quadrants. By considering the projection of Eq. (63) onto both $\mathbf{x}_2'$ and $\dot{\mathbf{x}}_2$, it is easily shown that

$$\mathrm{sign}\,(\mathbf{b}_1' \cdot \mathbf{x}_2') = \mathrm{sign}\,(\mathbf{b}_1' \cdot \dot{\mathbf{x}}_2) \ . \tag{67}$$

Thus, we see that indeed, $\mathbf{b}_1'$ does intersect the plane in opposite quadrants and hence will intersect $\mathbf{a}_1'$ in a significant proportion of cases.

It is also worth mentioning that the $|\dot{\mathbf{x}}_2| \ll 1$, or static brane limit, is the approximation taken in the quatisation of standard string theory. Since we are working with the DBI action, Eq. (24), which is derived as a low energy limit of string theory, we should for consistency restrict ourselves to this situation. The action given by Eq. (24) can be taken as a prototype for string junctions away from this limit, however in this case the motivation from string theory becomes less clear.

In addition, we wish to consider the phenomenology of, for example, brane inflation, in which it is expected that networks of F- and D- strings are produced, which can go on to form junctions. These two networks would be produced at the same energy scale and because the are independent we would generically expect the heavy D-strings to be moving slower than the light F-strings. Thus, the limit $|\dot{\mathbf{x}}_2| \ll 1$ would be satisfied early in the evolution of the networks. As the dynamics of the strings become coupled via the formation of junctions, it is no longer clear that this would be the case and simulations would be required to estimate when this approximation breaks down. However, as we have shown, cusps will remain significant away from this simplifying limit.

We find that a junction formed from an F-, D- and FD-string behaves as an F-string ending on a static (locally) straight D-string. We have restricted our attention to the case where the $\sigma_i = 0$ ends of the strings are attached to a D-brane, however if we take this D-brane to be a D-string, then our results show that it could be replaced with another F-string, D-string and FD-string junction. Thus, an F-string stretched between two three-string junctions behaves as an F-string between two D1-branes, to order $g_s$. In particular, Eq. (66) implies that the coordinate length of the F-string is constant, as required. As we showed in Section 2, we expect cusps to form on a significant fraction of such strings, in contrast to standard cosmic strings which can form cusps only on closed strings.

## 4. Phenomenology

At the end of brane inflation a network of D- and F- strings is expected to form. Although the intercommutation probability for two D-stings is less than one [29], in the weak string coupling limit ($g_s \to 0$) it is always possible for an F- and D- string to intercommute at a collision [15], forming a three string junction. As we have seen an F-string ending on two such three string junctions will generically have cusps, at least in the regime where the separation between the two D-strings is small, i.e. when the distance between the two three string junctions is small. We have seen that in this limit, the heavy D/FD-string behaves as a single infinitely long string that is unaffected by the dynamics of the F-string. At the level of approximation used here (of order $g_s$), there is no energetic difference between the D-string and the bound FD-string, however simulations [22] have shown that the FD-string will grow at the expense of the D string. Explicit calculations of the intercommutation probability of F-strings with multiple bound state strings has been done [29] and in the $g_s \to 0$ limit they are identical. Thus, an F-string can intercommute with an FD-string to form a multiple bound state, with, to order $g_s$, the same probability of intercommutation as with a D-string. In effect then, we have two types of strings evolving in such a network, heavy strings which are D or FD strings or multiple bound states, and light F strings, which move in the background of the heavy strings, but do not affect their dynamics.

For cusps to be significant to the dynamics of the F-string, we require that the typical separation of the heavy strings is small compared to the length of F-string stretched between them. This condition is met early in the evolution of the string network [4], where the typical inter-string distance is of the order of the symmetry breaking scale [6] and the string network is in the friction dominated regime. As the heavy strings move apart, the F-strings ending on them will stretch, increasing the inter-string distance and reducing the importance of cusps. This is in addition to the fact that as the heavy strings move beyond the horizon scale, any effects of cusps would be lost. Thus, phenomenological consequences of cusps from junctions on cosmic superstrings will be most significant at early times, close to the end of brane inflation. This is similar to the situation in cosmic strings when particle production from cusps on string loops is dominant in the friction dominated regime and was used as a baryogenesis mechanism in Ref. [30].

The radiative properties of the cusps from F-strings are similar to those of standard cosmic strings. In particular, the mechanism for the emission of gravitational and particle radiation from cusps on loops of cosmic string [23, 24, 25, 31] also applies to cusps on strings between junctions. As in the standard case the energy (and entropy) released by cusps goes into excitation of all available fields. In the cosmic string case these are just the standard model fields, however here they could include the dilaton [27], Ramond-Ramond [32, 16] and moduli fields and all other fields present in the low energy limit of string theory. In particular, it is possible that gravitinos and other stable supersymmetry particles may be produced, in addition to standard model particles.

Since this particle production would predominately occur at early times, cosmological events such as baryogenesis and big bang nucleosynthesis should be sensitive to the presence of such cusps. This is in addition to the fact that cusps on such a network would contribute to the gravitational wave background [23, 24, 25].

Under S-duality the rôle of F- and D-strings is reversed; however the results derived here apply with the heavy strings now being the F-strings and the bound FD-string states. In this case the cusps would be present on the light D-strings ending on three string junctions, which allows a connection to be made with the cosmic strings of supergravity that coincide with D-strings in this limit [33], enalbing their properties to be studied in the supergravity limit [34].

## 5. Conclusions

We have shown how to characterise classical solutions to low energy effective string actions in an intuitive geometric manner. We have shown how the boundary conditions for strings ending on D-branes can lead to cusps, points with luminous velocity, in such solutions, in an analogous manner to cusps of standard cosmic string loops. In particular, we have demonstrated that cusps would be a generic feature of an F-string ending on two (parallel and stationary) D-strings. We have then considered general three string junctions that are possible in DBI-actions and have shown that the boundary conditions of such a junction can similarly be characterised and understood in a geometric setting, for the case of an F- and D-string meeting an FD-string. We find that this system exactly reproduces the situation of an F-string ending on two D-strings and hence a pair of such junctions would generically include cusps. We have shown that this remains true even when the D-strings are moving.

The relevance of such a scenario is that networks of F- and D-strings are expected to form at the end of brane inflation. Collisions of such networks would lead to pairs of three string junctions, each of which would then be expected to have cusps, opening up a new energy loss mechanism for such networks. Our new feature is in addition to cusps on string loops. Importantly the formation and existence of cusps is expected to be most significant early in the evolution of the network. The fact that the radiation from such cusps should include all available fields present in the low energy string theory, makes it possible that signatures of their presence would appear in baryogenesis or big bang nucleosynthesis.

The extreme nature of cusps means that they are possible targets for observations of string networks and here we have shown that it is, in principle, possible to distinguish between standard cosmic strings formed during cosmological phase transitions and cosmic superstring, relics of brane inflation. The observation of three string junctions would provide strong evidence for string theory. In future it might be possible to observe the emission of radiation from cusps and the distribution of such cusps.

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