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NEW PROOFS FOR THE ABHYANKAR-GURJAR INVERSION FORMULA AND THE EQUIVALENCE OF THE JACOBIAN CONJECTURE AND THE VANISHING CONJECTURE

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Abstract. We first give a new proof and also a new formulation for the Abhyankar-Gurjar inversion formula for formal maps of affine spaces. We then use the reformulated Abhyankar-Gurjar formula to give a more straightforward proof for the equivalence of the *Jacobian conjecture* with a special case of the *vanishing conjecture* of (homogeneous) quadratic differential operators with constant coefficients.

1. Introduction

Let K be a field of characteristic zero and $z = (z_1, z_2, \ldots, z_n)$ n commutative free variables. We denote by $\mathcal{A}_K[z]$ (resp., $\mathcal{A}_K[[z]]$) the algebra over K of polynomials (resp., formal power series) in z. Set $\partial_i := \partial_{z_i}$ $(1 \le i \le n)$ and $\partial := (\partial_1, \partial_2, ..., \partial_n)$.

Recall that the *Jacobian conjecture* posed by O. H. Keller [K] in 1939 claims that *any polynomial map* $F = (F_1, F_2, ..., F_n)$ *of* K^n *with Jacobian* $\mathcal{J}F(z) := \det(\partial_j F_i) \equiv 1$ *must be an automorphism of* K^n. Despite intense study from mathematicians in the last seventy years, the conjecture is still open even for the case $n = 2$. In 1998, S. Smale [S] included the *Jacobian conjecture* in his list of 18 fundamental mathematical problems for the 21st century. For more history and known results on the *Jacobian conjecture*, see [BCW], [E] and references therein.

In the study of the *Jacobian conjecture*, the following two remarkable reductions play very important roles.

The first one is the so-called *homogeneous reduction*, which was achieved independently by Bass, Connell and Wright [BCW] and Jagžev

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[J]. The reduction says that in order to prove or disprove the *Jacobian conjecture*, it suffices to study of polynomial maps of the form $F = z - H(z)$ with each component $H_i(z)$ of $H(z) \in \mathcal{A}_K[z]^{\times n}$ being zero or homogeneous of degree three.

The second reduction is the so-called *symmetric reduction*, which was achieved independently by de Bondt and van den Essen [BE] and Meng [M]. This reduction says that the Jacobian matrix of the polynomial maps of the form above may be further assumed to be symmetric.

Based on the two reductions above and some results in [Z2] on a deformation of polynomial maps, the author showed in [Z3] and [Z4] that the *Jacobian conjecture* is actually equivalent to the following *vanishing conjecture* on quadratic differential operators of $\mathcal{A}_K[z]$, i.e., the differential operators which can be written (uniquely) as quadratic forms in ∂ over K.

Conjecture 1.1. (**The Vanishing Conjecture**) *Let $P(z) \in \mathcal{A}_K[z]$ and Λ a quadratic differential operator of $\mathcal{A}_K[z]$ with $\Lambda^m(P^m) = 0$ for all $m \geq 1$. Then we have $\Lambda^m(P^{m+1}) = 0$ when $m \gg 0$.*

Actually, as shown in [Z4], the *Jacobian conjecture* will follow if one can just show that Conjecture 1.1 holds for a (single) sequence of quadratic differential operators $\{\Lambda_n \mid n \in \mathbb{N}\}$ such that the rank $\operatorname{rk} \Lambda_n \to \infty$ as $n \to \infty$, where the *rank* $\operatorname{rk} \Lambda_n$ of quadratic differential operators is defined to be the rank of the corresponding quadratic forms. For the study of some more general forms of Conjecture 1.1, see [Z4], [EZ], [EWZ] and [Z5].

Another important result on the *Jacobian conjecture* is the Abhyankar-Gurjar inversion formula, which provides a nice formula for the formal inverse maps of polynomial maps (see Theorem 3.1). This formula was first proved by Gurjar (unpublished) and later by Abhyankar [A] in a simplified form (see also [BCW]).

One remark on the Abhyankar-Gurjar inversion formula is as follows. By comparing Eq. (3.1) with Eq. (3.5), we see that up to the factor $\mathcal{J}F$ (the Jacobian of the polynomial map F), the differential operator in the Abhyankar-Gurjar inversion formula coincides with the one obtained by applying the well-known anti-involution τ (see Eq. (2.16)) of the Weyl algebra to the differential operator in the Taylor series expansion of $u(F) = u(z - H)$ at z (see Eqs. (3.5)). In particular, for polynomial maps F that satisfies the Jacobian condition $\mathcal{J}F \equiv 1$, these two differential operators becomes identical! But, unfortunately, from the proof given in [A] (see also [BCW]), which to the author's best knowledge is also the only published proof in the literature, it is not easy to see why such a resemblance or connection exists at the first place.

In this paper, we first use some results in [Z1] to give a new proof for the Abhyankar-Gurjar inversion formula, which provides a good explanation for the resemblance or connection mentioned above. We then use a result in [Z6] to give a new formulation for the Abhyankar-Gurjar inversion formula, and also derive some of its consequences. Finally, by using this re-formulated Abhyankar-Gurjar inversion formula and some of its consequences derived in this paper, we give a new proof for the equivalence of the *Jacobian conjecture* with a special case of *vanishing conjecture*, Conjecture 1.1 (see Theorem 4.6).

Comparing with the proof of the equivalence of the *Jacobian conjecture* and Conjecture 1.1 given in [Z3] and [Z4], the proof given here is more straightforward. But, contrast to the proof in [Z3] and [Z4], it does not show the equivalence of the *Jacobian conjecture* with Conjecture 1.1 itself but only with a special case of it.

The arrangement of this paper is as follows.

In Section 2, we discuss some $\mathbb{Z}$-gradings and the associated completions of polynomial algebras and the Weyl algebras. The main aims of this section are, first, to lay down a firm ground setting for the arguments in the rest of paper and, second, to show that certain results on polynomial algebras and the Weyl algebras can be extended to their completions constructed in this section. These extended results such as Proposition 2.3 and Lemma 2.4 will play crucial roles in our later arguments.

In Section 3, we first give a new proof and also a new formulation for the Abhyankar-Gurjar inversion formula. We then derive some consequences from the re-formulated Abhyankar-Gurjar inversion formula, which will be needed in the next section.

In Section 4, we give a new proof for the equivalence of the *Jacobian conjecture* with a special case of the *vanishing conjecture*. In addition, a special case of a conjecture proposed in [Z3] is also proved here (see Conjecture 4.4 and Proposition 4.3).

Finally, one remark on the base field of this paper is as follows.

Unless stated otherwise, the base field K throughout the paper is assumed to be a field of characteristic zero (instead of the complex number field $\mathbb{C}$). This more general choice of the base field is necessary, or at least convenient, for some base field extension arguments in the paper. Note also that some of the results quoted in this paper were only proved over the complex number field $\mathbb{C}$ in the quoted references. But, either by applying Lefschetz's principle or by going through their proofs (which actually work equally well for all the fields of characteristic zero), the reader can easily see that these quoted results also hold as long as the base field has characteristic zero. The same remark will

not be repeated for each of these quoted results in the context of this paper.

2. Some $\mathbb{Z}$-gradings and Associated Completions of Polynomial Algebras and the Weyl Algebras

Let K be any field of characteristic zero and $\xi = (\xi_1, ..., \xi_n)$ and $z = (z_1, ..., z_n)$ $2n$ free commutative variables. We denote by $\mathcal{A}_K[\xi]$, $\mathcal{A}_K[z]$ and $\mathcal{A}_K[\xi, z]$ the polynomial algebras over K in ξ, z and (ξ, z), respectively. The corresponding formal power series algebras will be denoted respectively by $\mathcal{A}_K[[\xi]]$, $\mathcal{A}_K[[z]]$ and $\mathcal{A}_K[[\xi, z]]$. We denote by $\mathcal{D}_K[z]$ the differential operator algebra or the Weyl algebra of the polynomial algebra $\mathcal{A}_K[z]$.

In this section, we consider some $\mathbb{Z}$-gradings on the polynomial algebra $\mathcal{A}_K[\xi, z]$ and the Weyl algebra $\mathcal{D}_K[z]$, and also the associated completions of $\mathcal{A}_K[\xi, z]$ and $\mathcal{D}_K[z]$ with respected to these $\mathbb{Z}$-gradings. All formal power series and differential operators involved in this paper will lie in the completions constructed in this section for $\mathcal{A}_K[\xi, z]$ and $\mathcal{D}_K[z]$, respectively. We will also show that certain results on $\mathcal{A}_K[\xi, z]$ and $\mathcal{D}_K[z]$ can be extended to the associated completions of $\mathcal{A}_K[\xi, z]$ and $\mathcal{D}_K[z]$.

But, first, let us fix the following notations and conventions which along with those fixed in the previous section will be used throughout this paper.

(1) We denote by $\langle \cdot, \cdot \rangle$ the standard bi-linear paring of n-tuples of elements of algebras over K. For example, $\langle \xi, z \rangle = \sum_{i=1}^{n} \xi_i z_i$.

(2) We will freely use some standard multi-index notations. For instance, let $\alpha = (k_1, k_2, ..., k_n)$ and $\beta = (m_1, m_2, ..., m_n)$ be any n-tuples of non-negative integers, we have

$$|\alpha| = \sum_{i=1}^{n} k_i.$$

$$\alpha! = k_1! k_2! \cdots k_n!.$$

$$\binom{\alpha}{\beta} = \begin{cases} \frac{\alpha!}{\beta!(\alpha-\beta)!} & \text{if } k_i \geq m_i \text{ for all } 1 \leq i \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

(3) For any $k \geq 1$ and $h(z) = (h_1(z), h_2(z), \cdots, h_k(z)) \in \mathcal{A}_K[[z]]^{\times k}$, we define the *order*, denoted by $o(h(z))$, of $h(z)$ by

$$o(h(z)) := \min_{1 \leq i \leq k} o(h_i(z)),$$

and when $h(z) \in \mathcal{A}_K[z]^{\times k}$, the *degree* of $\deg h(z)$ of $h(z)$ by

$$\deg h(z) := \max_{1 \le i \le k} \deg h_i(z).$$

For any $h_t(z) \in \mathcal{A}_{K[t]}[z]^{\times k}$ or $\mathcal{A}_{K[t]}[z]^{\times k}$ $(k \ge 1)$ for some formal parameter or variable t, the notation $o(h_t(z))$ and $\deg h_t(z)$ stand for the *order* and the *degree* of $h_t(z)$ with respect to z, respectively.

(4) For any differential operator $\phi \in \mathcal{D}_K[z]$ and polynomial $u(z) \in \mathcal{A}_K[z]$, the notation $\phi\, u(z)$ usually denotes the composition of ϕ and the multiplication operator by $u(z)$. So $\phi\, u(z)$ is still a differential operator of $\mathcal{A}_K[z]$. The polynomial obtained by applying ϕ to $u(z)$ will be denoted by $\phi(u(z))$ or $\phi \cdot u(z)$.

Next, we recall some general facts on $\mathbb{Z}$-graded algebras and their associated completions.

Let V be a vector space over K. We say that V is $\mathbb{Z}$-*graded* by its subspaces V_m $(m \in \mathbb{Z})$ if $V = \oplus_{m \in \mathbb{Z}} V_m$, i.e., every $v \in V$ can be written uniquely as $v = \sum_{m \in \mathbb{Z}} v_m$ with $v_m \in V_m$ $(m \in \mathbb{Z})$ and all but finitely many $v_m = 0$.

For any $\mathbb{Z}$-graded vector V as above, we define the *associated completion* $\overline{V}$ to be the vector space of the elements of the form

$$u = \sum_{m \in \mathbb{Z}} u_m \tag{2.1}$$

such that

1) $u_m \in V_m$ for all $m \in \mathbb{Z}$;
2) $u_m = 0$ when m is large negative enough, i.e., there exists $N \in \mathbb{Z}$ (depending on u) such that $u_m = 0$ for all $m \le N$.

It is easy to see that the addition and the scalar product of the vector space V extend to $\overline{V}$ in the obvious way, with which $\overline{V}$ forms a vector space over K and contains V as a vector subspace.

For any $\mathbb{Z}$-graded vector space V (as above), we define the associated *grading function*, or simply the *associated grading* to be the function $\eta : \overline{V} \to \mathbb{Z} \cup \{+\infty\}$ such that $\eta(0) = +\infty$ and, for any $0 \ne u \in \overline{V}$ as in Eq. (2.1),

$$\eta(u) = \min \{m \in \mathbb{Z} \mid u_m \ne 0\}. \tag{2.2}$$

Note that for any $m \in \mathbb{Z}$ and $0 \ne u \in V_m$, we have $\eta(u) = m$. Therefore, we also say elements of V_m are η-*homogeneous* of η-grading m. The η-homogeneous elements u_m $(m \in \mathbb{Z})$ in Eq. (2.1) is called the η-*homogeneous part* of the element $u \in \overline{V}$ of η-grading m. Very often,

we also say that the vector space $\overline{V}$ is the *completion* of V with respect to the grading $\eta|_V : V \to \mathbb{Z} \cup \{+\infty\}$.

Now let $\mathcal{A}$ be any associative (but not necessarily commutative) algebra over K. We say that $\mathcal{A}$ is $\mathbb{Z}$-*graded* by its subspaces A_m $(m \in \mathbb{Z})$ if $\mathcal{A}$ as a vector space is $\mathbb{Z}$-graded by A_m $(m \in \mathbb{Z})$ and, for any $m, k \in \mathbb{Z}$, we further have

$$A_m \cdot A_k \subset A_{m+k}. \tag{2.3}$$

Let $\bar{\mathcal{A}}$ be the associated completion of $\mathcal{A}$ as a $\mathbb{Z}$-graded vector space. Then the product of $\mathcal{A}$ can be extended to $\bar{\mathcal{A}}$ in the obvious way and, with the extended product, $\bar{\mathcal{A}}$ forms an associative algebra which contains $\mathcal{A}$ as a subalgebra.

Let $\mathcal{A}$ be a $\mathbb{Z}$-graded algebra and $\mathcal{M}$ an $\mathcal{A}$-module. We say that $\mathcal{M}$ is a $\mathbb{Z}$-*graded* $\mathcal{A}$-module if $\mathcal{M}$ as a vector space is $\mathbb{Z}$-graded by some of its vector subspaces M_m $(m \in \mathbb{Z})$ such that for all $m, k \in \mathbb{Z}$, we have

$$A_m \cdot M_k \subset M_{m+k}. \tag{2.4}$$

Furthermore, let $\overline{M}$ be the associated completion of $\mathcal{M}$. Then the $\mathcal{A}$-module structure on $\mathcal{M}$ can be extended naturally to an $\mathcal{A}$-module structure on $\overline{M}$. This $\mathcal{A}$-module structure on $\overline{M}$ can be further extended in the obvious way to an $\bar{\mathcal{A}}$-module structure.

Example 2.1. *a) The polynomial algebra $\mathcal{A}_K[z]$ is $\mathbb{Z}$-graded by the degree of polynomials in $\mathcal{A}_K[z]$. More precisely, for any $m \in \mathbb{Z}$, set $A_m[z] = 0$ if $m < 0$, and $A_m[z]$ the subspace of homogeneous polynomials in z of degree m. Then it is easy to see that $\mathcal{A}_K[z]$ is $\mathbb{Z}$-graded by the subspaces $A_m[z]$ $(m \in \mathbb{Z})$ and the associated completion $\bar{\mathcal{A}}_K[z]$ of $\mathcal{A}_K[z]$ is the formal power series algebra $\mathcal{A}_K[[z]]$ in z over K.*

b) Let $\mathcal{A}_K[z, z^{-1}]$ denote the algebra of Laurent polynomials in z over K. Then $\mathcal{A}_K[z, z^{-1}]$ is also $\mathbb{Z}$-graded by the (generalized) degree of Laurent polynomial in z, i.e., counting the degree of z_i^{-1} by -1 for each $1 \le i \le n$. The associated completion of $\mathcal{A}_K[z, z^{-1}]$ with respect to this $\mathbb{Z}$-grading is the algebra of Laurent power series in z over K.

Besides the one in the example *a)* above, in this paper we also need to work over the following two $\mathbb{Z}$-graded algebras.

The first one is the polynomial algebra $\mathcal{A}_K[\xi, z]$ over K in the $2n$ variables ξ_i and z_i $(1 \le i \le n)$ fixed at the beginning of this section. For any $m \in \mathbb{Z}$, we set

$$A_m[\xi, z] := \mathrm{Span}_K\{\xi^\alpha z^\beta \mid \alpha, \beta \in \mathbb{N}^n; |\beta| - |\alpha| = m\}. \tag{2.5}$$

Then it is easy to see that $\mathcal{A}_K[\xi, z]$ is $\mathbb{Z}$-graded by the subspaces $A_m[\xi, z]$ $(m \in \mathbb{Z})$ and its associated completion, denoted by $\bar{\mathcal{A}}_K[\xi, z]$, is

a proper subalgebra of the formal power series $\mathcal{A}_K[[\xi, z]]$. Note that the associated completion $\bar{\mathcal{A}}_K[\xi, z]$ contains the formal power series algebra $\mathcal{A}_K[[z]]$ in z as a subalgebra. But it does not contain the formal power series algebra $\mathcal{A}_K[[\xi]]$ in ξ.

Let $\eta : \bar{\mathcal{A}}_K[\xi, z] \to \mathbb{Z} \cup \{+\infty\}$ be the associated $\mathbb{Z}$-grading of $\mathcal{A}_K[\xi, z]$. Then for any $\alpha, \beta \in \mathbb{N}^n$, we have

$$\eta(\xi^\alpha z^\beta) = |\beta| - |\alpha|. \tag{2.6}$$

More generally, for any $f(\xi, z) \in \bar{\mathcal{A}}[\xi, z]$,

$$\eta(f(\xi, z)) = \min\{|\beta| - |\alpha| \mid \alpha, \beta \in \mathbb{N}^n; [\xi^\alpha z^\beta] f(\xi, z) \neq 0\}, \tag{2.7}$$

where $[\xi^\alpha z^\beta] f(\xi, z)$ denotes the coefficient of the monomial $\xi^\alpha z^\beta$ in $f(\xi, z)$.

The second $\mathbb{Z}$-graded algebra that we will need later is the Weyl algebra $\mathcal{D}_K[z]$ of the polynomial algebra $\mathcal{A}_K[z]$.

Note first that it is well-known (e.g., by using Proposition 2.2 (p. 4) in [B1] or Theorem 3.1 (p. 58) in [C]) that the differential operators $z^\beta \partial^\alpha$ $(\alpha, \beta \in \mathbb{N}^n)$ form a linear basis of $\mathcal{D}_K[z]$. So do the differential operators $\partial^\alpha z^\beta$ $(\alpha, \beta \in \mathbb{N}^n)$.

For any $m \in \mathbb{Z}$, we set

$$D_m[z] := \mathrm{Span}_K\{z^\beta \partial^\alpha \mid \alpha, \beta \in \mathbb{N}^n; |\beta| - |\alpha| = m\}. \tag{2.8}$$

Then it is easy to see that the Weyl algebra $\mathcal{D}_K[z]$ is $\mathbb{Z}$-graded by the subspaces $D_m[z]$ $(m \in \mathbb{Z})$. Actually, this grading is the same as the grading in the Kashiwara-Malgrange V-filtration along the origin (see [B2], [D]).

We denote by $\overline{\mathcal{D}}_K[z]$ the associated completion of $\mathcal{D}_K[z]$ and $\nu : \overline{\mathcal{D}}_K[z] \to \mathbb{Z} \cup \{+\infty\}$ the associated $\mathbb{Z}$-grading of $\mathcal{D}_K[z]$. Then for any $\alpha, \beta \in \mathbb{N}^n$, we have

$$\nu(z^\beta \partial^\alpha) = |\beta| - |\alpha|. \tag{2.9}$$

Note also that for any $\alpha, \beta \in \mathbb{N}^n$, we also have

$$\nu(\partial^\alpha z^\beta) = |\beta| - |\alpha|. \tag{2.10}$$

This is because by the Leibniz rule, we have

$$\partial^\alpha z^\beta = \sum_{\gamma \in \mathbb{N}^n} \binom{\alpha}{\gamma} (\partial^\gamma (z^\beta)) \partial^{\alpha - \gamma},$$

and each differential operator $(\partial^\gamma (z^\beta)) \partial^{\alpha-\gamma}$ with a non-zero coefficient in the sum above has ν-grading $|\beta| - |\alpha|$.

With the $\mathbb{Z}$-grading of the polynomial algebra $\mathcal{A}_K[z]$ as in Example 2.1, *a*) and the $\mathbb{Z}$-grading of the Weyl algebra $\mathcal{D}_K[z]$ defined above, it

is easy to see that $\mathcal{A}_K[z]$ is a $\mathbb{Z}$-graded $\mathcal{D}_K[z]$-module. Therefore, the associated completion $\bar{\mathcal{A}}_K[z](=\mathcal{A}_K[[z]])$ of $\mathcal{A}_K[z]$ becomes a module of the associated completion $\overline{\mathcal{D}}_K[z]$ of the Weyl algebra $\mathcal{D}_K[z]$. In other words, we have a well-defined action of $\overline{\mathcal{D}}_K[z]$ on the formal power series algebra $\mathcal{A}_K[[z]]$.

Next, we consider the *right* and *left total symbols* of the differential operators in the associated completion $\overline{\mathcal{D}}_K[z]$ of the Weyl algebra $\mathcal{D}_K[z]$.

First let us recall the *right* and *left total symbols* of the differential operators in $\mathcal{D}_K[z]$, i.e., the differential operators of the polynomial algebra $\mathcal{A}_K[z]$.

For any $\phi \in \mathcal{D}_K[z]$, we can write ϕ uniquely as the following two finite sums:

$$\phi = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(z)\partial^\alpha = \sum_{\beta\in\mathbb{N}^n} \partial^\beta b_\beta(z) \tag{2.11}$$

where $a_\alpha(z), b_\beta(z) \in \mathcal{A}_K[z]$ but denote the multiplication operators by $a_\alpha(z)$ and $b_\beta(z)$, respectively.

For the differential operator $\phi \in \mathcal{D}_K[z]$ in Eq. (2.11), its *right* and *left total symbols* are defined to be the polynomials $\sum_{\alpha\in\mathbb{N}^n} a_\alpha(z)\xi^\alpha \in \mathcal{A}_K[\xi, z]$ and $\sum_{\beta\in\mathbb{N}^n} b_\beta(z)\xi^\beta \in \mathcal{A}_K[\xi, z]$, respectively. We denote by $\mathcal{R} : \mathcal{D}[z] \to \mathcal{A}_K[\xi, z]$ (resp., $\mathcal{L} : \mathcal{D}[z] \to \mathcal{A}_K[\xi, z]$) the linear map which maps any $\phi \in \mathcal{D}[z]$ to its right (resp., left) total symbol.

Furthermore, from the definitions of the $\mathbb{Z}$-gradings of $\mathcal{A}_K[\xi, z]$ and $\mathcal{D}_K[z]$, and also Eq. (2.10), it is easy to see that both $\mathcal{R}$ and $\mathcal{L}$ are $\mathbb{Z}$-grading preserving, i.e., for any $m \in \mathbb{Z}$, we have

$$\mathcal{R}(D_m[z]) \subset A_m[\xi, z]. \tag{2.12}$$

$$\mathcal{L}(D_m[z]) \subset A_m[\xi, z]. \tag{2.13}$$

Since $\mathcal{R}$ and $\mathcal{L}$ are isomorphisms, for each $m \in \mathbb{Z}$ the restrictions of $\mathcal{R}$ and $\mathcal{L}$ on $D_m[z]$ are also isomorphisms from $D_m[z]$ to $A_m[\xi, z]$. Hence, the following lemma holds.

Lemma 2.2. *$\mathcal{R}$ and $\mathcal{L}$ extend to $\mathbb{Z}$-grading preserving linear isomorphisms from the completion $\overline{\mathcal{D}}_K[z]$ of $\mathcal{D}_K[z]$ to the completion $\bar{\mathcal{A}}_K[\xi, z]$ of $\mathcal{A}_K[\xi, z]$.*

We will still denote by $\mathcal{R}$ and $\mathcal{L}$ the extended isomorphisms of $\mathcal{R}$ and $\mathcal{L}$, respectively. For any $\phi \in \overline{\mathcal{D}}_K[z]$, the images $\mathcal{R}(\phi)$ and $\mathcal{L}(\phi)$ will also be called respectively the *right* and *left total symbols* of ϕ.

Next we introduce the following linear automorphism Φ of $\mathcal{A}_K[\xi, z]$. Set

$$\Lambda := \sum_{i=1}^{n} \partial_{\xi_i}\partial_{z_i}, \tag{2.14}$$

$$\Phi := e^{\Lambda} = \sum_{m\geq 0} \frac{\Lambda^m}{m!}. \tag{2.15}$$

First, note that Φ is a well-defined linear map from $\mathcal{A}_K[\xi, z]$ to $\mathcal{A}_K[\xi, z]$. This is because Λ is locally nilpotent, i.e., for any $f(\xi, z) \in \mathcal{A}_K[\xi, z]$, $\Lambda^m f = 0$ when $m \gg 0$. Also, Φ is invertible with the inverse map given by $e^{-\Lambda}$.

Second, it is easy to see that Λ is a linear endomorphism of $\mathcal{A}_K[\xi, z]$ and preserves the η-grading of $\mathcal{A}_K[\xi, z]$ given in Eqs. (2.6) and (2.7). Hence, so does Φ. Therefore, Φ extends to an automorphism, still denoted by Φ, of the associated completion $\bar{\mathcal{A}}_K[\xi, z]$ of $\mathcal{A}_K[\xi, z]$.

Proposition 2.3. *For the extended isomorphisms Φ, $\mathcal{R}$ and $\mathcal{L}$ introduced above, we have $\Phi = \mathcal{R} \circ \mathcal{L}^{-1}$ (as linear automorphisms of $\bar{\mathcal{A}}_K[\xi, z]$).*

Proof: Note first that by Lemma 2.2 and the discussion before the proposition, we know that both Φ and $\mathcal{R}\circ\mathcal{L}^{-1}$ are η-grading preserving automorphisms of $\bar{\mathcal{A}}_K[\xi, z]$.

Second, by Theorem 3.1 in [Z6], we know that the restrictions of Φ and $\mathcal{R} \circ \mathcal{L}^{-1}$ on the subalgebra $\mathcal{A}_K[\xi, z] \subset \bar{\mathcal{A}}_K[\xi, z]$ coincide. In particular, their restrictions on the η-homogeneous subspaces $A_m[\xi, z]$ ($m \in \mathbb{Z}$) in Eq.(2.5) of $\mathcal{A}_K[\xi, z]$ coincide. Hence, as linear automorphisms of $\bar{\mathcal{A}}_K[\xi, z]$, Φ and $\mathcal{R} \circ \mathcal{L}^{-1}$ must also be same. $\square$

Note that even though the base field in [Z6] is the complex number field $\mathbb{C}$, the proof of Theorem 3.1 in [Z6] works equally well for all the fields of characteristic zero.

Finally, let us give the following lemma that will be needed later.

Let $\tau : \mathcal{D}_K[z] \to \mathcal{D}_K[z]$ be the linear map defined by setting

$$\tau(h(z)\partial^\alpha) = (-1)^{|\alpha|}\partial^\alpha h(z) \tag{2.16}$$

for each $h(z) \in \mathcal{A}_K[z]$ and $\alpha \in \mathbb{N}^n$.

Then it is well-known (e.g., see §16.2 in [C]) that τ is an anti-involution of the Weyl algebra $\mathcal{D}_K[z]$, i.e., $\tau^2 = id$ and, for any $\phi, \psi \in \mathcal{D}_K[z]$, we have

$$\tau(\phi\psi) = \tau(\psi)\tau(\phi). \tag{2.17}$$

Note that by Eqs. (2.9) and (2.10), we see that τ preserves the ν-grading of $\mathcal{D}_K[z]$. Hence, τ extends to an automorphism, still denoted by τ, of the associated completion $\overline{\mathcal{D}}_K[z]$ of the weyl algebra $\mathcal{D}_K[z]$. Furthermore, the following lemma can also be easily checked.

Lemma 2.4. *The extended τ is also an anti-involution of the completion $\overline{\mathcal{D}}_K[z]$ of the Weyl algebra $\mathcal{D}_K[z]$.*

3. The Abhyankar-Gurjar Inversion Formula Revisited

In this section, we first give a new proof and then a new formulation (see Theorem 3.2) for the Abhyankar-Gurjar inversion formula (see Theorem 3.1). Some consequences of the reformulated Abhyankar-Gurjar inversion formula will also be derived. But, first, let us fix the following notation.

Let $F(z) = (F_1, F_2, ..., F_n)$ be a homomorphism of the formal power series algebra $\mathcal{A}_K[[z]]$ such that $F(0) = 0$. Up to some conjugations by linear automorphisms, we may and will always assume F has the form $F(z) = z - H(z)$ and $H(z) = (H_1, H_2, ..., H_n) \in \mathcal{A}_K[[z]]^{\times n}$ with the order $o(H) \geq 2$.

With the assumption above on F, we have $(\mathcal{J}F)(0) = 1$. Therefore, F has a formal inverse map $G(z) = F^{-1}(z)$ which can be written as $G(z) = z + N(z)$ for some $N(z) = (N_1, N_2, ..., N_n) \in \mathcal{A}_K[[z]]^{\times n}$ with $o(N) \geq 2$.

With the setup above, the Abhyankar-Gurjar inversion formula can be stated as follows.

Theorem 3.1. (**Abhyankar-Gurjar**) *For any $u(z) \in \mathcal{A}_K[[z]]$, we have*

$$u(G(z)) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\,\partial^\alpha\,(uH^\alpha \mathcal{J}F) = \Big(\sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\,\partial^\alpha H^\alpha \mathcal{J}F\Big)\cdot u(z). \tag{3.1}$$

In particular, the formal inverse map G of F is given by

$$G(z) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial^\alpha\,(zH^\alpha \mathcal{J}F)\,. \tag{3.2}$$

Proof: First, by Proposition 2.1 and Lemma 2.4 in [Z1], we know that there exists a unique $a(z) = (a_1(z), a_2(z), ..., a_n(z)) \in \mathcal{A}_K[[z]]^{\times n}$ with $o(a(z)) \geq 2$, such that for any $u(z) \in \mathcal{A}_K[[z]]$,

$$e^{a(z)\partial}(u(z)) = u(F), \tag{3.3}$$

$$e^{-a(z)\partial}(u(z)) = u(G), \tag{3.4}$$

where $a(z)\partial := \sum_{i=1}^n a_i(z)\partial_i$.

On the other hand, by the Taylor series expansion of $u(F) = u(z-H)$ at z, we have

$$u(F) = \sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} H^\alpha (\partial^\alpha u)(z) = \left(\sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} H^\alpha \partial^\alpha\right) \cdot u(z). \tag{3.5}$$

Note that by the facts that $o(a) \geq 2$ and $o(H) \geq 2$, it is easy to see that both the differential operators $e^{a(z)\partial}$ and $\sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} H^\alpha(z)\partial^\alpha$ lie in the completion $\overline{\mathcal{D}}_K[z]$ of $\mathcal{D}_K[z]$ constructed in the previous section. Furthermore, since Eqs. (3.3) and (3.5) hold for all $u(z) \in \mathcal{A}_K[[z]]$, hence we have

$$e^{a(z)\partial} = \sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} H^\alpha \partial^\alpha. \tag{3.6}$$

Next, we consider the images of the differential operators in the equation above under the anti-involution τ of $\overline{\mathcal{D}}_K[z]$ in Lemma 2.4.

First, by Eq. (2.16) we have

$$\begin{aligned} \tau(a(z)\partial) &= -\sum_{i=1}^n \partial_i a_i(z) = -\sum_{i=1}^n (a_i(z)\partial_i + \partial_i(a_i(z)) \\ &= -a(z)\partial - \nabla a(z), \end{aligned} \tag{3.7}$$

where $\nabla a(z) := \sum_{i=1}^n (\partial_i a_i)(z)$, i.e., the *gradient* of the n-tuple $a(z)$, but it here denotes the multiplication operator by ∇a.

Second, by Lemma 2.4, τ is also an anti-involution of $\overline{\mathcal{D}}_K[z]$, whence we have

$$\tau(e^{a(z)\partial}) = e^{\tau(a(z)\partial)} = e^{-a(z)\partial - \nabla a}. \tag{3.8}$$

Now, apply τ to Eq. (3.6), by Eqs. (3.8) and (2.16), we have

$$e^{-a(z)\partial - \nabla a(z)} = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha H^\alpha. \tag{3.9}$$

Note that for any $u(z) \in \mathcal{A}_K[[z]]$, by Theorem 2.8, *b*) in [Z1] with $F(t,z)$ there replaced by $G(z)$, or equivalently, by letting $t = -1$, we have

$$e^{-a(z)\partial - \nabla a(z)}(u(z)) = \mathcal{J}G(z)u(G(z)). \tag{3.10}$$

Combining Eqs. (3.9) and (3.10), we have

$$\sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha (H^\alpha u(z)) = \mathcal{J}G(z)u(G(z)) \tag{3.11}$$

for all $u(z) \in \mathcal{A}_K[[z]]$.

By replacing $u(z)$ in Eq. (3.11) by $u(z)\mathcal{J}F(z)$, and by the identity $\mathcal{J}F(G)\mathcal{J}G(z) = \mathcal{J}(F(G)) = \mathcal{J}z = 1$, we have

$$\sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial^\alpha(H^\alpha \mathcal{J}F(z)u(z)) = \mathcal{J}G(z)\mathcal{J}F(G)u(G(z)) = u(G(z)),$$

which is Eq. (3.1). □

Next we give a reformulation of the Abhyankar-Gurjar inversion formula in terms of certain elements of $\bar{\mathcal{A}}_K[\xi, z]$ and the differential operator $\Phi = e^\Lambda$ defined in Eqs. (2.14) and (2.15).

Theorem 3.2. *Let H and N be as above. Then for any $q(z) \in \mathcal{A}_K[[z]]$, we have*

$$\Phi\left(q(z)(\mathcal{J}F)(z)e^{\langle\xi,H(z)\rangle}\right) = q(G)\, e^{\langle\xi,N(z)\rangle}. \tag{3.12}$$

Note that the formal power series $q(z)(\mathcal{J}F)(z)e^{\langle\xi,H(z)\rangle}$ on the left hand side of Eq. (3.12) lies in $\bar{\mathcal{A}}_K[\xi, z]$ due to the assumption $o(H) \geq 2$. So, by the discussion in the previous section, the left hand side of Eq. (3.12) makes sense and gives an element of $\bar{\mathcal{A}}_K[\xi, z]$. Similarly, due to the fact $o(N) \geq 2$, the element $q(G)e^{\langle\xi,N(z)\rangle}$ on right hand side of Eq. (3.12) also lies in $\bar{\mathcal{A}}_K[\xi, z]$.

Proof of Theorem 3.2: First, for any $u(z) \in \mathcal{A}_K[[z]]$, by Eq. (3.1) with $u(z)$ replaced by $q(z)u(z)$, we have

$$q(G)u(G) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial^\alpha\big(H^\alpha(z)\mathcal{J}F(z)q(z)u(z)\big). \tag{3.13}$$

Second, by the Taylor series of $u(G) = u(z + N)$ at z, we have

$$\begin{aligned} q(G)u(G) &= q(G)\sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}N^\alpha(z)\partial^\alpha(u(z)) \\ &= \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}q(G)N^\alpha(z)\partial^\alpha(u(z)). \end{aligned} \tag{3.14}$$

Therefore, as differential operators in $\overline{\mathcal{D}}_K[z]$, we have

$$\sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\,\partial^\alpha H^\alpha(z)\mathcal{J}F(z)q(z) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\,q(G)N^\alpha(z)\partial^\alpha. \tag{3.15}$$

In terms of the left and right total symbol maps $\mathcal{L}$ and $\mathcal{R}$ from $\overline{\mathcal{D}}_K[z]$ to $\bar{\mathcal{A}}_K[\xi, z]$ (see lemma 2.2), the equation above can be written further

as follows:

$$\mathcal{L}^{-1}\left(\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\,\xi^\alpha H^\alpha(z)(\mathcal{J}F)(z)q(z)\right)=\mathcal{R}^{-1}\left(\sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\,q(G)N^\alpha(z)\xi^\alpha\right).$$

Take sum over $\alpha\in\mathbb{N}^n$ and then apply $\mathcal{R}$ to the both sides, we get

$$\mathcal{L}^{-1}\left(q(z)(\mathcal{J}F)(z)e^{\langle\xi,H(z)\rangle}\right)=\mathcal{R}^{-1}\left(q(G)e^{\langle\xi,N(z)\rangle}\right).$$

$$(\mathcal{R}\circ\mathcal{L}^{-1})\left(q(z)(\mathcal{J}F)(z)e^{\langle\xi,H(z)\rangle}\right)=q(G)e^{\langle\xi,N(z)\rangle}. \tag{3.16}$$

But, by Proposition 2.3 we also have $\mathcal{R}\circ\mathcal{L}^{-1}=\Phi$. Hence, Eq. (3.12) follows. □

Note that by letting $q(z)=1$ in Eq. (3.12), we immediately have the following corollary.

Corollary 3.3. *Let F and H be fixed as before. Then*

$$\Phi\left(\mathcal{J}F(z)e^{\langle\xi,H(z)\rangle}\right)=e^{\langle\xi,N(z)\rangle}. \tag{3.17}$$

In particular, when $\mathcal{J}F\equiv 1$, we have

$$\Phi\left(e^{\langle\xi,H(z)\rangle}\right)=e^{\langle\xi,N(z)\rangle}. \tag{3.18}$$

Remark 3.4. *By going backward the proof of Theorem 3.2 with $q(z)=1$, it is easy to see that Eq. (3.1) also follows from Eq. (3.17). Therefore, Eq. (3.17) or Eq. (3.12) can be viewed as a reformulation of the Abhyankar-Gurjar inversion formula in Eq. (3.1).*

By using Theorem 3.2, we can also derive the following formula, which will play a crucial role in next section.

Theorem 3.5. *Let $P=\langle\xi,H\rangle$. Then for any $k\geq 0$ and $q(z)\in\mathcal{A}_K[z]$, we have*

$$q(G(z))\langle\xi,N(z)\rangle^k=k!\sum_{m\geq 0}\frac{\Lambda^m\left(P^{m+k}q(z)\,\mathcal{J}F\right)}{m!(m+k)!}. \tag{3.19}$$

In particular, by choosing $q(z)=1$, we get

$$\langle\xi,N(z)\rangle^k=k!\sum_{m\geq 0}\frac{\Lambda^m\left(P^{m+k}\mathcal{J}F\right)}{m!(m+k)!}. \tag{3.20}$$

Proof: We view the formal power series of ξ and z on the both sides of Eq. (3.12) as formal power series in ξ with coefficients in $\mathcal{A}_K[[z]]$. Then by comparing the homogeneous parts in ξ of degree k of both sides of Eq. (3.12) and noting that $\Phi=e^\Lambda$, we have

$$\sum_{m\geq 0}\frac{\Lambda^m\left(P^{m+k}q(z)\,\mathcal{J}F\right)}{m!(m+k)!}=\frac{1}{k!}\,q(G(z))\,\langle\xi,N(z)\rangle^k,$$

which gives us Eq. (3.19). □

Note also that by letting $k = 0$ in Eq. (3.19), we immediately have the following inversion formula.

Corollary 3.6. *Let P be as in Theorem 3.5. Then for any $q(z) \in \mathcal{A}_K[[z]]$, we have*

$$q(G(z)) = \sum_{m\geq 0} \frac{\Lambda^m \left(P^m q(z)\, \mathcal{J}F\right)}{(m!)^2}. \tag{3.21}$$

In particular, by choosing $q(z) = z$, we get

$$G(z) = \sum_{m\geq 0} \frac{\Lambda^m \left(z P^m \mathcal{J}F\right)}{(m!)^2}. \tag{3.22}$$

4. A New Proof For the Equivalence of the Jacobian Conjecture and the Vanishing Conjecture

In this section, we show the equivalence of the *Jacobian conjecture* with a special case of the *vanishing conjecture*, Conjecture 1.1. But, we first need to recall the following deformation of formal maps.

Let $F = z - H$ and $G = z + N$ as fixed in the previous section. Let t be a formal parameter which commutes with z_i $(1 \leq i \leq n)$. We set $F_t(z) := z - tH(z)$. Since $F_{t=1}(z) = F(z)$, $F_t(z)$ can be viewed as a deformation of the formal map $F(z)$.

We may view $F_t(z)$ as a formal map of $\mathcal{A}_{K(t)}[[z]]$, i.e., a formal map over the rational function field $K(t)$. Hence, with base field K replaced by $K(t)$, all results derived in the previous sections apply to the formal map $F_t(z)$.

Denote by $G_t(z)$ the formal inverse of $F_t(z)$. It is well known that $G_t(z)$ can actually be written as $G_t(z) = z + tN_t(z)$ for some $N_t(z) \in \mathcal{A}_{K[t]}[[z,t]]^{\times n}$ with $\mathrm{o}(N_t(z)) \geq 2$, i.e., all the coefficients of $N_t(z)$ are actually polynomials of t over K. For example, this can be easily seen by applying the Abhyankar-Gurjar inversion formula in Eq. (3.2) to the formal map $F_t(z)$.

Another remark is that since $F_{t=1}(z) = F(z)$, by the uniqueness of inverse maps and the fact $N_t(z) \in \mathcal{A}_{K[t]}[[z]]^{\times n}$ mentioned above, it is easy to see that we also have $G_{t=1}(z) = G(z)$.

Next, let us start with the following lemma.

Lemma 4.1. *JH is nilpotent iff $\mathcal{J}F_t \equiv 1$.*

Proof: First, since $\mathcal{A}_K[[z]]$ is an integral domain over K, as a matrix with entries in $\mathcal{A}_K[[z]]$, JH is nilpotent iff its characteristic polynomial $\det(\lambda I - JH) = \lambda^n$.

Second, $\mathcal{J}F_t = \det(I - tJH) = t^n \det(t^{-1}I - JH)$. Therefore, $\mathcal{J}F_t \equiv 1$ iff $\det(t^{-1}I - JH) = t^{-n}$ iff the characteristic polynomial of JH is equals to λ^n. Hence the lemma follows. □

Proposition 4.2. *Let $P(\xi, z) = \langle \xi, H(z) \rangle$. Then JH is nilpotent iff $\Lambda^m(P^m) = 0$ for all $m \geq 1$.*

Proof: Applying Eq. (3.12) to the formal map F_t with $q(z) = 1/\mathcal{J}F_t(z)$, we have

$$e^{\Lambda}(e^{t\langle \xi, H(z)\rangle}) = \mathcal{J}G_t(z) e^{t\langle \xi, N_t(z)\rangle}. \tag{4.1}$$

Applying the change of variables $z \to z$ and $\xi \to t^{-1}\xi$ in the equation above, and noting that $\Lambda \to t\Lambda$, we get

$$e^{t\Lambda}(e^{\langle \xi, H(z)\rangle}) = \mathcal{J}G_t(z) e^{\langle \xi, N_t(z)\rangle}. \tag{4.2}$$

By looking at the homogeneous part in ξ of degree zero of the equation above, we get

$$\sum_{m \geq 0} \frac{t^m \Lambda^m(P^m)}{(m!)^2} = \mathcal{J}G_t(z). \tag{4.3}$$

From the equation above, it is easy to see that $\mathcal{J}G_t(z) \equiv 1$ iff $\Lambda^m(P^m) = 0$ for all $m \geq 1$.

On the other hand, since $\mathcal{J}F_t(z) = 1/(\mathcal{J}G_t)(F_t)$, we also have that $\mathcal{J}F_t(z) \equiv 1$ iff $\mathcal{J}G_t(z) \equiv 1$.

Hence, we have $\mathcal{J}F_t(z) \equiv 1$ iff $\Lambda^m(P^m) = 0$ for all $m \geq 1$. Then the proposition follows immediately from this fact and Lemma 4.1 □

When $F = z - H$ is a polynomial map, the ($\Leftarrow$) part of Proposition 4.2 can actually be improved as follows.

Proposition 4.3. *Let $F = z - H$ and P as in Proposition 4.2 but with $H(z) \in \mathcal{A}_K[z]^{\times n}$. Assume that $\Lambda^m(P^m) = 0$ when $m \gg 0$. Then JH is nilpotent.*

Proof: First, by Eq. (4.3) and the assumption of the proposition, we see that $\mathcal{J}G_t(z)$ is actually a polynomial in z and t, i.e., $\mathcal{J}G_t(z) \in \mathcal{A}_K[z, t]$.

Second, since $H(z) \in \mathcal{A}_K[z]^{\times n}$ and $F_t = z - tH$, both $\mathcal{J}F_t = \det(I - tJH)$ and the composition $(\mathcal{J}G_t)(F_t)$ are also in $\mathcal{A}_K[z, t]$.

Since the polynomial algebra $\mathcal{A}_K[z, t]$ is an integral domain, from the identity $(\mathcal{J}G_t)(F_t)\,\mathcal{J}F_t = 1$ we get $\mathcal{J}F_t = 1$. Then by Lemma 4.1, JH is nilpotent. □

Note that combining with Proposition 4.2, Proposition 4.3 gives a positive answer for a special case of the following conjecture.

Conjecture 4.4. *For any $P(\xi, z) \in \mathcal{A}_K[\xi, z]$ with $\Lambda^m(P^m) = 0$ when $m \gg 0$, we have $\Lambda^m(P^m) = 0$ for all $m \geq 1$.*

Note that up to changes of variables, the conjecture above is actually equivalent to Conjecture 4.4 in [Z3], whose general case is still open.

Proposition 4.5. *Let $F = z - H$ and P as above. Assume further that F is a polynomial map with JH nilpotent. Then the following two statements are equivalent.*

(a) $G_t(z)$ is a polynomial map over $K[t]$.

(b) $\Lambda^m(P^{m+1}) = 0$ when $m \gg 0$.

Proof: Note that since JH is nilpotent, $\mathcal{J}F_t(z) = \det(I - tJH) \equiv 1$. Applying Eq. (3.20) with $k = 1$ to F_t, we have

$$\langle \xi, tN_t(z)\rangle = \sum_{m\geq 0} \frac{\Lambda^m(\langle \xi, tH\rangle^{m+1})}{m!(m+1)!} = \sum_{m\geq 0} \frac{t^{m+1}\Lambda^m(P^{m+1})}{m!(m+1)!}.$$

Hence, we have

$$\langle \xi, N_t(z)\rangle = \sum_{m\geq 0} \frac{t^m\Lambda^m(P^{m+1})}{m!(m+1)!}. \tag{4.4}$$

From the equation above, it is easy to see that, $N_t(z)$ is a n-tuple of polynomials in t with coefficients in $\mathcal{A}_K[z]$ iff $\Lambda^m(P^{m+1}) = 0$ when $m \gg 0$.

On the other hand, as pointed out at the beginning of this section, $N_t(z) \in \mathcal{A}_{K[t]}[[z]]^{\times n}$. Therefore, $N_t(z)$ is a n-tuple of polynomials in t with coefficients in $\mathcal{A}_K[z]$ iff $N_t(z)$ is a n-tuple of polynomials in z with coefficients in $K[t] = \mathcal{A}_K[t]$.

Combining the two equivalences above, we get the equivalence of the statements (a) and (b) in the proposition. □

Now we are ready to formulate and prove the main result of this section.

Theorem 4.6. *The following two statements are equivalent to each other.*

(a) The Jacobian conjecture holds for all $n \geq 1$.

(b) For any $n \geq 1$ and $H(z) \in \mathcal{A}_K[z]^{\times n}$ with $o(H) \geq 2$, Conjecture 1.1 holds for the quadratic differential operator $\Lambda = \sum_{i=1}^n \partial_{\xi_i}\partial_{z_i}$ and the polynomial $P(\xi, z) := \langle \xi, H\rangle$.

Proof: $(a) \Rightarrow (b)$: Let $F_t = z - tH$ and assume that $\Lambda^m(P^m) = 0$ for all $m \geq 1$. Then by Proposition 4.2, we know that JH is nilpotent and, by Lemma 4.1, $\mathcal{J}F_t(z) \equiv 1$.

Since we have assumed that the *Jacobian conjecture* holds, the formal inverse map $G_t = z + tN_t$ of F_t is also a polynomial map over $K[t]$. Then by Proposition 4.5, we have $\Lambda^m(P^{m+1}) = 0$ when $m \gg 0$.

$(b) \Rightarrow (a)$: Let F be any polynomial map with $\mathcal{J}F \equiv 1$. By the well-known homogeneous reduction in [BCW] and [J], we may assume that F has the form $F = z - H$ such that H is homogeneous (of degree 3).

Under the homogeneous assumption on H, it is easy to check that the condition $\mathcal{J}F \equiv 1$ implies (actually is equivalent to) the statement that JH is nilpotent.

Then by Proposition 4.2, we have $\Lambda^m(P^m) = 0$ for all $m \geq 1$. Since we have assumed that the *vanishing conjecture*, Conjecture 1.1, holds for Λ and $P(\xi, z)$, we have $\Lambda^m(P^{m+1}) = 0$ when $m \gg 0$. Then by Proposition 4.5, we know that the formal inverse map $G_t(z)$ is also a polynomial map over $K[t]$. Hence so is $G(z) = G_{t=1}(z)$. $\square$

Remark 4.7. *Note that it has been shown in* [Z3] *and* [Z4] *that the Jacobian conjecture is actually equivalent to the vanishing conjecture, Conjecture 1.1, without any assumption on* $P(\xi, z) \in \mathcal{A}_K[\xi, z]$.

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