arXiv:0907.3990v2 [math.AC] 18 Jan 2010

A DEFORMATION OF COMMUTATIVE POLYNOMIAL ALGEBRAS IN EVEN NUMBERS OF VARIABLES

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Abstract. We introduce and study a deformation of commutative polynomial algebras in even numbers of variables. We also discuss some connections and applications of this deformation to the generalized Laguerre orthogonal polynomials and the interchanges of right and left total symbols of differential operators of polynomial algebras. Furthermore, a more conceptual re-formulation for the *image* conjecture [Z3] is also given in terms of the deformed algebras. Consequently, the well-known *Jacobian* conjecture [Ke] is reduced to an open problem on this deformation of polynomial algebras.

1. Introduction

Let $\xi = (\xi_1, \xi_2, ..., \xi_n)$ and $z = (z_1, z_2, ..., z_n)$ be $2n$ commutative free variables. Throughout this paper, we denote by $\mathbb{C}[\xi, z]$, $\mathbb{C}[z]$ and $\mathbb{C}[\xi]$ the vector spaces (without any algebra structures) over $\mathbb{C}$ of polynomials in (ξ, z), in z and in ξ, respectively. The corresponding polynomial algebras will be denoted respectively by $\mathcal{A}[\xi, z]$, $\mathcal{A}[z]$ and $\mathcal{A}[\xi]$.

For any $1 \leq i \leq n$, we set $\partial_i := \partial_{z_i}$ and $\delta_i = \partial_{\xi_i}$. Denote by $\partial = (\partial_1, \partial_2, ..., \partial_n)$ and $\delta = (\delta_1, \delta_2, ..., \delta_n)$. We also occasionally use ∂_z and ∂_ξ to denote ∂ and δ, respectively.

Set $\Omega := \sum_{i=1}^{n}(\partial_i \otimes \delta_i + \delta_i \otimes \partial_i)$. For any $t \in \mathbb{C}$, we define a new product $*_t$ for the vector space $\mathbb{C}[\xi, z]$ by setting, for any $f, g \in \mathbb{C}[\xi, z]$,

$$f *_t g := \mu\left(e^{-t\,\Omega}(f \otimes g)\right), \tag{1.1}$$

where $\mu : \mathbb{C}[\xi, z] \otimes \mathbb{C}[\xi, z] \to \mathbb{C}[\xi, z]$ denotes the product map of the polynomial algebra $\mathcal{A}[\xi, z]$.

Denote by $\mathcal{B}_t[\xi, z]$ $(t \in \mathbb{C})$ the new algebra $(\mathbb{C}[\xi, z], *_t)$. For the case $t = 1$, we also introduce the following short notation:

$$* := *_{t=1}. \tag{1.2}$$

$$\mathcal{B}[\xi, z] := \mathcal{B}_{t=1}[\xi, z]. \tag{1.3}$$

2000 *Mathematics Subject Classification.* 33C45, 32W99, 14R15.

Key words and phrases. The generalized Laguerre polynomials, total symbols of differential operators, the image conjecture, the Jacobian conjecture.

Note that, when $t=0$, the algebra $\mathcal{B}_{t=0}[\xi,z]$ coincides with the usual polynomial algebra $\mathcal{A}[\xi,z]$.

In this paper, we first show that $\mathcal{B}_t[\xi,z]$ $(t\in\mathbb{C})$ gives a deformation of the polynomial algebra $\mathcal{A}[\xi,z]$. Actually, it is a trivial deformation in the sense of deformation theory. To be more precise, set

$$\Lambda := \sum_{i=1}^{n}\delta_i\partial_i. \tag{1.4}$$

$$\Phi_t := e^{t\Lambda} = \sum_{m\geq 0}\frac{t^m\Lambda^m}{m!}. \tag{1.5}$$

$$\Phi := \Phi_{t=1}. \tag{1.6}$$

Note that, Φ_t for any $t\in\mathbb{C}$ is a well-defined bijective linear map from $\mathbb{C}[\xi,z]$ to $\mathbb{C}[\xi,z]$, whose inverse map is given by $\Phi_{-t}=e^{-t\Lambda}$. This is because the differential operator Λ of $\mathbb{C}[\xi,z]$ is locally nilpotent, i.e. for any $f(\xi,z)\in\mathbb{C}[\xi,z]$, $\Lambda^m f(\xi,z)=0$ when $m\gg 0$.

With the notation fixed above, we will show that, for any $t\in\mathbb{C}$, $\Phi_t:\mathcal{B}_t[\xi,z]\to\mathcal{A}[\xi,z]$ actually is an isomorphism of $\mathbb{C}$-algebras (See Proposition 2.1 and Corollary 2.2).

Note that, from the point view of deformation theory, the deformation $\mathcal{B}_t[\xi,z]$ $(t\in\mathbb{C})$ is not interesting at all. But, surprisingly, as we will show in this paper, the algebra $\mathcal{B}_t[\xi,z]$ and the isomorphism Φ_t are actually closely related with the generalized Laguerre polynomials (See [Sz], [PS] and [AAR]) and the interchanges of right and left total symbols of differential operators of polynomial algebras.

Furthermore, as we will show in Section 4, the algebras $\mathcal{B}_t[\xi,z]$ $(t\in\mathbb{C})$ and the isomorphism Φ_t via their connections with the *image* conjecture proposed in [Z3] are also related with the *Jacobian* conjecture which was first proposed by O. H. Keller [Ke] in 1939 (See also [BCW] and [E]). Actually, the *Jacobian* conjecture can be viewed as a conjecture which, in some sense, just claims that the algebra $\mathcal{B}_t[\xi,z]$ $(t\neq 0)$ should not differ or change too much from the polynomial algebra $\mathcal{A}[\xi,z]=\mathcal{B}_{t=0}[\xi,z]$. Therefore, from this point of view, the triviality of the deformation $\mathcal{B}_t[\xi,z]$ $(t\in\mathbb{C})$ (in the sense of deformation theory) can be viewed as a fact in favor of the *Jacobian* conjecture. For another interesting application of the isomorphism Φ to the *Jacobian* conjecture, see [Z5].

Considering the length of the paper, below we give a more detailed description for the contents and the arrangement of the paper.

In Subsection 2.1, we prove some simple properties of the deformation $\mathcal{B}_t[\xi,z]$ $(t\in\mathbb{C})$ and the isomorphism $\Phi_t:\mathcal{B}_t[\xi,z]\to\mathcal{A}[\xi,z]$, which

will be needed for the rest of this paper. In particular, in this subsection the triviality of the deformation $\mathcal{B}_t[\xi, z]$ ($t \in \mathbb{C}$) in the sense of deformation theory is proved in Proposition 2.1. and Corollary 2.2.

In Subsection 2.2, we show that, for different $t \in \mathbb{C}$, the ℓ-adic topologies induced by $\mathcal{B}_t[\xi, z]$ on the common base vector space $\mathbb{C}[\xi, z]$ are different. But they are all homeomorphic to the ℓ-adic topology induced by the polynomial algebra $\mathcal{A}[\xi, z]$ under the isomorphism $\Phi_t : \mathcal{B}_t[\xi, z] \to \mathcal{A}[\xi, z]$ (viewed as an automorphism of $\mathbb{C}[\xi, z]$). See Proposition 2.9 and also Corollary 2.10 for the precise statements.

In Subsection 2.3, we study the induced isomorphism $(\Phi_t)_*$ ($t \in \mathbb{C}$) of Φ_t from the differential operator algebra, or the Weyl algebra $\mathcal{D}_t[\xi, z]$ of $\mathcal{B}_t[\xi, z]$ to the Weyl algebra $\mathcal{D}[\xi, z]$ of $\mathcal{A}[\xi, z]$. The main results of this subsection are Propositions 2.11 and 2.13. Proposition 2.11 says that the derivations ∂_{z_i} and ∂_{ξ_i} ($1 \leq i \leq n$) of $\mathcal{A}[\xi, z]$ are also derivations of $\mathcal{B}_t[\xi, z]$ for all $t \in \mathbb{C}$ and are fixed by the isomorphism $(\Phi_t)_*$. Proposition 2.13 gives explicitly the images under $(\Phi_t)_*$ of the multiplication operators with respect to the product $*_t$ of $\mathcal{B}_t[\xi, z]$.

In Section 3, by using some results derived in Section 2, we show in Theorem 3.1 that $\Phi = \Phi_{t=1}$ (resp. $\Phi_{t=-1}$) as an automorphism of $\mathbb{C}[\xi, z]$ actually coincides with the linear map which changes left (resp. right) total symbols of differential operators of $\mathcal{A}[z]$ to their right (resp. left) total symbols. Consequently, the products $*_{t=\pm 1}$ appear naturally when one derives left or right total symbols of certain differential operators of $\mathcal{A}[z]$ (See Corollary 3.2). The results derived in this subsection also play some important roles in [Z5] in which among some other results a more straightforward proof for the equivalence of the *Jacobian* conjecture and the *vanishing* conjecture (See [Z1] and [Z2]) will be given.

In Subsection 4.1, we study the Taylor series expansion of polynomials in $\mathbb{C}[\xi, z]$ with respect to the new product $*_t$ and use it to give a more conceptual proof for the expansion of polynomials given in Eq. (4.6). This expansion was first proved in [Z3] and played a crucial role there in the proof of the implication of the *Jacobian* conjecture from the *image* conjecture (See Conjecture 4.3).

In Subsection 4.2, we first recall the notion of the so-called *Mathieu* subspaces of commutative algebras (See Definition 4.2), which was first introduced in [Z4], and also the *image* conjecture (See Conjecture 4.3) for the differential operators $\xi_i - t\partial_i$ ($1 \leq i \leq n$) in terms of the notion of Mathieu subspaces. We then give a re-formulation of Conjecture 4.3 in terms of the algebra $\mathcal{B}_t[\xi, z]$ ($t \in \mathbb{C}$) (See Conjecture 4.5) and show in Theorem 4.6 that these two conjectures are equivalent to each other. Since it has been shown in [Z3] that Conjecture 4.3 implies *Jacobian* conjecture, hence so does Conjecture 4.5.

Consequently, via its connections with Conjecture 4.5, the *Jacobian* conjecture is reduced to an open problem on the deformation $\mathcal{B}_t[\xi, z]$ ($t \in \mathbb{C}$) of the polynomial algebra $\mathcal{A}[\xi, z]$. The open problem asks if the ideal $\xi\mathbb{C}[\xi, z]$ of $\mathcal{A}[\xi, z]$ generated by ξ will remain to be a Mathieu subspace in the algebra $\mathcal{B}_t[\xi, z]$ for any $t \neq 0$. Note that any ideal is automatically a Mathieu subspace, but not conversely. Therefore, the triviality (in the sense of deformation theory) of the deformation $\mathcal{B}_t[\xi, z]$ ($t \in \mathbb{C}$) proved in Proposition 2.1 can be viewed as a fact in favor of the *Jacobian* conjecture.

Section 5 is mainly on a connection of the algebra $\mathcal{B}[\xi, z]$, especially, its product $*$ with the multi-variable generalized Laguerre polynomials, and also some of the applications of this connection to both $\mathcal{B}[\xi, z]$ and the generalized Laguerre polynomials.

In Subsection 5.1, we very briefly recall the definition of the (generalized) Laguerre polynomials $L_\alpha^{[\mathbf{k}]}(z)$ ($\mathbf{k}, \alpha \in \mathbb{N}^n$) (See Eqs. (5.1)–(5.3)) and also the orthogonal property (See Theorem 5.1) of these polynomials.

In Subsection 5.2, we show in Theorem 5.2 that, for any $\mathbf{k}, \alpha \in \mathbb{N}^n$, we have

$$L_\alpha^{[\mathbf{k}]}(\xi z) = \frac{(-1)^{|\alpha|}}{\alpha!}\, \xi^{-\mathbf{k}}(\xi^{\alpha+\mathbf{k}} * z^\alpha) = \frac{(-1)^{|\alpha|}}{\alpha!}\, z^{-\mathbf{k}}(\xi^\alpha * z^{\alpha+\mathbf{k}}). \tag{1.7}$$

Consequently, the generalized Laguerre polynomials $L_\alpha^{[\mathbf{k}]}(z)$ ($\mathbf{k}, \alpha \in \mathbb{N}^n$) can be obtained by evaluating the polynomials $\xi^{-\mathbf{k}}(\xi^{\alpha+\mathbf{k}} * z^\alpha)$ or $z^{-\mathbf{k}}(\xi^\alpha * z^{\alpha+\mathbf{k}})$ at $\xi = (1, 1, ..., 1)$. Note that the evaluation map at $\xi = (1, 1, ..., 1)$ is not an algebra homomorphism from $\mathcal{B}[\xi, z]$ to $\mathcal{A}[\xi, z]$. Otherwise, the generalized Laguerre polynomials would be trivialized.

In the first part of Subsection 5.3, we use certain results of the generalized Laguerre polynomials and the connection in Eq. (1.7) above to derive more properties on the polynomials $\xi^\alpha * z^\alpha$ which, by Proposition 2.6, (c), are actually the monomials of ξ and z in the new algebra $\mathcal{B}[\xi, z]$.

For example, by using the connection in Eq. (1.7) and I. Schur's irreducibility theorem [Sc1] of the Laguerre polynomials in one variable, we immediately have that, when $n = 1$, the monomials $\xi^m * z^m$ ($m \geq 2$) of $\mathcal{B}[\xi, z]$ are actually irreducible over $\mathbb{Q}$ (See Theorem 5.9). Furthermore, by using I. Schur's irreducibility theorem [Sc2] and M. Filaseta and T.-Y. Lam's irreducibility theorem [FL] on the generalized Laguerre polynomials, we have that, all but finitely many of the polynomials $\xi^{-k}(\xi^{m+k} * z^m)$ and $z^{-k}(\xi^m * z^{m+k})$ ($m, k \in \mathbb{N}$) are irreducible over $\mathbb{Q}$ (See Theorem 5.10).

In the second part of Subsection 5.3, we use the connection given in Eq. (1.7) and certain results of $\mathcal{B}[\xi, z]$ derived in Section 2 to give new proofs, first, for some recurrent formulas of the generalized Laguerre polynomials (See Proposition 5.11) and, second, for the fact that the generalized Laguerre polynomials satisfy the so-called *associate Laguerre differential equation* (See Theorem 5.12). At the end of this subsection, we draw the reader's attention to a conjecture, Conjecture 5.13, on the generalized Laguerre polynomials, which is still open even for the classical Laguerre polynomials in one variable.

Acknowledgment The author would like to thank the anonymous referees for pointing out many typos, minor errors of the previous version of this paper, and also for suggesting the new proof of Lemma 5.8 without the condition that the base filed K has infinitely many elements.

2. The Deformation $\mathcal{B}_t[\xi, z]$ of the Polynomial Algebra $\mathcal{A}[\xi, z]$

In this section, we first derive in Subsection 2.1 some properties and identities for the algebra $\mathcal{B}_t[\xi, z]$ ($t \in \mathbb{C}$). In Subsection 2.2, we show that, for different $t \in \mathbb{C}$, the ℓ-adic topologies induced by the algebras $\mathcal{B}_t[\xi, z]$ ($t \in \mathbb{C}$) on the common base vector space $\mathbb{C}[\xi, z]$ are different. But they are all homeomorphic under the isomorphism $\Phi_t : \mathcal{B}_t[\xi, z] \to \mathcal{A}[\xi, z]$ to the ℓ-adic topology on $\mathbb{C}[\xi, z]$ induced by the polynomial algebra $\mathcal{A}[\xi, z]$ (See Proposition 2.9 and also Corollary 2.10).

In Subsection 2.3, we study the isomorphism $(\Phi_t)_*$ induced by Φ_t from the Weyl algebra of $\mathcal{B}_t[\xi, z]$ to the Weyl algebra of $\mathcal{A}[\xi, z]$. The main results in this subsection are Propositions 2.11 and 2.13.

2.1. Some Properties of the Algebras $\mathcal{B}_t[\xi, z]$.

First, one remark on notation and convention is that, we will freely use throughout this paper some commonly used multi-index notations and conventions. For instance, for n-tuples $\alpha = (k_1, k_2, ..., k_n)$ and $\beta = (m_1, m_2, ..., m_n)$ of non-negative integers, we have

$$|\alpha| = \sum_{i=1}^{n} k_i.$$

$$\alpha! = k_1! k_2! \cdots k_n!.$$

$$\binom{\alpha}{\beta} = \begin{cases} \frac{\alpha!}{\beta!(\alpha-\beta)!} & \text{if } k_i \geq m_i \text{ for all } 1 \leq i \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

The notation and convention fixed in the previous section will also be used throughout this paper.

The first main result of this section is the following proposition.

Proposition 2.1. *For any $t \in \mathbb{C}$ and $f, g \in \mathbb{C}[\xi, z]$, we have*

$$\Phi_t(f *_t g) = \Phi_t(f)\Phi_t(g). \tag{2.1}$$

Proof: We first set

$$f *'_t g := \Phi_t^{-1}\big(\Phi_t(f)\Phi_t(g)\big) = \Phi_{-t}\big(\Phi_t(f)\Phi_t(g)\big). \tag{2.2}$$

for any $f, g \in \mathbb{C}[\xi, z]$.

We view t as a formal parameter which commutes with ξ and z. Then, by Eqs. (1.1), (2.2) and the fact that the differential operators Λ and Ω are locally nilpotent on $\mathbb{C}[\xi, z]$ and $\mathbb{C}[\xi, z] \otimes \mathbb{C}[\xi, z]$, respectively, we see that $f *_t g$ and $f *'_t g$ are polynomials in t with coefficients in $\mathbb{C}[\xi, z]$. Furthermore, by setting $t = 0$ in Eqs. (1.1) and (2.2), we see that the constant terms (with respect to t) of $f *_t g$ and $f *'_t g$ are both $fg \in \mathbb{C}[\xi, z]$. In other words, we have

$$f *_t g \mid_{t=0} = f *'_t g \mid_{t=0} = fg. \tag{2.3}$$

From Eq. (1.1), we have,

$$\begin{aligned} \frac{\partial}{\partial t}(f *_t g) &= -\mu\left(e^{-t\Omega}(\Omega(f \otimes g))\right) \\ &= -\sum_{i=1}^{n} \mu\left(e^{-t\Omega}\left((\delta_i f) \otimes (\partial_i g) + (\partial_i f) \otimes (\delta_i g)\right)\right) \\ &= -\sum_{i=1}^{n} \left((\delta_i f) *_t (\partial_i g) + (\partial_i f) *_t (\delta_i g)\right). \end{aligned} \tag{2.4}$$

On the other hand, from Eq. (2.2), we have,

$$\begin{aligned} \frac{\partial}{\partial t}(f *'_t g) &= \frac{\partial}{\partial t}\left(e^{-t\Lambda}((e^{t\Lambda} f)\,(e^{t\Lambda} g))\right) \\ &= e^{-t\Lambda}\left(-\Lambda((e^{t\Lambda} f)\,(e^{t\Lambda} g)) + (e^{t\Lambda}\Lambda f)\,(e^{t\Lambda} g) + (e^{t\Lambda} f)\,(e^{t\Lambda}\Lambda g)\right). \end{aligned}$$

Note that, for any $u, v \in \mathbb{C}[\xi, z]$, it is easy to check that we have the following identity:

$$\Lambda(uv) = (\Lambda u)v + u(\Lambda v) + \sum_{i=1}^{n} \left((\delta_i u)(\partial_i v) + (\partial_i u)(\delta_i v) \right).$$

By the last two equations above and also Eq. (2.2), we have

$$\frac{\partial}{\partial t}(f *'_t g) = -\sum_{i=1}^{n} e^{-t\Lambda}\left(((e^{t\Lambda}\delta_i f)\,(e^{t\Lambda}\partial_i g)) + (e^{t\Lambda}\partial_i f)\,(e^{t\Lambda}\delta_i g) \right) \tag{2.5}$$

$$= -\sum_{i=1}^{n} \left((\delta_i f) *'_t (\partial_i g) + (\partial_i f) *'_t (\delta_i g) \right).$$

Next, we use the induction on $(\deg f + \deg g)$ to show Eq. (2.1). First, when $\deg f + \deg g = 0$, i.e. both f and g have degree zero, it is easy to see from Eqs. (1.1) and (2.2) that $f *_t g = f *'_t g = fg$ in this case.

In general, by Eqs. (2.4), (2.5) and also the induction assumption, we have

$$\frac{\partial}{\partial t}(f *_t g) = \frac{\partial}{\partial t}(f *'_t g). \tag{2.6}$$

Since $f *_t g$ and $f *'_t g$ are polynomials in t with coefficients in $\mathbb{C}[\xi, z]$ and both satisfy Eqs. (2.3) and (2.6), it is easy to see that they must be equal to each other. Hence, Eq. (2.1) holds. □

Corollary 2.2. *For any $t \in \mathbb{C}$, $\Phi_t : \mathcal{B}_t[\xi, z] \to \mathcal{A}[\xi, z]$ is an isomorphism of algebras. Therefore, in the sense of deformation theory, the deformation $\mathcal{B}_t[\xi, z]$ is a trivial deformation of the commutative polynomial algebra $\mathcal{A}[\xi, z]$.*

Next we derive some properties of the algebras $\mathcal{B}_t[\xi, z]$ $(t \in \mathbb{C})$, which will be needed for the rest of this paper.

Lemma 2.3. *For any $f, g \in \mathbb{C}[\xi, z]$, we have*

$$f *_t g = \sum_{\alpha,\beta \in \mathbb{N}^n} \frac{(-t)^{|\alpha|+|\beta|}}{\alpha!\beta!} (\delta^\beta \partial^\alpha f)(\partial^\beta \delta^\alpha g). \tag{2.7}$$

Proof: Note first that, for any $1 \le i, j \le n$, $\partial_i \otimes \delta_i$ and $\delta_j \otimes \partial_j$ commute with each other. So we have

$$e^{-t\,\Omega} = e^{-t\sum_{i=1}^n \delta_i \otimes \partial_i}\, e^{-t\sum_{i=1}^n \partial_i \otimes \delta_i}, \tag{2.8}$$

$$\begin{aligned} e^{-t\sum_{i=1}^n \partial_i \otimes \delta_i} &= \prod_{i=1}^n e^{-t(\partial_i \otimes \delta_i)} = \prod_{i=1}^n \sum_{k_i \ge 0} \frac{(-t)^{k_i}}{k_i!} (\partial_i^{k_i} \otimes \delta_i^{k_i}) \\ &= \sum_{\alpha \in \mathbb{N}^n} \frac{(-t)^{|\alpha|}}{\alpha!} (\partial^\alpha \otimes \delta^\alpha). \end{aligned} \tag{2.9}$$

Similarly,

$$e^{-t\sum_{i=1}^n \delta_i \otimes \partial_i} = \sum_{\beta \in \mathbb{N}^n} \frac{(-t)^{|\beta|}}{\beta!} (\delta^\beta \otimes \partial^\beta). \tag{2.10}$$

Then it is easy to see that Eq. (2.7) follows directly from Eq. (1.1) and the last three equations above. □

Proposition 2.4. *(a) For any $\lambda_i(\xi) \in \mathbb{C}[\xi]$, $p_i(z) \in \mathbb{C}[z]$ $(i = 1, 2)$, we have*

$$\lambda_1(\xi) *_t \lambda_2(\xi) = \lambda_1(\xi)\lambda_2(\xi). \tag{2.11}$$

$$p_1(z) *_t p_2(z) = p_1(z)p_2(z). \tag{2.12}$$

$$\lambda(\xi) *_t p(z) = \sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|}t^{|\alpha|}}{\alpha!}(\delta^\alpha\lambda(\xi))(\partial^\alpha p(z)). \tag{2.13}$$

(b) For any $\lambda(\xi) \in \mathbb{C}[\xi]$, $p(z) \in \mathbb{C}[z]$ and $g(\xi, z) \in \mathbb{C}[\xi, z]$, we have

$$\lambda(\xi) *_t g(\xi, z) = \lambda(\xi - t\partial)g(\xi, z). \tag{2.14}$$

$$p(z) *_t g(\xi, z) = p(z - t\delta)g(\xi, z). \tag{2.15}$$

Note that the components $\xi_i - t\partial_i$ $(1 \le i \le n)$ of the n-tuple $\xi - t\partial$ in Eq. (2.14) commute with one another. So the substitution $\lambda(\xi - t\partial)$ of $\xi - t\partial$ into the polynomial $\lambda(\xi)$ is well-defined. Similarly, the substitution $p(z - t\delta)$ in Eq. (2.15) is also well-defined.

Proof: Eqs. (2.11)–(2.13) follow directly from Eq. (2.7).
To show Eq. (2.14), first, by Eq. (2.7), we have

$$\lambda(\xi) *_t g(\xi, z) = \sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|}t^{|\alpha|}}{\alpha!}(\delta^\alpha\lambda(\xi))(\partial^\alpha g(\xi, z)). \tag{2.16}$$

Second, note that the multiplication operators by ξ_i $(1 \le i \le n)$ and the derivations ∂_j $(1 \le j \le n)$ commute. By using the Taylor series expansion of $\lambda(\xi - t\partial)$ at ξ, we have

$$\begin{aligned}\lambda(\xi - t\partial)g(\xi, z) &= \left(\sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|}t^{|\alpha|}}{\alpha!}(\delta^\alpha\lambda)(\xi)\partial^\alpha\right) g(\xi, z) \\ &= \sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|}t^{|\alpha|}}{\alpha!}(\delta^\alpha\lambda(\xi))(\partial^\alpha g(\xi, z)).\end{aligned} \tag{2.17}$$

Hence, Eq. (2.14) follows from the last two equations. Eq. (2.15) can be proved similarly. □

Lemma 2.5. *For any $t \in \mathbb{C}$, $\lambda(\xi) \in \mathbb{C}[\xi]$ and $p(z) \in \mathbb{C}[z]$, we have*

$$\Phi_t(\lambda(\xi)) = \lambda(\xi). \tag{2.18}$$

$$\Phi_t(p(z)) = p(z). \tag{2.19}$$

$$\Phi_t(\lambda(\xi)p(z)) = \lambda(\xi) *_{-t} p(z). \tag{2.20}$$

Proof: Since $\Lambda(\lambda(\xi)) = \Lambda(p(z)) = 0$, $\Phi_t = e^{t\Lambda}$ fixes $\lambda(\xi)$ and $p(z)$. Hence we have Eqs. (2.18) and (2.19).

To show Eq. (2.20), by Eqs. (2.1), (2.18) and (2.19), we have

$$\begin{aligned}\lambda(\xi) *_t p(z) &= \Phi_{-t}\left(\Phi_t(\lambda(\xi))\Phi_t(p(z)) \right) \\ &= \Phi_{-t}(\lambda(\xi)p(z)).\end{aligned}$$

Replacing t be $-t$ in the equation above, we get Eq. (2.20). □

Proposition 2.6. *For any $t \in \mathbb{C}$, the following statements hold.*

(a) The subspaces $\mathbb{C}[\xi]$ and $\mathbb{C}[z]$ of $\mathcal{B}_t[\xi, z]$ are closed under the product $*_t$ and hence, are actually subalgebras of $\mathcal{B}_t[\xi, z]$.

(b) As associative algebras, $(\mathbb{C}[\xi], *_t)$ and $(\mathbb{C}[z], *_t)$ are identical as the usual polynomial algebras $\mathcal{A}[\xi]$ and $\mathcal{A}[z]$ in ξ and z, respectively.

(c) $\mathcal{B}_t[\xi, z]$ is a commutative free algebra generated freely by ξ_i and z_i $(1 \le i \le n)$. The set of the monomials generated by ξ_i and z_i $(1 \le i \le n)$ in $\mathcal{B}_t[\xi, z]$ is given by $\{\xi^\alpha *_t z^\beta \mid \alpha, \beta \in \mathbb{N}^n\}$.

Proof: Note that (a) and (b) follow immediately from Eqs. (2.11) and (2.12).

To show (c), first, by Eqs. (2.18) and (2.19), we know that the algebra isomorphism $\Phi_{-t} = \Phi_t^{-1} : \mathcal{A}[\xi, z] \to \mathcal{B}_t[\xi, z]$ as a linear map from $\mathbb{C}[\xi, z]$ to $\mathbb{C}[\xi, z]$ fixes ξ_i and z_i $(1 \le i \le n)$. Hence, $\mathcal{B}_t[\xi, z]$ is a commutative free algebra generated freely by ξ_i and z_i $(1 \le i \le n)$.

The second part of (c) follows from Eqs. (2.11), (2.12) and the fact that the product $*_t$ is associative and commutative. □

The next two lemmas will be needed in Subsection 5.3.

Lemma 2.7. *For any $t \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{N}$,*

$$(z\partial - \xi\delta)(\xi^\alpha *_t z^\beta) = (|\beta| - |\alpha|)(\xi^\alpha *_t z^\beta), \tag{2.21}$$

where $z\partial - \xi\delta := \sum_{i=1}^n (z_i\partial_i - \xi_i\delta_i)$.

Proof: First, by Euler's lemma, we have

$$(z\partial - \xi\delta)(\xi^\alpha z^\beta) = (|\beta| - |\alpha|)(\xi^\alpha z^\beta). \tag{2.22}$$

Second, note that $z\partial - \xi\delta$ commutes with Λ, hence also with Φ_t for any $t \in \mathbb{C}$. Apply Φ_{-t} to Eq. (2.22), we get

$$(z\partial - \xi\delta)\Phi_{-t}(\xi^\alpha z^\beta) = (|\beta| - |\alpha|)\Phi_{-t}(\xi^\alpha z^\beta).$$

Then, by Eq. (2.20) with t replaced by $-t$, Eq. (2.21) follows from the equation above. □

Lemma 2.8. *For any $\lambda_i(\xi) \in \mathbb{C}[\xi]$ and $p_i(z) \in \mathbb{C}[z]$ $(i = 1, 2)$, we have*

$$(\lambda_1(\xi)p_1(z)) *_t (\lambda_2(\xi)p_2(z)) = (\lambda_1(\xi) *_t p_2(z))\,(\lambda_2(\xi) *_t p_1(z)). \tag{2.23}$$

Proof: First, by Eq. (2.7), we have

$$\begin{aligned}&(\lambda_1(\xi)p_1(z)) *_t (\lambda_2(\xi)p_2(z))\\ &= \sum_{\alpha,\beta\in\mathbb{N}^n} \frac{(-t)^{|\alpha|+|\beta|}}{\alpha!\beta!}\big((\delta^\alpha\lambda_1(\xi))(\partial^\beta p_1(z))\big)\big((\delta^\beta\lambda_2(\xi))(\partial^\alpha p_2(z))\big)\end{aligned}$$

Taking sum over $\alpha \in \mathbb{N}^n$ and applying Eq. (2.13):

$$= (\lambda_1(\xi) *_t p_2(z)) \sum_{\beta\in\mathbb{N}^n} \frac{(-t)^{|\beta|}}{\beta!}(\partial^\beta p_1(z))(\delta^\beta\lambda_2(\xi))$$

Taking sum over $\beta \in \mathbb{N}^n$ and applying Eq. (2.13):

$$= (\lambda_1(\xi) *_t p_2(z))(\lambda_2(\xi) *_t p_1(z)).$$

Hence we get Eq. (2.23). □

2.2. **The ℓ-adic Topologies Induced by $\mathcal{B}_t[\xi, z]$ on $\mathbb{C}[\xi, z]$.** We have seen that the algebras $\mathcal{B}_t[\xi, z]$ $(t \in \mathbb{C})$ share the same base vector space $\mathbb{C}[\xi, z]$ and, by Proposition 2.6, (c), they are all commutative free algebras generated freely by ξ and z. Therefore, we may talk about the ℓ-adic topologies on $\mathbb{C}[\xi, z]$ induced by the algebras $\mathcal{B}_t[\xi, z]$ $(t \in \mathbb{C})$, which are defined as follows.

For any $t \in \mathbb{C}[\xi, z]$ and $m \geq 0$, set $U_{t,m}$ to be the subspace of $\mathbb{C}[\xi, z]$ spanned by the monomials $\xi^\alpha *_t z^\beta$ of $\mathcal{B}_t[\xi, z]$ with $\alpha, \beta \in \mathbb{N}^n$ and $|\alpha + \beta| \geq m$. The ℓ-adic topology induced from the algebra $\mathcal{B}_t[\xi, z]$ is the topology whose open subsets are the subsets generated by $U_{t,m}$ $(m \in \mathbb{N})$ and their translations by elements of $\mathcal{B}_t[\xi, z]$. We denote by $\mathcal{T}_t$ this topology on $\mathbb{C}[\xi, z]$.

The main result of this subsection is the following proposition.

Proposition 2.9. (a) *For any $s \neq t \in \mathbb{C}$, we have $\mathcal{T}_s \neq \mathcal{T}_t$.*

(b) *For any $t \in \mathbb{C}$, the algebra isomorphism $\Phi_t : (\mathcal{B}_t[\xi, z], \mathcal{T}_t) \to (\mathcal{A}[\xi, z], \mathcal{T}_0)$ is also a homeomorphism of topological spaces. Consequently, $(\mathcal{B}_t[\xi, z], \mathcal{T}_t)$ $(t \in \mathbb{C})$ as topological spaces are all homeomorphic.*

Proof: (a) Let $\{\alpha_m \in \mathbb{N}^n \,|\, m \geq 1\}$ be any sequence of elements of $\mathbb{N}^n$ such that $|\alpha_m| = m$ for any $m \geq 1$.

Set $u_m := \xi^{\alpha_m} *_t z^{\alpha_m}$ for any $m \geq 1$. Then, by the definition of $\mathcal{T}_t$, we see that the sequence $\{u_m\}$ converges to $0 \in \mathbb{C}[\xi, z]$ with respect to the topology $\mathcal{T}_t$.

But, on the other hand, set $r := s - t \neq 0$. Then, by Eq. (2.14), we have

$$\begin{aligned} u_m &= \xi^{\alpha_m} *_t z^{\alpha_m} = (\xi - t\partial)^{\alpha_m} z^{\alpha_m} = ((\xi - s\partial) + r\partial)^{\alpha_m} z^{\alpha_m} \\ &= \sum_{\substack{\beta,\gamma\in\mathbb{N}^n \\ \beta+\gamma=\alpha_m}} \binom{\alpha_m}{\gamma} (\xi - s\partial)^{\gamma} (\partial^{\beta} z^{\alpha_m}) \\ &= \sum_{\substack{\beta,\gamma\in\mathbb{N}^n \\ \beta+\gamma=\alpha_m}} \binom{\alpha_m}{\gamma} \xi^{\gamma} *_s (\partial^{\beta} z^{\alpha_m}) \equiv \alpha_m! \mod (U_{s,0}). \end{aligned}$$

From the equation above, we see that the sequence $\{u_m\}$ does not converge to $0 \in \mathbb{C}[\xi, z]$ with respect to the topology $\mathcal{T}_s$. Hence $\mathcal{T}_s \neq \mathcal{T}_t$.

(b) Note that $\mathcal{B}_{t=0}[\xi, z]$ is the usual polynomial algebra $\mathcal{A}[\xi, z]$ and $\Phi_t : \mathcal{B}_t[\xi, z] \to \mathcal{A}[\xi, z]$ is an algebra isomorphism. Furthermore, from Eqs. (2.1), (2.18) and (2.19), we have

$$\Phi_t(\xi^{\alpha} *_t z^{\beta}) = \xi^{\alpha} z^{\beta} \tag{2.24}$$

for any $\alpha, \beta \in \mathbb{N}^n$.

Therefore, for any $m \geq 0$, we have, $\Phi_t(U_{t,m}) = U_{0,m}$ and $\Phi_t^{-1}(U_{0,m}) = U_{t,m}$. Hence, we have (b). $\square$

Actually, the proof above also shows that Proposition 2.9 also holds for the following topologies on $\mathbb{C}[\xi, z]$ induced by the free algebras $\mathcal{B}_t[\xi, z]$ $(t \in \mathbb{C})$.

For any $t \in \mathbb{C}[\xi, z]$ and $m \geq 0$, set

$$U'_{t,m} := \sum_{\substack{\alpha\in\mathbb{N}^n; \\ |\alpha|\geq m}} \xi^{\alpha} *_t \mathbb{C}[z]. \tag{2.25}$$

Denote by $\mathcal{T}'_t$ the topology on $\mathbb{C}[\xi, z]$ generated by $U'_{t,m}$ and their translations (as open subsets). Then, by a similar argument as in the proof of Proposition 2.9, it is easy to see that the following corollary also holds.

Corollary 2.10. (a) *For any $s \neq t \in \mathbb{C}$, we have ${\mathcal{T}_s}' \neq {\mathcal{T}_t}'$.*

(b) *For any $t \in \mathbb{C}$, the algebra isomorphism $\Phi_t : (\mathcal{B}_t[\xi, z], {\mathcal{T}_t}') \to (\mathcal{A}[\xi, z], {\mathcal{T}_0}')$ is also a homeomorphism of topological spaces.*

Note that, due to the symmetric roles played by ξ and z, the corollary above also holds if ξ in Eq. (2.25) is replaced by z.

2.3. **The Induced Isomorphism $(\Phi_t)_*$ on Differential Operator Algebras.** For any $t \in \mathbb{C}$, denote by $\mathcal{D}_t[\xi, z]$ the differential operator algebra or the Weyl algebra of $\mathcal{B}_t[\xi, z]$, i.e. the associative algebra generated by the $\mathbb{C}$-derivations and the multiplication operators of the algebra $\mathcal{B}_t[\xi, z]$. Since $\Phi_t : \mathcal{B}_t[\xi, z] \to \mathcal{A}[\xi, z]$ is an algebra isomorphism (See Corollary 2.2), it induces an algebra isomorphism, denoted by $(\Phi_t)_* : \mathcal{D}_t[\xi, z] \to \mathcal{D}[\xi, z]$, from the Weyl algebra $\mathcal{D}_t[\xi, z]$ of $\mathcal{B}_t[\xi, z]$ to the Weyl algebra $\mathcal{D}[\xi, z]$ of $\mathcal{A}[\xi, z]$.

Recall that the induced map $(\Phi_t)_*$ is defined by setting

$$(\Phi_t)_*(\psi) = \Phi_t \circ \psi \circ \Phi_t^{-1} = \Phi_t \circ \psi \circ \Phi_{-t} \tag{2.26}$$

for any $\psi \in \mathcal{D}_t[\xi, z]$.

The main result of this subsection are the following two propositions, even though their proofs are very simple.

Proposition 2.11. *For any $t \in \mathbb{C}$, the following statements hold.*

(a) ∂_i and δ_i $(1 \le i \le n)$ are also derivations of $\mathcal{B}_t[\xi, z]$.

(b) For any $1 \le i \le n$, we have

$$(\Phi_t)_*(\partial_i) = \partial_i, \tag{2.27}$$

$$(\Phi_t)_*(\delta_i) = \delta_i, \tag{2.28}$$

Proof: Note first that ∂_i and δ_i $(1 \le i \le n)$ commute with Λ, hence also with Φ_t for any $t \in \mathbb{C}$. Then, Eqs. (2.27) and (2.28) follows immediately from this fact and the definition of $(\Phi_t)_*$ given in Eq. (2.26).

(a) follows from the general fact that the induced map of any algebra isomorphism maps derivations to derivations. It can also be checked directly as follows.

For any $f, g \in \mathcal{B}_t[\xi, z]$, by Eq. (2.1) and the fact that ∂_i $(1 \le i \le n)$ commute with Φ_t $(t \in \mathbb{C})$, we have

$$\begin{aligned}
\partial_i(f *_t g) &= \partial_i\Big(\Phi_{-t}\big(\Phi_t(f)\Phi_t(g)\big)\Big) = \Phi_{-t}\Big(\partial_i\big(\Phi_t(f)\Phi_t(g)\big)\Big) \\
&= \Phi_{-t}\big((\partial_i\Phi_t(f))\Phi_t(g)\big) + \Phi_{-t}\big(\Phi_t(f)(\partial_i\Phi_t(g))\big) \\
&= \Phi_{-t}\big(\Phi_t(\partial_i f)\Phi_t(g)\big) + \Phi_{-t}\big(\Phi_t(f)\Phi_t(\partial_i g)\big) \\
&= (\partial_i f) *_t g + f *_t (\partial_i g).
\end{aligned}$$

Similarly, we can show that δ_i $(1 \le i \le n)$ are also derivations of $\mathcal{B}_t[\xi, z]$. $\square$

Corollary 2.12. *For any $\alpha, \beta, \gamma \in \mathbb{N}^n$, we have*

$$\partial^\gamma(\xi^\alpha *_t z^\beta) = \gamma! \binom{\beta}{\gamma} (\xi^\alpha *_t z^{\beta-\gamma}), \tag{2.29}$$

$$\delta^\gamma(\xi^\alpha *_t z^\beta) = \gamma! \binom{\alpha}{\gamma} (\xi^{\alpha-\gamma} *_t z^\beta). \tag{2.30}$$

Proof: Note that, by Eqs. (2.11) and (2.12), we know that, for any $\alpha, \beta \in \mathbb{N}^n$, $\xi^\alpha *_t z^\beta$ will remain the same if we replace the (usual) product of $\mathcal{A}[\xi, z]$ in the factors ξ^α and z^β by the product $*_t$ of $\mathcal{B}_t[\xi, z]$. By Proposition 2.11, (a), we know that ∂_i and δ_i $(1 \le i \le n)$ are also the derivations of $\mathcal{B}_t[\xi, z]$. From these two facts, it is easy to see that both equations in the corollary hold. □

Proposition 2.13. *For any $t \in \mathbb{C}$ and $f(\xi, z) \in \mathbb{C}[\xi, z]$, $(\Phi_t)_*$ maps the multiplication operator of $\mathcal{B}_t[\xi, z]$ by $f(\xi, z)$ (with respect to the product $*_t$) to the multiplication operator of $\mathcal{A}[\xi, z]$ by $\Phi_t(f(\xi, z))$ (with respect to the product of $\mathcal{A}[\xi, z]$).*

Proof: We denote by ψ_f the multiplication operator of $\mathcal{B}_t[\xi, z]$ by $f(\xi, z)$ (with respect to the product $*_t$). Then for any $u(\xi, z) \in \mathbb{C}[\xi, z]$, by Eqs. (2.26) and (2.1) we have

$$\begin{aligned}(\Phi_t)_*(\psi_f)u(\xi, z) &= (\Phi_t \circ \psi_f \circ \Phi_t^{-1})u(\xi, z) = \Phi_t\big(f(\xi, z) *_t \Phi_t^{-1}(u(\xi, z))\big) \\ &= \Phi_t(f(\xi, z))\, \Phi_t\big(\Phi_t^{-1}(u(\xi, z))\big) = \Phi_t(f(\xi, z))\, u(\xi, z).\end{aligned}$$

Hence, the proposition follows. □

By the proposition above and Eqs. (2.18) and (2.19), we also have the following corollary.

Corollary 2.14. *For any $t \in \mathbb{C}$, $\lambda(\xi) \in \mathbb{C}[\xi]$ and $p(z) \in \mathbb{C}[z]$, $(\Phi_t)_*$ maps the multiplication operators of $\mathcal{B}_t[\xi, z]$ by $\lambda(\xi)$ and $p(z)$ (with respect to the product $*_t$) to the multiplication operators of $\mathcal{A}[\xi, z]$ by $\lambda(\xi)$ and $p(z)$ (with respect to the product of $\mathcal{A}[\xi, z]$), respectively.*

Note that, as pointed out before, the algebras $\mathcal{B}_t[\xi, z]$ $(t \in \mathbb{C})$ share the same base vectors space $\mathbb{C}[\xi, z]$. Therefore, their Weyl algebras $\mathcal{D}_t[\xi, z]$ $(t \in \mathbb{C})$ are all subalgebras of the algebra of linear endomorphisms of $\mathbb{C}[\xi, z]$. The following corollary says that all these subalgebras turn out to be same, i.e. they do not depend on the parameter $t \in \mathbb{C}$.

Corollary 2.15. *For any $t \in \mathbb{C}$, as subalgebras of the algebra of linear endomorphisms of $\mathbb{C}[\xi, z]$, $\mathcal{D}_t[\xi, z] = \mathcal{D}[\xi, z]$.*

Proof: By Proposition 2.6, (c), we know that $\mathcal{B}_t[\xi, z]$ is a commuative free algebra generated freely by ξ and z. By Proposition 2.11, (a), we know that ∂_i and δ_i $(1 \le i \le n)$ are also derivations of $\mathcal{B}_t[\xi, z]$. Therefore, the Weyl algebra $\mathcal{D}_t[\xi, z]$ as an associative algebra over $\mathbb{C}$

is generated by the derivations ∂_i, δ_i $(1 \leq i \leq n)$ and the multiplication operators (with respect to the product $*_t$) by $\xi_i, z_i \in \mathcal{B}_t[\xi, z]$ $(1 \leq i \leq n)$.

By Eqs. (2.14) and (2.15), we see that the multiplication operators by $\xi_i, z_i \in \mathcal{B}_t[\xi, z]$ $(1 \leq i \leq n)$ are same as the operators $\xi_i - t\partial_i$ and $z_i - t\delta_i$ which lie in $\mathcal{D}[\xi, z]$. Hence we have $\mathcal{D}_t[\xi, z] \subseteq \mathcal{D}[\xi, z]$.

To show $\mathcal{D}[\xi, z] \subseteq \mathcal{D}_t[\xi, z]$, by Proposition 2.11, (a), it will be enough to show that the multiplication operators (with respect to the product of $\mathcal{A}[\xi, z]$) by $\xi_i, z_i \in \mathcal{A}[\xi, z]$ $(1 \leq i \leq n)$ also belong to $\mathcal{D}_t[\xi, z]$.

But, for any $f(\xi, z) \in \mathbb{C}[\xi, z]$, by Eqs. (2.14) and (2.15), we have

$$\xi_i f(\xi, z) = (\xi_i - t\partial_i) f(\xi, z) + t\partial_i f(\xi, z) = \xi_i *_t f(\xi, z) + t\partial_i f(\xi, z),$$
$$z_i f(\xi, z) = (z_i - t\delta_i) f(\xi, z) + t\delta_i f(\xi, z) = z_i *_t f(\xi, z) + t\delta_i f(\xi, z).$$

From the equations above, we see that the multiplication operators (with respect to the product of $\mathcal{A}[\xi, z]$) by $\xi_i, z_i \in \mathcal{A}[\xi, z]$ $(1 \leq i \leq n)$ do belong to $\mathcal{D}_t[\xi, z]$. $\square$

3. Connections with Interchanges of Right and Left Total Symbols of Differential Operators

In this section, we show in Theorem 3.1 that the isomorphisms Φ_t with $t = \pm 1$ coincide with the interchanges between total left and right symbols of differential operators of the polynomial algebra $\mathcal{A}[z]$.

First, let us fix the following notation and convention for the differential operators of $\mathcal{A}[z]$.

We denote by $\mathcal{D}[z]$ the differential operator algebra or the Weyl algebra of $\mathcal{A}[z]$. For any differential operator $\phi \in \mathcal{D}[z]$ and polynomial $u(z) \in \mathcal{A}[z]$, the notation $\phi\, u(z)$ usually denotes the composition of ϕ and the multiplication operator by $u(z)$. So $\phi\, u(z)$ is still a differential operator of $\mathcal{A}[z]$. The polynomial obtained by applying ϕ to $u(z)$ will be denoted by $\phi(u(z))$.

Next, let us recall the right and left total symbols of differential operators of the polynomial algebra $\mathcal{A}[z]$.

For any $\phi \in \mathcal{D}[z]$, it is well-known (e.g. see Proposition 2.2 (pp. 4) in [B] or Theorem 3.1 (pp. 58) in [C]) that ϕ can be written uniquely as the following two finite sums:

$$\phi = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(z) \partial^\alpha = \sum_{\beta \in \mathbb{N}^n} \partial^\beta b_\beta(z) \tag{3.1}$$

where $a_\alpha(z), b_\beta(z) \in \mathbb{C}[z]$ but denote the multiplication operators by $a_\alpha(z)$ and $b_\beta(z)$, respectively.

For the differential operator $\phi \in \mathcal{D}[z]$ in Eq. (3.1), the *right* and *left total symbols* are defined to be the polynomials $\sum_{\alpha\in\mathbb{N}^n} a_\alpha(z)\xi^\alpha \in \mathbb{C}[\xi, z]$ and $\sum_{\beta\in\mathbb{N}^n} b_\beta(z)\xi^\beta \in \mathbb{C}[\xi, z]$, respectively. We denote by $\mathcal{R} : \mathcal{D}[z] \to \mathbb{C}[\xi, z]$ (resp. $\mathcal{L} : \mathcal{D}[z] \to \mathbb{C}[\xi, z]$) the linear map which maps any $\phi \in \mathcal{D}[z]$ to its right total symbol (resp. left total symbol).

Note that, by the uniqueness of the expressions in Eq. (3.1), both $\mathcal{R}$ and $\mathcal{L}$ are isomorphisms of vector spaces over $\mathbb{C}$. The interchange of the left (resp. right) total symbol of differential operators to their right (resp. left) total symbols is given by the isomorphism $\mathcal{R} \circ \mathcal{L}^{-1}$ (resp. $\mathcal{L} \circ \mathcal{R}^{-1}$) from $\mathbb{C}[\xi, z]$ to $\mathbb{C}[\xi, z]$.

The main result of this section is the following theorem.

Theorem 3.1. *As linear maps from $\mathbb{C}[\xi, z]$ to $\mathbb{C}[\xi, z]$, we have*

$$\Phi = \mathcal{R} \circ \mathcal{L}^{-1}. \tag{3.2}$$

$$\Phi_{t=-1} = \mathcal{L} \circ \mathcal{R}^{-1}. \tag{3.3}$$

Proof: Note first that, Eq. (3.3) follows from Eq. (3.2) and the fact that $\Phi_{t=-1} = \Phi_{t=1}^{-1} = \Phi^{-1}$.

To show Eq. (3.2), since both Φ and $\mathcal{R} \circ \mathcal{L}^{-1}$ are linear maps, it is enough to show that, for any $\alpha, \beta \in \mathbb{N}^n$, we have

$$\Phi(\xi^\alpha z^\beta) = (\mathcal{R} \circ \mathcal{L}^{-1})(\xi^\alpha z^\beta). \tag{3.4}$$

Since

$$(\mathcal{R} \circ \mathcal{L}^{-1})(\xi^\alpha z^\beta) = \mathcal{R}(\partial^\alpha z^\beta), \tag{3.5}$$

so we have to find the right total symbol of the differential operator $\partial^\alpha z^\beta \in \mathcal{D}[z]$.

Note that, for any dummy $u(z) \in \mathbb{C}[z]$, by the Leibniz rule, we have

$$\begin{aligned} \partial^\alpha(z^\beta u(z)) &= \sum_{\gamma\in\mathbb{N}^n} \binom{\alpha}{\gamma} (\partial^\gamma z^\beta)(\partial^{\alpha-\gamma} u(z)) \\ &= \left(\sum_{\gamma\in\mathbb{N}^n} \binom{\alpha}{\gamma} (\partial^\gamma z^\beta)\partial^{\alpha-\gamma} \right) u(z). \end{aligned} \tag{3.6}$$

Therefore, the right total symbol of the differential operator $\partial^\alpha z^\beta \in \mathcal{D}[z]$ is given by

$$\begin{aligned} \mathcal{R}(\partial^\alpha z^\beta) &= \sum_{\gamma\in\mathbb{N}^n} \binom{\alpha}{\gamma} (\partial^\gamma z^\beta)\xi^{\alpha-\gamma} = \sum_{\gamma\in\mathbb{N}^n} \binom{\alpha}{\gamma} \xi^{\alpha-\gamma}(\partial^\gamma z^\beta) \\ &= \sum_{\gamma\in\mathbb{N}^n} \frac{1}{\gamma!} (\delta^\gamma \xi^\alpha)(\partial^\gamma z^\beta) \end{aligned}$$

Combining the equation above with Eqs. (2.13) and (2.20) with $t = -1$, we have

$$\mathcal{R}(\partial^\alpha z^\beta) = \xi^\alpha *_{t=-1} z^\beta = \Phi_{t=1}(\xi^\alpha z^\beta) = \Phi(\xi^\alpha z^\beta). \tag{3.7}$$

Hence, we have proved Eq. (3.4) and also the theorem. □

Corollary 3.2. *For any $\lambda(\xi) \in \mathbb{C}[\xi]$ and $p(z) \in \mathbb{C}[z]$, we have*

$$\mathcal{R}(\lambda(\partial)p(z)) = \lambda(\xi) *_{t=-1} p(z). \tag{3.8}$$

$$\mathcal{L}(p(z)\lambda(\partial)) = \lambda(\xi) * p(z). \tag{3.9}$$

Proof: By Eqs. (3.2) and (2.20) with $t = 1$, we have

$$\begin{aligned}\mathcal{R}(\lambda(\partial)p(z)) &= \mathcal{R}(\mathcal{L}^{-1}(\lambda(\xi)p(z))) = (\mathcal{R} \circ \mathcal{L}^{-1})(\lambda(\xi)p(z)) \\ &= \Phi_{t=1}(\lambda(\xi)p(z)) = \lambda(\xi) *_{t=-1} p(z).\end{aligned}$$

So we have Eq. (3.8). Eq. (3.9) can be proved similarly by using Eqs. (3.3) and (2.20) with $t = -1$. □

Finally, we end this section with the following one-variable example.

Example 3.3. *Let $n = 1$ and $\phi = z^2\partial^3$. Then,*

$$\begin{aligned}\mathcal{R}(\phi) &= \mathcal{R}(z^2\partial^3) = \xi^3 z^2. \\ \mathcal{L}(\phi) &= \mathcal{L}(z^2\partial^3) = \xi^3 * z^2 = (z-\delta)^2\xi^3 \\ &= (z^2 - 2z\delta + \delta^2)\xi^3 = \xi^3 z^2 - 6\xi^2 z + 6\xi.\end{aligned}$$

Therefore, we have

$$\phi = z^2\partial^3 = \partial^3 z^2 - 6\partial^2 z + 6\partial.$$

4. A Re-formulation of the Image Conjecture on Commuting Differential Operators of Order One with Constant Leading Coefficients

In this section, we show that the algebra $\mathcal{B}_t[\xi, z]$ ($t \in \mathbb{C}$) is closely related with a theorem (See Theorem 4.1) first proved in [Z3] and also with the so-called *image* conjecture (See Conjecture 4.3) proposed in [Z3] on the differential operators $\xi - t\partial$ ($t \in \mathbb{C}$).

In Subsection 4.1, we use certain Taylor series expansion of elements of $\mathcal{B}_t[\xi, z]$ to give a new and more conceptual proof for Theorem 4.1. In Subsection 4.2, we first give a new formulation (See Conjecture 4.5) for Conjecture 4.3 in terms of the algebra $\mathcal{B}_t[\xi, z]$ and the notion of *Mathieu subspaces* (see Definition 4.2) introduced in [Z4], and then show in Theorem 4.6 that the new formulation is indeed equivalent to Conjecture 4.3.

4.1. **The Taylor Series with Respect to the Product $*_t$.** First, let us recall the following elementary fact on polynomials in ξ and z.

For any $f(\xi, z) \in \mathcal{A}[\xi, z]$, we may view $f(\xi, z)$ as a polynomial in ξ with coefficients in $\mathcal{A}[z]$. Then it has the following Taylor series expansion

$$f(\xi, z) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!}\, \xi^\alpha c_\alpha(z) \tag{4.1}$$

for some $c_\alpha(z) \in \mathcal{A}[z]$.

Let $ev_0 : \mathcal{A}[\xi, z] \to \mathcal{A}[z]$ be the *evaluation* map of $\mathcal{A}[\xi, z]$ at $\xi = 0$, i.e. for any $u(\xi, z) \in \mathcal{A}[\xi, z]$, $ev_0(u) := u(0, z)$. Then, the $c_\alpha(z)$ ($\alpha \in \mathbb{N}^n$) in Eq. (4.1) are given by

$$c_\alpha(z) = ev_0(\delta^\alpha f). \tag{4.2}$$

Note that another characterization of the evaluation map ev_0 is that ev_0 is the (unique) algebra homomorphism from $\mathcal{A}[\xi, z]$ to $\mathcal{A}[z]$ with $ev_0(\xi_i) = 0$ and $ev_0(z_i) = z_i$ for any $1 \le i \le n$.

Now, come back to our algebras $\mathcal{B}_t[\xi, z]$ ($t \in \mathbb{C}$). By Proposition 2.6, (c), we know that $\mathcal{B}_t[\xi, z]$ is also a commutative free algebra generated freely by ξ and z with the same base vector space $\mathbb{C}[\xi, z]$. Hence, we should expect similar expansions as in Eq. (4.1) for polynomials $f(\xi, z) \in \mathbb{C}[\xi, z]$ with respect to the product $*_t$.

But, in order to formulate the expected expansions precisely, we need first to introduce the analogue of the evaluation map ev_0 for the algebra $\mathcal{B}_t[\xi, z]$.

Note that, by Proposition 2.6, (b), the subalgebra of $\mathcal{B}_t[\xi, z]$ generated by z is also $\mathcal{A}[z] \subset \mathbb{C}[\xi, z]$. Parallel to the second characterization of the evaluation map ev_0 mentioned above, we let $\mathcal{E}_t$ be the unique algebra homomorphism from $\mathcal{B}_t[\xi, z] \to \mathcal{A}[z]$ such that $\mathcal{E}_t(\xi_i) = 0$ and $\mathcal{E}_t(z_i) = z_i$ for any $1 \le i \le n$.

Note also that, by Eqs. (2.18) and (2.19), the algebra isomorphism $\Phi_t : \mathcal{B}_t[\xi, z] \to \mathcal{A}[\xi, z]$ maps ξ_i (resp. z_i) to ξ_i (resp. z_i) for any $1 \le i \le n$. Hence the composition $ev_0 \circ \Phi_t : \mathcal{B}_t[\xi, z] \to \mathcal{A}[z]$ has the same characterizing property of $\mathcal{E}_t$. Therefore, we have

$$\mathcal{E}_t = ev_0 \circ \Phi_t. \tag{4.3}$$

Furthermore, we can also derive a more explicit formula for $\mathcal{E}_t$ as follows.

For any $\alpha \in \mathbb{N}^n$ and $p(z) \in \mathbb{C}[z]$, consider

$$\mathcal{E}_t(\xi^\alpha p(z)) = ev_0(\Phi_t(\xi^\alpha p(z))) \tag{4.4}$$

Applying Eq. (2.20) and then Eq. (2.14) with t replaced by $-t$:

$$\begin{aligned} &= ev_0(\xi^\alpha *_{-t} p(z))) \\ &= ev_0((\xi + t\partial)^\alpha (p(z))) = t^{|\alpha|}\partial^\alpha(p(z)). \end{aligned}$$

From the formula above, we see that, for any $g(z,\xi) \in \mathbb{C}[z,\xi]$, $\mathcal{E}_t(g(z,\xi)) \in \mathbb{C}[z]$ can be obtained by, first, writing each monomial of $g(z,\xi)$ as $\xi^\beta z^\gamma$ ($\beta,\gamma \in \mathbb{N}^n$), i.e. putting the free variables ξ_i's to the most left in each monomial of $g(z,\xi)$, and then replacing the part ξ^β by the differential operator $t^{|\beta|}\partial^\beta$ and applying it to the other part z^γ of the monomial. For examples, we have

$$\begin{aligned} \mathcal{E}_t(1) &= 1; \\ \mathcal{E}_t(z^\alpha) &= (t\partial)^0(z^\alpha) = z^\alpha && \textit{for any } \alpha \in \mathbb{N}^n; \\ \mathcal{E}_t(\xi^\alpha) &= t^{|\alpha|}\partial^\alpha(1) = 0 && \textit{for any } 0 \neq \alpha \in \mathbb{N}^n; \\ \mathcal{E}_t(z_1^m\xi_1^2) &= t^2\partial_1^2(z_1^m) = m(m-1)t^2 z_1^{m-2} && \textit{for any } m \geq 2. \end{aligned}$$

Now we are ready to formulate and prove the expected expansion of polynomials with respect to the new product $*_t$, which is parallel to the Taylor expansion in Eq. (4.1).

Theorem 4.1. *For any $t \in \mathbb{C}$ and $f(\xi,z) \in \mathbb{C}[\xi,z]$, we have*

$$f(\xi,z) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\, \xi^\alpha *_t a_\alpha(z), \tag{4.5}$$

$$f(\xi,z) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\, (\xi - t\partial_z)^\alpha a_\alpha(z), \tag{4.6}$$

where, for any $\alpha \in \mathbb{N}^n$,

$$a_\alpha(z) = \mathcal{E}_t(\delta^\alpha f). \tag{4.7}$$

Furthermore, the expansions of the forms in Eqs. (4.5) and (4.6) for $f(\xi,z)$ are unique.

Proof: Note first that, by Eq. (2.14) in Proposition 2.4, Eq. (4.5) and Eq. (4.6) are actually equivalent. So we will focus only on Eq. (4.5).

The uniqueness of the expansion in Eq. (4.5) follows directly from Proposition 2.6, (a)-(c).

To show that Eq. (4.5) with $a_\alpha(z)$ ($\alpha \in \mathbb{N}^n$) given in Eq. (4.7) does hold, we first write the Taylor series expansion of $\Phi_t(f(\xi,z))$ as in Eq. (4.1):

$$\Phi_t(f(\xi,z)) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\, \xi^\alpha a_\alpha(z) \tag{4.8}$$

where $a_\alpha(z) \in \mathbb{C}[z]$ ($\alpha \in \mathbb{N}^n$) are given by

$$a_\alpha(z) = ev_0(\delta^\alpha \Phi_t(f)). \tag{4.9}$$

Applying Φ_{-t} to Eq.(4.8) and, by Eq. (2.20) with t replaced by $-t$, we get Eq. (4.5).

Next, note that δ^α ($\alpha \in \mathbb{N}^n$) commute with Λ, hence they also commute with $\Phi_t = e^{t\Lambda}$. Then, by Eqs. (4.9) and (4.3), we have

$$a_\alpha(z) = ev_0(\Phi_t(\delta^\alpha f)) = (ev_0 \circ \Phi_t)(\delta^\alpha f) = \mathcal{E}_t(\delta^\alpha f).$$

Therefore, Eq. (4.7) also holds. □

Several remarks on Theorem 4.1 and the proof above are as follows.

First, Theorem 4.1 with $t = 1$ was first proved in [Z3]. The proof in [Z3] is more straightforward. It does not use the algebra $\mathcal{B}_t[\xi, z]$ and the product $*_t$. But the proof given here is more conceptual. For example, the expansion in Eq. (4.6) becomes much more natural after we show here that it is just the usual Taylor series expansion of polynomials as in Eq. (4.1) but in the new context of the algebra $\mathcal{B}_t[\xi, z]$.

Second, Eq. (4.7) can also be derived directly from Eq. (4.6) as in [Z3]. Namely, apply δ^α to Eq. (4.6) and then replace ξ by $t\partial$ in both sides of the resulting equation.

Third, not all formal power series $f(\xi, z) \in \mathcal{A}[[\xi, z]]$ can be expanded in the form of Eq. (4.5) or (4.6). For example, let $n = 1$ and $f(\xi, z) = e^{\xi z}$ and assume that (4.6) holds for $f(\xi, z)$. Then, by the argument in the previous paragraph, we see that $a_m(z)$ ($m \geq 0$) must be given by Eq. (4.7). But, for the series $\delta^m f(\xi, z) = z^m \sum_{k\geq 0} \frac{(\xi z)^k}{k!}$, $\mathcal{E}_t$ is not well-defined, which is a contradiction.

Another way to look at the fact above is as follows. Even though $\mathcal{B}_t[\xi, z]$ ($t \neq 0$) and $\mathcal{A}[\xi, z]$ share the same base vector space $\mathbb{C}[\xi, z]$, by Proposition 2.9, we know that they induce different ℓ-adic topologies on $\mathbb{C}[\xi, z]$. Therefore, their completions with respect to the different ℓ-adic topologies will be different. In other words, the formal power series algebras with respect to the product $*_t$ ($t \neq 0$) and the usual formal power series algebra $\mathcal{A}[[\xi, z]]$ do not share the same base vector space anymore.

For the example $f(\xi, z) = e^{\xi z}$ above, we have $f(\xi, z) \in \mathcal{A}[[\xi, z]]$. But, by the argument in the proof of Proposition 2.9 with α_m ($m \geq 1$) replaced by m, it is easy to see that, for any $t \neq 0$, $f(\xi, z) = e^{\xi z}$ does not lie in the completion of $\mathcal{B}_t[\xi, z]$ with respect to the ℓ-adic topology on $\mathbb{C}[\xi, z]$ induced by $\mathcal{B}_t[\xi, z]$. Therefore, $f(\xi, z) = e^{\xi z}$ can not be written as a formal power series with respect to the product $*_t$ as in Eq. (4.5).

4.2. **Re-Formulation of the Image Conjecture in Terms of the Algebra $\mathcal{B}_t[\xi, z]$.** First let us recall the following notion introduced recently in [Z4].

Definition 4.2. *Let R be any commutative ring and $\mathcal{A}$ a commutative R-algebra. We say that an R-subspace $\mathcal{M}$ of $\mathcal{A}$ is a Mathieu subspace of $\mathcal{A}$ if the following property holds: if $a \in \mathcal{A}$ satisfies $a^m \in \mathcal{M}$ for all $m \geq 1$, then, for any $b \in \mathcal{A}$, we have $ba^m \in \mathcal{M}$ for all $m \gg 0$, i.e. there exists $N \geq 1$ (depending on a and b) such that $ba^m \in \mathcal{M}$ for all $m \geq N$.*

From the definition above, it is easy to see that any ideal of $\mathcal{A}$ is automatically a Mathieu subspace of $\mathcal{A}$, but not conversely (See [Z4] for some examples of Mathieu subspaces which are not ideals). Therefore, the notion of Mathieu subspaces can be viewed as a generalization of the notion of ideals.

Next, for any $t \in \mathbb{C}$, set

$$\mathrm{Im}\,(\xi - \mathrm{t}\partial) := \sum_{\mathrm{i}=1}^{\mathrm{n}} (\xi_{\mathrm{i}} - \mathrm{t}\partial_{\mathrm{i}})\mathbb{C}[\xi, \mathrm{z}]. \tag{4.10}$$

We call $\mathrm{Im}\,(\xi - \mathrm{t}\partial)$ the *image* of the commuting differential operators $(\xi_i - t\partial_i)$ $(1 \leq i \leq n)$.

With the notion and notation fixed above, the *image* conjecture proposed in [Z4] for the commuting differential operators $(\xi - t\partial)$ can be re-stated as follows.

Conjecture 4.3. *For any $t \in \mathbb{C}$, $\mathrm{Im}\,(\xi - \mathrm{t}\partial)$ is a Mathieu subspace of the polynomial algebra $\mathcal{A}[\xi, z]$.*

One of the motivations of the conjecture above is the following theorem proved in [Z3].

Theorem 4.4. *Conjecture 4.3 implies the Jacobian conjecture.*

Actually, it has been shown in [Z3] that the *Jacobian* conjecture is equivalent to some very special cases of Conjecture 4.3. For more detail, see [Z3].

The main result of this subsection is to show that the conjecture above can actually be re-formulated as follows.

Conjecture 4.5. *Set $\xi\mathbb{C}[\xi, z] := \sum_{i=1}^{m} \xi_i \mathbb{C}[\xi, z]$. Then, for any $t \in \mathbb{C}$, $\xi\mathbb{C}[\xi, z]$ as a subspace of $\mathcal{B}_t[\xi, z]$ is a Mathieu subspace of $\mathcal{B}_t[\xi, z]$.*

Theorem 4.6. *Conjecture 4.5 is equivalent to Conjecture 4.3.*

Proof: First, denote by $\xi *_t \mathbb{C}[\xi, z]$ the ideal of $\mathcal{B}_t[\xi, z]$ generated by ξ_i $(1 \leq i \leq n)$. View $\xi *_t \mathbb{C}[\xi, z]$ as a subspace of $\mathcal{A}[\xi, z]$ and apply Eqs. (4.10) and (2.14), we have

$$\mathrm{Im}\,(\xi - \mathrm{t}\partial) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \xi_{\mathrm{i}} *_{\mathrm{t}} \mathbb{C}[\xi, \mathrm{z}] = \xi *_{\mathrm{t}} \mathbb{C}[\xi, \mathrm{z}]. \tag{4.11}$$

Second, by Eqs. (2.1) and (2.18), we have

$$\Phi_t(\xi *_t \mathbb{C}[\xi, z]) = \Phi_t(\xi)\Phi_t(\mathbb{C}[\xi, z]) = \xi\mathbb{C}[\xi, z]$$

Hence, we also have

$$\xi *_t \mathbb{C}[\xi, z] = \Phi_t^{-1}(\xi\mathbb{C}[\xi, z]) = \Phi_{-t}(\xi\mathbb{C}[\xi, z]). \tag{4.12}$$

Combine Eqs. (4.11) and (4.12), we get

$$\Phi_{-t}(\xi\mathbb{C}[\xi, z]) = \mathrm{Im}\,(\xi - \mathrm{t}\partial). \tag{4.13}$$

Third, by Proposition 4.9 in [Z4], we know that pre-images of Mathieu subspaces under algebra homomorphisms are still Mathieu subspaces, from which it is easy to check that Mathieu subspaces are preserved by algebra isomorphisms. By using this fact (on the algebra isomorphism $\Phi_{-t} : \mathcal{B}_{-t}[\xi, z] \to \mathcal{A}[\xi, z]$) and also Eq. (4.13), we see that, $\xi\mathbb{C}[\xi, z]$ is a Mathieu subspace of $\mathcal{B}_{-t}[\xi, z]$ iff $\mathrm{Im}\,(\xi - \mathrm{t}\partial)$ is a Mathieu subspace of $\mathcal{A}[\xi, z]$.

Replacing t by $-t$ in the equivalence above, we have that, $\xi\mathbb{C}[\xi, z]$ is a Mathieu subspace of $\mathcal{B}_t[\xi, z]$ for any $t \in \mathbb{C}$ iff $\mathrm{Im}\,(\xi + \mathrm{t}\partial)$ is a Mathieu subspace of $\mathcal{A}[\xi, z]$ for any $t \in \mathbb{C}$ iff $\mathrm{Im}\,(\xi - \mathrm{t}\partial)$ is a Mathieu subspace of $\mathcal{A}[\xi, z]$ for any $t \in \mathbb{C}$. Hence, we have proved the theorem. □

From Theorems 4.4 and 4.6, we immediately have the following corollary.

Corollary 4.7. *Conjecture 4.5 implies the Jacobian conjecture.*

Remark 4.8. *Note that, when $t = 0$, Conjecture 4.5 is trivial since $\xi\mathbb{C}[\xi, z]$ is an ideal of the algebra $\mathcal{B}_{t=0}[\xi, z] = \mathcal{A}[\xi, z]$. In general, Conjecture 4.5 in some sense just claims that the algebras $\mathcal{B}_t[\xi, z]$ $(t \in \mathbb{C})$ do not differ or change too much from $\mathcal{A}[\xi, z]$ so that the vector subspace $\xi\mathbb{C}[\xi, z]$ still remains as a Mathieu subspace of $\mathcal{B}_t[\xi, z]$.*

From this point of view, the triviality of the deformation $\mathcal{B}_t[\xi, z]$ $(t \in \mathbb{C})$ of the polynomial algebra $\mathcal{A}[\xi, z]$ given in Corollary 2.2 may be viewed as a fact in favor of Conjecture 4.5, hence also to the Jacobian conjecture via the implication in Corollary 4.7.

Remark 4.9. *Conjecture 4.5 and also the Jacobian conjecture can be viewed as problems caused by the following fact. Namely, due to the*

change of the algebra structure from $\mathcal{A}[\xi, z]$ to $\mathcal{B}_t[\xi, z]$, the evaluation map at $\xi = 0$, which is an algebra homomorphism from $\mathcal{A}[\xi, z]$ to $\mathcal{A}[z]$, is not an algebra homomorphism from $\mathcal{B}_t[\xi, z]$ to $\mathcal{A}[z]$ if $t \neq 0$. Therefore, its kernel $\xi\mathbb{C}[\xi, z]$ does not remain to be an ideal of $\mathcal{B}_t[\xi, z]$ anymore.

But, on the other hand, as we will see later in Subsection 5.2 (See Theorem 5.2 and Remark 5.4), the same fact for the evaluation map at $\xi = 1$, i.e. $\xi_i = 1$ $(1 \leq i \leq n)$, in some sense also causes something truely remarkable, namely, the generalized Laguerre polynomials.

5. Connections with the Generalized Laguerre Polynomials

In this section, we study some connections and interactions of the monomials of the algebra $\mathcal{B}[\xi, z]$ in ξ and z with the generalized Laguerre polynomials in one or more variables.

In Subsection 5.1, we briefly recall the definition and the orthogonal property of the generalized Laguerre polynomials. In Subsection 5.2, we show that the generalized Laguerre polynomials can be obtained from certain monomials of the algebra $\mathcal{B}[\xi, z]$ in ξ and z (See Theorem 5.2 and Corollary 5.3).

In Subsection 5.3, we study some applications of the connection given in Theorem 5.2. We first use certain properties of the generalized Laguerre polynomials to derive some results on some monomials of $\mathcal{B}[\xi, z]$ in ξ and z. We then use some results derived in Section 2 on the algebra $\mathcal{B}[\xi, z]$ to give new proofs for some important properties of the generalized Laguerre polynomials (see Proposition 5.11 and Theorem 5.12).

5.1. The Generalized Laguerre Orthogonal Polynomials.

First, let us recall the generalized Laguerre orthogonal polynomials in one variable.

For any $k \in \mathbb{R}$ and $m \in \mathbb{N}$, the generalized Laguerre polynomial $L_m^{[k]}(z)$ in one variable is given by

$$L_m^{[k]}(z) = \sum_{j=0}^{m} \binom{m+k}{m-j} \frac{(-z)^j}{j!}. \tag{5.1}$$

Here we are only interested in the case that $k \in \mathbb{N}$. For any fixed $k \in \mathbb{N}$, the generating function of the generalized Laguerre polynomials $L_m^{[k]}(z)$ $(m \geq 0)$ is given by

$$\frac{\exp(-\frac{zu}{1-u})}{(1-u)^{k+1}} = \sum_{m=0}^{+\infty} L_m^{[k]}(z)\, u^m, \tag{5.2}$$

where u above denotes a formal variable which commutes with z.

The multi-variable generalized Laguerre polynomials are defined as follows.

Let $\mathbf{k} = (k_1, k_2, ..., k_n) \in \mathbb{N}^n$ and $\alpha = (a_1, a_2, ..., a_n) \in \mathbb{N}^n$. The generalized Laguerre polynomials in n-variable $z = (z_1, z_2, ..., z_n)$ is defined by

$$L_\alpha^{[\mathbf{k}]}(z) := L_{a_1}^{[k_1]}(z_1) L_{a_2}^{[k_2]}(z_2) \cdots L_{a_n}^{[k_n]}(z_n). \tag{5.3}$$

The polynomials $L_\alpha(z) := L_\alpha^{[0]}(z)$ ($\alpha \in \mathbb{N}^n$) are the so-called the (classical) *Laguerre* polynomials. They were named after Edmond. N. Laguerre [L]. The generalized Laguerre polynomials were introduced much later by G. Pólya and G. Szegö [PS] in 1976.

One of the most important properties of the generalized Laguerre polynomials is the following theorem.

Theorem 5.1. *For any* $\mathbf{k}, \alpha, \beta \in \mathbb{N}^n$*, we have*

$$\int_{(\mathbb{R}_{>0})^n} L_\alpha^{[\mathbf{k}]}(z) L_\beta^{[\mathbf{k}]}(z)\, w(z)\, dz = \delta_{\alpha,\beta}\, \frac{(\alpha + \mathbf{k})!}{\alpha!}, \tag{5.4}$$

where $\delta_{\alpha,\beta}$ *is the Kronecker delta function and* $w(z)$ *given by*

$$w(z) := z^{\mathbf{k}} e^{-\sum_{i=1}^n z_i}. \tag{5.5}$$

The function $w(z)$ above is called the *weight* function of the generalized Laguerre polynomials $L_\alpha^{[\mathbf{k}]}(z)$ ($\alpha \in \mathbb{N}^n$).

Consequently, with any fixed $\mathbf{k}$, the generalized Laguerre polynomials $L_\alpha^{[\mathbf{k}]}(z)$ ($\alpha \in \mathbb{N}^n$) form an orthogonal basis of $\mathbb{C}[z]$ with respect to the Hermitian form defined by

$$(f, g) = \int_{(\mathbb{R}_{>0})^n} f(z) \bar{g}(z)\, w(z)\, dz, \tag{5.6}$$

where $\bar{g}(z)$ denotes the complex conjugation of the polynomial $g(z) \in \mathbb{C}[z]$.

There are many other interesting and important properties of the generalized Laguerre polynomials. We refer the reader to [Sz], [PS], [AAR] and [DX] for very thorough study on this family of orthogonal polynomials. See also the *Wolfram Research* web sources [W1] and [W2] for over one hundred formulas and identities on the (generalized) Laguerre polynomials.

5.2. **The Generalized Laguerre Polynomials in Terms of the Product $*$.** The main result of this subsection is the following theorem.

Theorem 5.2. *For any* $\mathbf{k}, \alpha \in \mathbb{N}^n$*, we have*

$$L_\alpha^{[\mathbf{k}]}(\xi z) = \frac{(-1)^{|\alpha|}}{\alpha!}\, \xi^{-\mathbf{k}}(\xi^{\alpha+\mathbf{k}} * z^\alpha). \tag{5.7}$$

$$L_\alpha^{[\mathbf{k}]}(\xi z) = \frac{(-1)^{|\alpha|}}{\alpha!}\, z^{-\mathbf{k}}(\xi^{\alpha} * z^{\alpha+\mathbf{k}}), \tag{5.8}$$

where $\xi z := (\xi_1 z_1, \xi_2 z_2, ..., \xi_n z_n)$.

In particular, for the Laguerre polynomials, we have

$$L_\alpha(\xi z) = \frac{(-1)^{|\alpha|}}{\alpha!}\, \xi^\alpha * z^\alpha. \tag{5.9}$$

Proof: We first prove Eq. (5.9). Note first that, as pointed out in Subsection 2.1 [Z4], the Laguerre polynomials $L_m(z)$ ($m \in \mathbb{N}$) in one variable can be obtained as

$$L_m(z) = \frac{1}{m!}(\partial - 1)^m(z^m). \tag{5.10}$$

Changing the variable $z \to \xi z$ in the equation above, we get

$$\begin{aligned} L_m(\xi z) &= \frac{1}{m!}(\xi^{-1}\partial - 1)^m(\xi^m z^m) = \frac{1}{m!}\xi^{-m}(\partial - \xi)^m(\xi^m z^m) \\ &= \frac{1}{m!}(\partial - \xi)^m(z^m) = \frac{(-1)^m}{m!}(\xi - \partial)^m(z^m). \end{aligned}$$

By Eq. (5.3) with $\mathbf{k} = 0$ and the equation above, we see that the multi-variable Laguerre polynomials $L_\alpha(z)$ ($\alpha \in \mathbb{N}^n$) can be given by

$$L_\alpha(\xi z) = \frac{(-1)^{|\alpha|}}{\alpha!}(\xi - \partial)^\alpha(z^\alpha). \tag{5.11}$$

Then, apply Eq. (2.14) with $\lambda(\xi) = \xi^\alpha$ and $t = 1$, we get Eq. (5.9).

To show Eq. (5.7), recall that we have the following well-known identity for the one-variable generalized Laguerre polynomials, which can be easily derived from the generating functions of the generalized Laguerre polynomials in Eq. (5.2):

$$L_m^{[k]}(z) = (-1)^k \partial^k L_{m+k}(z). \tag{5.12}$$

Now, by Eq. (5.3) and the equation above, we see that the multi-variable generalized Laguerre polynomials can be given by

$$L_\alpha^{[\mathbf{k}]}(z) = (-1)^{|\mathbf{k}|}\partial^{\mathbf{k}} L_{\alpha+\mathbf{k}}(z). \tag{5.13}$$

Changing the variable $z \to \xi z$ in the equation above, we get

$$\begin{aligned} L_\alpha^{[\mathbf{k}]}(\xi z) &= (-1)^{|\mathbf{k}|}(\partial^{\mathbf{k}} L_{\alpha+\mathbf{k}})(\xi z) \\ &= (-1)^{|\mathbf{k}|}\xi^{-\mathbf{k}}\partial^{\mathbf{k}}(L_{\alpha+\mathbf{k}}(\xi z)) \end{aligned} \tag{5.14}$$

Applying Eq. (5.9) and then Eq.(2.29):

$$= \frac{(-1)^{|\alpha|}}{(\alpha+\mathbf{k})!}\xi^{-\mathbf{k}}\partial^{\mathbf{k}}(\xi^{\alpha+\mathbf{k}} * z^{\alpha+\mathbf{k}}).$$
$$= \frac{(-1)^{|\alpha|}}{\alpha!}\xi^{-\mathbf{k}}(\xi^{\alpha+\mathbf{k}} * z^{\alpha}).$$

Hence, we get Eq. (5.7). Switching ξ and z in Eq. (5.7) and using the commutativity of the product $*$, we get Eq. (5.8). □

Corollary 5.3. *For any $\mathbf{k}, \alpha \in \mathbb{N}^n$, we have*

$$L_\alpha(z) = \frac{(-1)^{|\alpha|}}{\alpha!} \ (\xi^\alpha * z^\alpha)|_{\xi=1};$$

$$L_\alpha^{[\mathbf{k}]}(z) = \frac{(-1)^{|\alpha|}}{\alpha!} \ (\xi^{\alpha+\mathbf{k}} * z^\alpha)\big|_{\xi=1};$$

$$L_\alpha^{[\mathbf{k}]}(z) = \frac{(-1)^{|\alpha|}}{\alpha!} \ z^{-\mathbf{k}}(\xi^\alpha * z^{\alpha+\mathbf{k}})\big|_{\xi=1},$$

where $|_{\xi=1}$ denotes the evaluation map from $\mathbb{C}[\xi, z]$ to $\mathbb{C}[z]$ by setting $\xi_i = 1$ for any $1 \le i \le n$.

Remark 5.4. *Note that, the evaluation map $|_{\xi=1}$ viewed as a linear map from $\mathcal{A}[\xi, z]$ to $\mathcal{A}[z]$ is a homomorphism of algebras. But, as a linear map from the algebra $\mathcal{B}[\xi, z]$ to the polynomial algebra $\mathcal{A}[z]$, it is not a homomorphism of algebras anymore. In particular, we have*

$$(\xi^\alpha * z^\alpha)|_{\xi=1} \neq 1 * z^\alpha = z^\alpha.$$

Otherwise the generalized Laguerre polynomials would be trivialized.

Therefore, in some sense, the fact that the evaluation map $|_{\xi=1}$: $\mathcal{B}[\xi, z] \to \mathcal{A}[z]$ fails to be an algebra homomorphism causes the non-trivial, actually truly remarkable, generalized Laguerre polynomials. But, on the other hand, as we have discussed in Subsection 4.2 (See Remark 4.9), the same fact for the evaluation map at $\xi = 0$ also causes some extremely difficult open problems such as Conjecture 4.5 and the Jacobian conjecture.

Another immediate consequence of Theorem 5.2 is the following corollary.

Corollary 5.5. *For any $\alpha, \beta \in \mathbb{N}^n$, we have*

$$\xi^\beta(\xi^\alpha * z^{\alpha+\beta}) = z^\beta(\xi^{\alpha+\beta} * z^\alpha). \tag{5.15}$$

Note that the corollary follows immediately from Eqs. (5.7) and (5.8) with $\mathbf{k} = \beta$. But here we also give a more straightforward proof.

Proof: Consider

$$\begin{aligned}\xi^\beta(\xi^\alpha * z^{\alpha+\beta}) &= (\xi - \partial + \partial)^\beta(\xi^\alpha * z^{\alpha+\beta}) \\ &= \sum_{\gamma\in\mathbb{N}^n} \binom{\beta}{\gamma} (\xi - \partial)^{\beta-\gamma}\partial^\gamma(\xi^\alpha * z^{\alpha+\beta})\end{aligned}$$

Applying Eq. (2.29) and then Eq.(2.14):

$$\begin{aligned}&= \sum_{\gamma\in\mathbb{N}^n} \binom{\beta}{\gamma} \frac{(\alpha+\beta)!}{(\alpha+\beta-\gamma)!} (\xi - \partial)^{\beta-\gamma}(\xi^\alpha * z^{\alpha+\beta-\gamma}) \\ &= \sum_{\gamma\in\mathbb{N}^n} \binom{\beta}{\gamma} \frac{(\alpha+\beta)!}{(\alpha+\beta-\gamma)!} \xi^{\beta-\gamma} * (\xi^\alpha * z^{\alpha+\beta-\gamma}) \\ &= \sum_{\gamma\in\mathbb{N}^n}^{\beta} \binom{\beta}{\gamma} \frac{(\alpha+\beta)!}{(\alpha+\beta-\gamma)!} (\xi^{\alpha+\beta-\gamma} * z^{\alpha+\beta-\gamma}).\end{aligned}$$

By switching $\xi \leftrightarrow z$ in the argument above and using the commutativity of the product $*$, it is easy to see that we also have

$$z^\beta(\xi^{\alpha+\beta} * z^\alpha) = \sum_{\gamma\in\mathbb{N}^n}^{\beta} \binom{\beta}{\gamma} \frac{(\alpha+\beta)!}{(\alpha+\beta-\gamma)!} (\xi^{\alpha+\beta-\gamma} * z^{\alpha+\beta-\gamma}).$$

Hence Eq. (5.15) follows. □

5.3. **Some Applications of Theorem 5.2.** First, let us derive some identities for the exponential series $\exp_*(\cdot) = e_*^{\{\cdot\}}$ of the algebra $\mathcal{B}[\xi, z]$, i.e. the usual exponential series but with the product replaced by $*$.

Proposition 5.6. *Let $u = (u_1, u_2, ..., u_n)$ be n free commutative variables. Set $\xi * z := (\xi_1 * z_1,\ \xi_2 * z_2, ...,\ \xi_n * z_n)$ and $(\xi * z)u := \sum_{i=1}^n (\xi_i * z_i)u_i$. Then, for any $\mathbf{k} = (k_1, k_2, ..., k_n) \in \mathbb{N}^n$, we have*

$$\xi^{-\mathbf{k}}\left(\xi^{\mathbf{k}} * e_*^{-(\xi*z)u}\right) = \prod_{i=1}^n \frac{\exp(-\frac{(\xi_i z_i)\, u_i}{1-u_i})}{(1-u_i)^{k_i+1}}. \tag{5.16}$$

$$z^{-\mathbf{k}}\left(z^{\mathbf{k}} * e_*^{-(\xi*z)u}\right) = \prod_{i=1}^n \frac{\exp(-\frac{(\xi_i z_i)\, u_i}{1-u_i})}{(1-u_i)^{k_i+1}}. \tag{5.17}$$

In particular, when $\mathbf{k} = 0$*, we have the following expression of the exponential* $\exp_*(-(\xi * z)u)$*:*

$$\exp_*(-(\xi * z)u) = \prod_{i=1}^{n} \frac{\exp(-\frac{(\xi_i z_i)\, u_i}{1-u_i})}{(1-u_i)}. \tag{5.18}$$

Proof: We give a proof for Eq. (5.16). The proof of Eq. (5.17) is similar.

First, by the commutativity and associativity of the product $*$ and also by Proposition 2.6, (b), it is easy to see that, for any $\alpha, \beta \in \mathbb{N}^n$, we have

$$(\xi^\alpha * z^\alpha) * (\xi^\beta * z^\beta) = \xi^{\alpha+\beta} * z^{\alpha+\beta}. \tag{5.19}$$

$$(\xi * z)^{*\alpha} = (\xi^\alpha * z^\alpha), \tag{5.20}$$

where $(\xi * z)^{*\alpha}$ denotes the "α^{th}" power of $(\xi * z)$ with respect to the new product $*$.

By the last two equations above and Eq. (5.7), we have

$$\begin{aligned} \xi^{-\mathbf{k}} \left(\xi^{\mathbf{k}} * \exp_*^{-(\xi * z)u}\right) &= \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!}\, \xi^{-\mathbf{k}} (\xi^{\alpha+\mathbf{k}} * z^\alpha) u^\alpha \\ &= \sum_{\alpha \in \mathbb{N}^n} L_\alpha^{[\mathbf{k}]}(\xi z) u^\alpha. \end{aligned} \tag{5.21}$$

On the other hand, by Eqs. (5.2) and (5.3), we see that the generating function of the multi-variable generalized Laguerre polynomials $L_\alpha^{[\mathbf{k}]}(z)$ $(\alpha \in \mathbb{N}^n)$ is given by

$$\prod_{i=1}^{n} \frac{\exp(-\frac{z_i\, u_i}{1-u_i})}{(1-u_i)^{k_i+1}} = \sum_{\alpha \in \mathbb{N}^n} L_\alpha^{[\mathbf{k}]}(z) u^\alpha. \tag{5.22}$$

Replacing z by ξz in the equation above, we get

$$\prod_{i=1}^{n} \frac{\exp(-\frac{(\xi_i z_i)\, u_i}{1-u_i})}{(1-u_i)^{k_i+1}} = \sum_{\alpha \in \mathbb{N}^n} L_\alpha^{[\mathbf{k}]}(\xi z) u^\alpha. \tag{5.23}$$

Combining Eqs. (5.21) and (5.23), we get Eq. (5.16). □

Next we use the connection given in Theorem 5.2 to derive more properties on the monomials in ξ and z with respect to the product $*$ from certain results on the generalized Laguerre polynomials.

For convenience, for any $\alpha \in \mathbb{N}^n$, we set

$$L_\alpha(z;\xi) := \xi^\alpha * z^\alpha. \tag{5.24}$$

Note that, by Eqs. (5.1) and (5.9), the polynomials $L_\alpha(z;\xi)$ ($\alpha \in \mathbb{N}^n$) are polynomials with coefficients in $\mathbb{Q}$. In particular, for any fixed $\xi \in (\mathbb{R}_{>0})^n$, by Eqs. (5.1) and (5.9), it is easy to see that the polynomials $L_\alpha(z;\xi)$ ($\alpha \in \mathbb{N}^n$) are polynomials in z with real coefficients and form a linear basis of $\mathbb{C}[z]$.

The next proposition says that this basis is also orthogonal with respect to the following *weight* function:

$$w_\xi(z) := e^{-\langle \xi, z\rangle} \prod_{i=1}^n \xi_i, \tag{5.25}$$

Proposition 5.7. *For any $\alpha, \beta \in \mathbb{N}^n$, we have*

$$\int_{(\mathbb{R}_{>0})^n} L_\alpha(z;\xi) L_\beta(z;\xi)\, w_\xi(z)\, dz = (\alpha!)^2 \delta_{\alpha,\beta}. \tag{5.26}$$

Proof: Note that, under the change of variables $z_i \to \xi_i z_i$ ($1 \le i \le n$), by Eqs. (5.9) and (5.24) the Laguerre polynomials $L_\alpha(z)$ will be changed to

$$L_\alpha(z) \to L_\alpha(\xi z) = \frac{(-1)^{|\alpha|}}{\alpha!} L_\alpha(z;\xi). \tag{5.27}$$

By Eq. (5.25) and also Eq. (5.5) with $\mathbf{k} = 0$, the weight function $w(z)$ of the Laguerre polynomials is changed to

$$w(z) \to w_\xi(z) \prod_{i=1}^n \xi_i^{-1}. \tag{5.28}$$

Now, apply the same changing of the variables to the integral in Eq. (5.4) with $\mathbf{k} = 0$, by the last two equations above, we get

$$\delta_{\alpha,\beta} = \frac{(-1)^{|\alpha+\beta|}}{\alpha!\beta!} \int_{(\mathbb{R}_{>0})^n} L_\alpha(z;\xi) L_\beta(z;\xi)\, w_\xi(z)\, dz \tag{5.29}$$

Hence Eq. (5.26) follows. □

Denote by $\mathcal{A}_{\mathbb{Q}}[\xi, z]$ the polynomial algebra in ξ and z over $\mathbb{Q}$. Next we assume $n = 1$ and consider the irreducibility of the polynomial $L_\alpha(z;\xi)$ ($\alpha \in \mathbb{N}^n$) as elements of $\mathcal{A}_{\mathbb{Q}}[\xi, z]$. But, first, we need to prove the following lemma.

Lemma 5.8. *Let ξ and z be two commutative free variables and K any field. Then, for any $f(z) \in K[z]$ with $\deg f \ge 2$, $f(z)$ is irreducible over K iff $f(\xi z) \in K[\xi, z]$ (as a polynomial in two variables) is irreducible over K.*

Proof: The ($\Leftarrow$) part of the lemma is trivial. We use the contradiction method to show the ($\Rightarrow$) part of the lemma.

Assume that $f(\xi z)$ is reducible in $K[\xi, z]$. Write

$$f(\xi z) = g(\xi, z)h(\xi, z) \tag{5.30}$$

for some $g(\xi, z), h(\xi, z) \in K[\xi, z]$ with $\deg g, \deg h \geq 1$.

Setting $\xi = 1$ in the equation above, we also have

$$f(z) = g(1, z)h(1, z). \tag{5.31}$$

Let $\bar{K}$ be the algebraic closure of K. Write $f(z) = b\prod_{i=1}^{d}(z - a_i)$ for some $b \in K\backslash\{0\}$ and $a_i \in \bar{K}$ $(1 \leq i \leq d)$. Then we have

$$f(\xi z) = b\prod_{i=1}^{d}(\xi z - a_i). \tag{5.32}$$

Since $f(z)$ is irreducible over K and $\deg f \geq 2$ by the assumption, we have $a_i \neq 0$ $(1 \leq i \leq d)$. Hence, for each i, $\xi z - a_i$ is irreducible in $\bar{K}[\xi, z]$. Then by Eqs. (5.30) and (5.32), we have

$$g(\xi, z) = c\prod_{k=1}^{m}(\xi z - a_{i_k}) \tag{5.33}$$

for some $c \in \bar{K}\backslash\{0\}$, $1 \leq m < d$ and $1 \leq i_1 < i_2 < \cdots < i_m \leq d$.

However, the equation above implies $g(1, z) = c\prod_{k=1}^{m}(z - a_{i_k})$. Since $g(\xi, z) \in K[\xi, z]$, we also have $g(1, z) \in K[z]$. Then by Eq. (5.31), $g(1, z)$ is a divisor of $f(z)$ in $K[z]$ with $1 \leq \deg g(1, z) = m < d = \deg f(z)$, which contradicts to the assumption that $f(z)$ is irreducible in $K[z]$. $\square$

Theorem 5.9. *Let ξ and z be two commutative free variables. For any $m \geq 2$, $L_m(z; \xi) = \xi^m * z^m$ is irreducible in $\mathcal{A}_{\mathbb{Q}}[\xi, z]$.*

Proof: By a theorem proved by I. Schur [Sc1], we know that, for any $m \geq 1$, the Laguerre polynomials $L_m(z)$ in one variable is irreducible over $\mathbb{Q}$. Hence, by Eq. (5.9) and Lemma 5.8, the theorem holds. $\square$

Note that I. Schur also proved in [Sc2] that the generalized Laguerre polynomials $L_m^{[1]}(z)$ $(m \geq 0)$ in one variable are also irreducible over $\mathbb{Q}$. Furthermore, M. Filaseta and T.-Y. Lam proved in [FL] that, for any non-negative $k \in \mathbb{Q}$, all but finitely many of the generalized Laguerre polynomials $L_m^{[k]}(z)$ $(m \geq 0)$ in one variable are irreducible over $\mathbb{Q}$. Hence, by a similar argument as for Theorem 5.9, we also have the following theorem.

Theorem 5.10. *Let ξ and z be two commutative free variables. Then, for any $k \in \mathbb{N}$, all but only finitely many of the polynomials $z^{-k}(\xi^m * z^{m+k})$ and $\xi^{-k}(\xi^{m+k} * z^m)$ $(m \in \mathbb{N})$ are irreducible over $\mathbb{Q}$.*

Next, we re-prove some important properties of the generalized Laguerre polynomials by using their expressions given in Theorem 5.2. For simplicity, we here only consider the one-variable case. Similar results for the multi-variable generalized Laguerre polynomials can be simply derived from the one-variable case via Eq. (5.3).

First, let us look at the following recurrent formulas of the Laguerre polynomials in one variable.

Proposition 5.11. *For any $m \geq 1$, we have*

$$(m+1)L_{m+1}(z) = (2m+1-z)L_m(z) - mL_{m-1}(z), \tag{5.34}$$

$$zL'_m(z) = m(L_m(z) - L_{m-1}(z)) \tag{5.35}$$

Proof: Note first that, for any $m \geq 1$, by Eqs. (2.14) and (2.15), we have

$$\xi * z^m = (\xi - \partial)z^m = \xi z^m - mz^{m-1},$$
$$z * \xi^m = (z - \delta)\xi^m = z\xi^m - m\xi^{m-1}.$$

Hence, we also have

$$\xi * z = \xi z - 1,$$
$$\xi z^m = \xi * z^m + mz^{m-1},$$
$$z\xi^m = z * \xi^m + m\xi^{m-1}.$$

By the last three equations above and also Eq. (2.23), we have

$$\begin{aligned}(\xi z - 1)(\xi^m * z^m) &= (\xi * z)(\xi^m * z^m) = (z\xi^m) * (\xi z^m)\\ &= (z * \xi^m + m\xi^{m-1}) * (\xi * z^m + mz^{m-1})\\ &= \xi^{m+1} * z^{m+1} + 2m\,\xi^m * z^m + m^2\xi^{m-1} * z^{m-1}.\end{aligned}$$

Multiply $(-1)^m/m!$ to the equation above and then apply Eq. (5.9), we get

$$(\xi z - 1)L_m(\xi z) = -(m+1)L_{m+1}(\xi z) + 2mL_m(\xi z) - mL_{m-1}(\xi z).$$

Replace ξz by z in the equation above, we get

$$(z-1)L_m(z) = -(m+1)L_{m+1}(z) + 2mL_m(z) - mL_{m-1}(z),$$

Hence Eq. (5.34) follows.

To show Eq. (5.35), by Eqs. (2.15) and (2.30), we have,

$$\xi^m * z^m = z * (\xi^m * z^{m-1}) = (z - \delta)(\xi^m * z^{m-1})$$

$$= z(\xi^m * z^{m-1}) - m(\xi^{m-1} * z^{m-1})$$
$$= \frac{1}{m} z\partial(\xi^m * z^m) - m(\xi^{m-1} * z^{m-1}).$$

Multiply $(-1)^m/m!$ to the equation above and then apply Eq. (5.9), we get

$$L_m(\xi z) = \frac{1}{m} z\partial(L_m(\xi z)) + L_{m-1}(\xi z) = \frac{1}{m}\xi z L'_m(\xi z) + L_{m-1}(\xi z).$$

Replace ξz by z in the equation above, we get

$$L_m(z) = \frac{1}{m} z L'_m(z) + L_{m-1}(z).$$

Hence Eq. (5.35) follows. □

Next, we give a new proof for the following important property of the generalized Laguerre polynomials in one variable.

Theorem 5.12. *For any $k, m \in \mathbb{N}$, $L_m^{[k]}(z)$ solves the following so-called associated Laguerre differential equation:*

$$zf''(z) + (k+1-z)f'(z) + mf(z) = 0. \tag{5.36}$$

Proof: First, by Eq. (5.1), we have $L_0^{[k]}(z) = 1$. It is easy to see that the theorem holds for this case.

Assume $m \geq 1$. Then, by Eq. (2.15), we have

$$\xi(\xi^{m+k} * z^m) = \xi(z * (\xi^{m+k} * z^{m-1}))$$
$$= \xi(z-\delta)(\xi^{m+k} * z^{m-1})$$
$$= \xi z(\xi^{m+k} * z^{m-1}) - \xi\delta(\xi^{m+k} * z^{m-1}).$$

Add $z\partial(\xi^{m+k} * z^{m-1})$ to the equation above and apply Eq. (2.21), we have

$$\xi(\xi^{m+k} * z^m) + z\partial(\xi^{m+k} * z^{m-1})$$
$$= \xi z(\xi^{m+k} * z^{m-1}) - (\xi\delta - z\partial)(\xi^{m+k} * z^{m-1})$$
$$= (\xi z - k - 1)(\xi^{m+k} * z^{m-1}).$$

By Eq. (2.29), we may re-write the equation above as

$$\xi(\xi^{m+k} * z^m) + \frac{1}{m} z\partial^2(\xi^{m+k} * z^m) = \frac{1}{m}(\xi z - k - 1)\partial(\xi^{m+k} * z^m).$$

Multiply $\frac{(-1)^m \xi^{-k-1}}{(m-1)!}$ to both sides of the equation above and then apply Eq. (5.7), we have

$$mL_m^{[k]}(\xi z) + z\xi^{-1}\partial^2(L_m^{[k]}(\xi z)) = (\xi z - k - 1)\xi^{-1}\partial(L_m^{[k]}(\xi z)).$$

By the Chain rule, the equation above is same as

$$mL_m^{[k]}(\xi z) + z\xi(\partial^2 L_m^{[k]})(\xi z) = (\xi z - k - 1)(\partial L_m^{[k]})(\xi z).$$

Replace ξz by z, or z by $\xi^{-1}z$ in the equation above, we get

$$mL_m^{[k]}(z) + z\partial^2 L_m^{[k]}(z) = (z - k - 1)\partial L_m^{[k]}(z).$$

Hence we have proved the theorem. $\square$

Finally, let us point out the following conjecture on the generalized Laguerre polynomials, which is a special case of Conjecture 3.5 in [Z4] for all the classical orthogonal polynomials.

Conjecture 5.13. *For any* $\mathbf{k} \in \mathbb{N}^n$*, the subspace* $\mathcal{M}$ *of the polynomial algebra* $\mathcal{A}[z]$ *spanned by the generalized Laguerre polynomials* $L_\alpha^{[\mathbf{k}]}(z)$ $(0 \neq \alpha \in \mathbb{N}^n)$ *is a Mathieu subspace of* $\mathcal{A}[z]$*.*

Despite the vast amount of known results on the generalized Laguerre polynomials in the literature, the conjecture above is even still open for the classical Laguerre polynomials, (i.e. the case with $\mathbf{k} = 0$) in one variable.

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