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MATHIEU SUBSPACES OF ASSOCIATIVE ALGEBRAS

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Abstract. Motivated by *the Mathieu conjecture* [Ma], *the image conjecture* [Z3] and the well-known *Jacobian conjecture* [K] (see also [BCW] and [E1]), the notion of *Mathieu subspaces* as a natural generalization of the notion of ideals has been introduced recently in [Z4] for associative algebras. In this paper, we first study algebraic elements in the radicals of Mathieu subspaces of associative algebras over fields and prove some properties and characterizations of Mathieu subspaces with algebraic radicals. We then give some characterizations or classifications for *strongly simple algebras* (the algebras with no non-trivial Mathieu subspaces) over arbitrary commutative rings, and for *quasi-stable algebras* (the algebras all of whose subspaces that do not contain the identity element of the algebra are Mathieu subspaces) over arbitrary fields. Furthermore, co-dimension one Mathieu subspaces and the minimal non-trivial Mathieu subspaces of the matrix algebras over fields are also completely determined.

1. Introduction

1.1. **Background and Motivation.** Let R be an arbitrary commutative ring and $\mathcal{A}$ an associative but not necessarily commutative algebra over R. Then we have the following notion introduced recently by the author in [Z4].

Definition 1.1. *Let M be a R-submodule or R-subspace of $\mathcal{A}$. We say M is a left (resp., right) Mathieu subspace of $\mathcal{A}$ if the following property holds: let $a \in \mathcal{A}$ such that $a^m \in M$ for all $m \geq 1$. Then for any $b \in \mathcal{A}$, we have $ba^m \in M$ (resp., $a^m b \in M$) for all $m \gg 0$, i.e., there exists $N \geq 1$ (depending on a and b) such that $ba^m \in M$ (resp., $a^m b \in M$) for all $m \geq N$.*

A R-subspace M of $\mathcal{A}$ is said to be a *pre-two-sided Mathieu subspace* of $\mathcal{A}$ if it is both left and right Mathieu subspace of $\mathcal{A}$. Note

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that the *pre-two-sided* Mathieu subspaces were called *two-sided* Mathieu subspace or *Mathieu subspaces* in [Z4]. The change of the name here is due to the following family of two-sided Mathieu subspaces, which were not discussed in [Z4] but are more entitled to be called (two-sided) Mathieu subspaces.

Definition 1.2. *A R-subspace M of a R-algebra $\mathcal{A}$ is said to be a two-sided Mathieu subspace, or simply a Mathieu subspace, of $\mathcal{A}$ if the following property holds: let $a \in \mathcal{A}$ such that $a^m \in M$ for all $m \geq 1$. Then for any $b, c \in \mathcal{A}$, we have $ba^m c \in M$ for all $m \gg 0$, i.e., there exists $N \geq 1$ (depending on a, b and c) such that $ba^m c \in M$ for all $m \geq N$.*

Three remarks are as follows. First, all the algebras $\mathcal{A}$ involved in this paper are assumed to unital. For these algebras, it is easy to see that every (two-sided) Mathieu subspace is a pre-two-sided (and hence, also one-sided) Mathieu subspace.

Second, from Definitions 1.1 and 1.2 it is also easy to see that every left (resp., right) ideal of $\mathcal{A}$ is a left (resp., right) Mathieu subspace of $\mathcal{A}$, and every (two-sided) ideal of $\mathcal{A}$ is a (two-sided) Mathieu subspace and hence, also a pre-two-sided Mathieu subspace of $\mathcal{A}$. But the converse is not true (see [DK], [Z4], [EWZ1], [FPYZ], [WZ] for some examples of Mathieu subspaces which are not ideals). Therefore, the notion of Mathieu subspaces can be viewed as a generalization of the notion of ideals.

Third, just like the notion of ideals which has a generalization for modules of algebras, namely, the notion of submodules, the notion of Mathieu subspaces can also be generalized to modules of associative algebras. For more discussions in this direction, see [Z6].

The introduction of the notion of Mathieu subspaces in [Z4] was mainly motivated by the studies of *the Mathieu conjecture* [Ma], *the vanishing conjecture* [Z1], [Z2], [Z5] and more recently, *the image conjecture* [Z3], and also the well-known *Jacobian conjecture* [K] (see also [BCW], [E1]). Actually, both *the Mathieu conjecture* and *the image conjecture* imply *the Jacobian conjecture*, and both are (open) problems on whether or not certain subspaces of some algebras are Mathieu subspaces (see [Ma], [Z3] and [Z4] for more detailed discussions). The notion was named after Olivier Mathieu due to his conjecture mentioned above.

There are also several other open problems and conjectures that are directly or indirectly related with Mathieu subspaces. For example, *the Dixmier conjecture* [D] as shown first by Y. Tsuchimoto [T] in 2005, and later by A. Belov and M. Kontsevich [BK] and P. K. Adjamagbo

and A. van den Essen [AE] in 2007 is actually equivalent to *the Jacobian conjecture*; and *the vanishing conjecture* [Z1], [Z2] on differential operators with constant coefficients, which now becomes a special case of *the image conjecture*, also implies *the Jacobian conjecture*.

Furthermore, it has also been proposed in Conjecture 3.2 in [Z4] that the subspace of polynomials in $n \geq 1$ variables with complex coefficients whose integrals over a fixed open subset of $\mathbb{R}^n$ with a positive measure are equal to zero should be a Mathieu subspace of the polynomial algebra in n variables over $\mathbb{C}$. In particular, by choosing some open subsets of $\mathbb{R}^n$ and positive measures properly, this conjecture is equivalent to saying that every family of classical orthogonal polynomials (see [Sz], [C], [DX]) in one or more variables with positive degrees should also span a co-dimension one Mathieu subspaces of the polynomial algebra (see Conjecture 3.5 and the related discussions in [Z4]). For some recent developments on the latter conjecture, see [EWZ2], [FPYZ] and [EZ]. For a recent survey on *the image conjecture* and its relations with *the vanishing conjecture*, *the Jacobian conjecture* and also the conjectures mentioned above, see [E2].

Surprisingly, the conjecture on integrals of polynomials mentioned above is also related with the so-called *polynomial moment problem* proposed by M. Briskin, J.-P. Francoise and Y. Yomdin in the series of papers [BFY1]-[BFY5], which was mainly motivated by the center problem for the complex Abel equation. For some recent studies on the *polynomial moment problem* in one or more variables, see [PM], [Pa], [Z4] and [FPYZ].

Currently, it is also under investigations by the author and some of his colleagues whether or not images of all locally nilpotent derivations, locally finite derivations and divergence-zero derivations of polynomial algebras over fields of characteristic zero are Mathieu subspaces of the polynomial algebras. For example, it has been shown recently in [EWZ1] that this is indeed the case for all locally finite derivations of polynomial algebras in two variables. It has also been shown in [EWZ1] that for the two-variable case the same problem for the divergence-zero derivations having 1 in the image is actually equivalent to the two-dimensional *Jacobian problem*. Furthermore, some Mathieu subspaces of the group algebras of finite groups have also been studied recently in [WZ].

Due to their connections with the various open problems or conjectures mentioned above, especially their connections with *the Jacobian conjecture* and *the Dixmier conjecture*, the seemingly familiar but still very mysterious Mathieu subspaces deserve much more attentions from

mathematicians. It is important and also necessary to study Mathieu subspaces in a separate and abstract setting.

1.2. **Contents and Arrangements.** Before we proceed, one remark is in order. Even though most of the results on Mathieu spaces in this paper are stated and proved for all the four types (*left*, *right*, *pre-two-sided* and *two-sided*) of Mathieu subspaces, for simplicity, in this subsection we only discuss the results for the *two-sided* case, i.e., only for Mathieu subspaces.

In this paper, we first study some properties of the *radicals* of arbitrary subspaces and Mathieu subspaces of (associative) algebras, where for any R-subspace V of a R-algebras $\mathcal{A}$, the *radical* of V, denoted by $\sqrt{V}$ or $\mathfrak{r}(V)$, is defined to be the set of the elements $a \in \mathcal{A}$ such that $a^m \in V$ when $m \gg 0$. We then prove some properties and characterizations for the Mathieu subspaces with algebraic radicals for algebras over fields.

One crucial result derived in this paper (see Theorem 3.10) is that when the base ring R is a field K, for algebraic elements $a \in \mathcal{A}$, the positive integers N in Definitions 1.1 and 1.2 actually can be chosen in a way that does not depend on the element $b, c \in \mathcal{A}$. Another crucial result for K-algebras $\mathcal{A}$ is Theorem 4.2 which gives a characterization for Mathieu subspaces V with algebraic radicals in terms of the idempotents contained in V. Consequently, for algebraic K-algebras, the Mathieu subspaces have an equivalent formulation that is much more similar to the definition of ideals (see Remark 4.4).

By using some results derived in this paper, we also give characterizations or classifications for *strongly simple algebras* (see Definition 6.1) over arbitrary commutative rings, and for *quasi-stable algebras* (see Definition 7.1) over arbitrary fields (see Theorems 6.2, Proposition 6.7 and Theorem 7.6). Furthermore, the co-dimension one Mathieu subspaces and the minimal non-trivial Mathieu subspaces of all types are also completely classified for (finite dimensional) matrix algebras over fields (see Theorem 5.1 and Proposition 5.5).

Considering the length of this paper, below we give a more detailed description for the arrangements of the paper.

In Section 2, we first fix some notations and conventions that will be used throughout this paper. We then study certain properties of the radicals of Mathieu subspaces or arbitrary R-subspaces of $\mathcal{A}$. A formally stronger but equivalent definition of Mathieu subspaces is also given in Proposition 2.1.

In Section 3, we study the algebraic elements of the radicals of arbitrary subspaces V and Mathieu subspaces M of K-algebras $\mathcal{A}$. The

main results of this section are Theorems 3.5, 3.9 and 3.10. Theorem 3.5 says that $\sqrt{V}$ has no non-trivial idempotents of $\mathcal{A}$ iff all algebraic elements of $\sqrt{V}$ are either nilpotent or invertible. Theorem 3.10 gives a characterization for algebraic elements in the radicals of Mathieu subspaces M of $\mathcal{A}$, namely, for each algebraic $a \in \mathcal{A}$, $a \in \sqrt{M}$ iff the principal ideal $(a^N) \subseteq M$ for some $N \geq 1$. Under the condition that $a^m \in M$ for all $m \geq 1$, Theorem 3.9 says that one can actually choose the integer N above to be the multiplicity of $0 \in K$ as a root of the minimal polynomial of the algebraic element $a \in \mathcal{A}$.

In Section 4, we use the results derived in Sections 2 and 3 to study various properties of Mathieu subspaces M with algebraic radicals. For convenience, for any K-algebra $\mathcal{A}$, we denote by $\mathcal{G}(\mathcal{A})$ (resp., $\mathcal{E}(\mathcal{A})$) the set of K-subspaces (resp., Mathieu subspaces) V of $\mathcal{A}$ such that $\sqrt{V}$ is algebraic over K, i.e., every element of $\sqrt{V}$ is algebraic over K.

In Subsection 4.1, we give a characterization for Mathieu subspaces $V \in \mathcal{E}(\mathcal{A})$ in terms of idempotents of $\mathcal{A}$ (see Theorem 4.2). Namely, a K-subspace $V \in \mathcal{G}(\mathcal{A})$ is a Mathieu subspace of $\mathcal{A}$ iff it contains the ideals of $\mathcal{A}$ generated by the idempotents contained in V. In particular, the Mathieu subspaces of simple algebraic K-algebras can be characterized as K-subspaces of $\mathcal{A}$ which do not contain any nonzero idempotents (see Proposition 4.5). Furthermore, the one-dimensional Mathieu subspaces of all K-algebras have been characterized in Proposition 4.8. This proposition will play some important roles in the later Sections 5-7.

In Subsection 4.2, we study the relations between the radical of $M \in \mathcal{E}(\mathcal{A})$ and the radical of the maximum ideal I_M contained in M. In Lemma 4.9, and more generally in Theorem 4.10, we show that these two radicals actually coincide with each other. In Theorem 4.12, we show that when $\mathcal{A}$ is commutative, a K-subspace $V \in \mathcal{G}(\mathcal{A})$ is a Mathieu subspace of $\mathcal{A}$ iff its radical $\sqrt{V}$ is an ideal of $\mathcal{A}$.

In Subsection 4.3, we first show in Proposition 4.16 that the intersection of any family of Mathieu subspaces in $\mathcal{E}(\mathcal{A})$ is still a Mathieu subspace of $\mathcal{A}$. We then show in Proposition 4.18 that the union of any ascending sequence of Mathieu subspaces is also a Mathieu subspace of $\mathcal{A}$ provided that the radical of the union is algebraic over K. Combining Propositions 4.16 and 4.18 with Zorn's lemma, we get existences of maximal or minimal elements in certain collections of Mathieu subspaces of algebraic K-algebras (see Proposition 4.20, Theorem 4.21 and Corollary 4.22).

In Section 5, we show in Theorem 5.1 that the only possible codimension one Mathieu subspace in the matrix algebra $M_n(K)$ $(n \geq 1)$

over a field K is the subspace H of the trace-zero matrices. More precisely, if $char.\, K = p \leq n$, $M_n(K)$ has no co-dimension one Mathieu subspace; and if $char.\, K = 0$ or $char.\, K = p > n$, H is the only co-dimension one Mathieu subspace (of any type) of $M_n(K)$. In Proposition 5.5) we show that the set of the nonzero minimal Mathieu subspaces is the same as the set of all dimension one K-subspaces of $M_n(K)$, which are not spanned by idempotent matrices.

In Section 6, we study the so-called *strongly simple algebras* $\mathcal{A}$ over arbitrary commutative rings R, i.e., the R-algebras $\mathcal{A}$ whose only Mathieu subspaces are 0 and $\mathcal{A}$ itself. Note that every strongly simple algebra is a simple algebra since any ideal of $\mathcal{A}$ is a Mathieu subspace of $\mathcal{A}$. Under the convenient assumption $R \subseteq \mathcal{A}$, we first show in Theorem 6.2 that if a R-algebra $\mathcal{A}$ is strongly simple, then the base ring R must be an integral domain and $\mathcal{A} \simeq K_R$ as R-algebras, where K_R denotes the field of fractions of R. In particular, for any field K, there are no strongly simple K-algebras except K itself.

We then show in Lemma 6.6 that for every integral domain R such that $R \neq K_R$ and K_R has a real-valued additive valuation $\nu : K_R \to \mathbb{R}$ satisfying $\nu(r) \geq 0$ for all $r \in R$, there is no strongly simple R-algebras. Note that this is the case for all Krull domains and Noetherian domains which are not fields (see Proposition 6.7). Consequently, all (commutative or noncommutative) rings except the finite fields $\mathbb{Z}_p$ (for all primes p) are strongly simple $\mathbb{Z}$-algebras (see Proposition 6.8).

In Section 7, we first introduce the notions of (*quasi-*)*stable algebras* in Definition 7.1. We show in Proposition 7.4 that every integral R-algebra $\mathcal{A}$, all of whose elements are either invertible or nilpotent, is quasi-stable. Consequently, every left or right integral Artinian local R-algebra is quasi-stable (see Corollary 7.5).

We then give a classification in Theorem 7.6 for the *quasi-stable algebras* over fields K. More precisely, we show that a K-algebra $\mathcal{A}$ is *quasi-stable* iff either $\mathcal{A} \simeq K \dot{+} K$ or $\mathcal{A}$ is an algebraic local K-algebra. Note that by Corollary 3.8, the latter holds iff $\mathcal{A}$ is algebraic and every element of $\mathcal{A}$ is either nilpotent or invertible iff $\mathcal{A}$ is algebraic and has no non-trivial idempotents.

The motivation of the study of quasi-stable algebras is given in Proposition 7.2 and Corollary 7.3. An application of Theorem 7.6 via Corollary 7.3 to commutative K-algebras is given in Corollary 7.12. Finally, for the completeness and also for the purpose of comparison, we also classify in Proposition 7.13 the *stable* K-algebras, i.e., the K-algebras $\mathcal{A}$ such that every K-subspace $V \subset \mathcal{A}$ with $1 \notin V$ is an ideal of $\mathcal{A}$.

2. Mathieu Subspaces and Their Radicals

In this section, we study some general properties of Mathieu subspaces and the radicals of subspaces of associative algebras. Most of the results derived in this section will be needed in the later sections.

First, let's fix the following conventions and notations that will be used throughout this paper.

Unless stated otherwise, R and K always stand for an arbitrary commutative ring and an arbitrary field, respectively. $\mathcal{A}$ stands for an arbitrary associative (but not necessarily commutative) algebra over R or K. Although most of the results in this paper also hold for non-unital algebras $\mathcal{A}$, for convenience we assume that all rings and algebras in this paper have the identity elements which will be uniformly denoted by 1, when no confusions occur. All algebra homomorphisms are assumed to preserve the identity elements. The ring or algebra with a single element 0 will be excluded in this paper.

Moreover, the following terminologies and notations for R-algebras $\mathcal{A}$ will also be in force throughout this paper.

1) The sets of units or invertible elements of R and $\mathcal{A}$ will be denoted by $R^\times$ and $\mathcal{A}^\times$, respectively.
2) A R-subspace V of $\mathcal{A}$ is said to be *proper* if $V \neq \mathcal{A}$, and *non-trivial* if $V \neq 0$ or $\mathcal{A}$.
3) An element $a \in \mathcal{A}$ is said to be an *idempotent* if $a^2 = a$, and a *quasi-idempotent* if $a^2 = ra$ for some $r \in R^\times$. An idempotent $a \in \mathcal{A}$ is said to be *non-trivial* if $a \neq 0$ or $1 \in \mathcal{A}$.
4) For any subset S of a R-algebra $\mathcal{A}$, we say S is *integral* or *algebraic* (when R is a field) over R if every element $a \in S$ is integral over R (i.e., a is a root of a monic polynomial with coefficients in R).
5) For any subset $S \subseteq \mathcal{A}$, we define the *radical* of S, denoted by $\sqrt{S}$ or $\mathfrak{r}(S)$, to be the set of all the elements $a \in \mathcal{A}$ such that $a^m \in S$ when $m \gg 0$. The subset of the elements in the radical $\sqrt{S}$ which are integral over R will be denoted by $\mathfrak{r}'(S)$.
6) The radical $\sqrt{0}$ of the zero ideal will also be denoted by $\mathrm{nil}\,(\mathcal{A})$. Note that when $\mathcal{A}$ is commutative, $\mathrm{nil}\,(\mathcal{A})$ is the *nilradical* of $\mathcal{A}$.
7) Let $\mathcal{A}$ and $\mathcal{B}$ be R-algebras. We denote by $\mathcal{A}\dot{+}\mathcal{B}$ the R-algebra with the base R-space $\mathcal{A} \times \mathcal{B}$ and the algebra product defined componentwise.

Note that for both Mathieu subspaces and ideals, we have several different cases: *left, right* and (*pre-*)*two-sided*. Very often, it is necessary and important to treat all these cases. For simplicity, we introduce the short terminology *ϑ-Mathieu subspaces* for Mathieu subspaces, where

ϑ stands for *left, right*, *pre-two-sided*, or *two-sided*. Similarly, we introduce the terminology *ϑ-ideals* for ideals, except for the specification $\vartheta =$ *"pre-two-sided"*, we also set *ϑ-ideals* to mean *two-sided* ideals.

In other words, the reader should read the letter ϑ as an index or a variable with four possible choices or "values". However, to avoid repeating the phrase "*for every specification of ϑ*" or *"for every ϑ"* infinitely many times, we will simply leave ϑ unspecified for the statements or propositions which hold for all the four specifications of ϑ.

Note that with the short terminologies fixed above, we immediately have the implication: *any ϑ-ideal of $\mathcal{A}$ is a ϑ-Mathieu subspace of $\mathcal{A}$*, which by the convention fixed above actually means four implications (corresponding to the four specifications of ϑ).

Finally, we fix the following notations.

For any $a \in \mathcal{A}$ and any $\vartheta \neq$ *"pre-two-sided"*, we let $(a)_\vartheta$ denote the ϑ-ideal of $\mathcal{A}$ generated by a. For the case $\vartheta =$ *"pre-two-sided"*, we set $(a)_\vartheta := aA + Aa$, i.e., the sum of the left ideal and the right ideal generated by a. Moreover, for the two-sided case, the commonly used notation (a) will also be freely used, i.e., $(a) = (a)_\vartheta$ with $\vartheta =$ *"two-sided"*.

Now let's start with the following formally stronger but equivalent definition of ϑ-Mathieu subspaces, which says that the condition "$a^m \in M$ *for all* $m \geq 1$" in Definitions 1.1 and 1.2 may be replaced by the condition "$a \in \sqrt{M}$".

Proposition 2.1. *Let $\mathcal{A}$ be a R-algebra and M a R-subspace of $\mathcal{A}$. Then M is a ϑ-Mathieu subspace of $\mathcal{A}$ iff the following property holds: for any $a \in \sqrt{M}$ and $b, c \in \mathcal{A}$, we have*

i) $ba^m \in M$ *when* $m \gg 0$, *if* $\vartheta =$ *"left";*
ii) $a^m c \in M$ *when* $m \gg 0$, *if* $\vartheta =$ *"right";*
iii) ba^m, $a^m c \in M$ *when* $m \gg 0$, *if* $\vartheta =$ *"pre-two-sided";*
iv) $ba^m c \in M$ *when* $m \gg 0$, *if* $\vartheta =$ *"two-sided".*

Proof: The ($\Leftarrow$) part is trivial. To show the ($\Rightarrow$) part, note first that since $a \in \sqrt{M}$, there exists $N \in \mathbb{N}$ such that $a^m \in M$ for all $m \geq N$. Set $x := a^N$. Then $x^m = a^{Nm} \in M$ for all $m \geq 1$.

Assume that M is a (two-sided) Mathieu subspace of $\mathcal{A}$. Then for any $b, c \in \mathcal{A}$, by Definition 1.2 it is easy to see that for the (finitely many) elements $ba^r \in \mathcal{A}$ ($0 \leq r \leq N-1$), there exists $N_1 \in \mathbb{N}$ such that

$$ba^{Nm+r}c = (ba^r)x^m c \in M$$

for all $0 \leq r \leq N-1$ and $m \geq N_1$.

From the equation above, it is easy to see that for all $k \geq NN_1$, we have $ba^kc \in M$. Therefore, the theorem holds for (two-sided) Mathieu subspaces.

By letting $c = 1$ (resp., $b = 1$) in the arguments above, we see that the theorem also holds for left (resp., right) Mathieu subspaces, whence the pre-two-sided case also follows. $\square$

Next, we use a similar argument as in the proof above to show the following lemma on the radicals of ϑ-Mathieu subspaces.

Lemma 2.2. *Let $\mathcal{A}$ be a R-algebra and S a subset of $\mathcal{A}$. Then the following statements hold.*

i) $\sqrt{S} \subseteq \mathfrak{r}(\sqrt{S})$.

ii) Assume further that S is a ϑ-Mathieu subspace of $\mathcal{A}$. Then $\sqrt{S} = \mathfrak{r}(\sqrt{S})$.

Proof: *i)* Let $a \in \sqrt{S}$. Then we have $a^m \in S$ when $m \gg 0$. Hence, for any $k \geq 1$, we also have $(a^k)^m = a^{km} \in S$ when $m \gg 0$, whence $a^k \in \sqrt{S}$. Therefore, $a \in \mathfrak{r}(\sqrt{S})$ and hence, the statement follows.

ii) Let $a \in \mathfrak{r}(\sqrt{S})$. Then $a^m \in \sqrt{S}$ when $m \gg 0$, i.e., there exists $N \geq 1$ such that $a^N \in \sqrt{S}$. Since S is a ϑ-Mathieu subspace of $\mathcal{A}$, by Proposition 2.1 there exists $N_1 \geq 1$ such that $a^{Nm+r} = (a^N)^m a^r \in S$ for all $0 \leq r \leq N-1$ when $m \geq N_1$. From this fact it is easy to see that for all $k \geq NN_1$, we have $a^k \in S$. Therefore, $a \in \sqrt{S}$ and hence, $\mathfrak{r}(\sqrt{S}) \subseteq \sqrt{S}$. Then by *i)*, the equality in *ii)* follows. $\square$

Note that the statement *ii)* in Lemma 2.2 is parallel to the fact in commutative algebra that the radicals of ideals are radical. Of course, in general the radicals of ϑ-Mathieu subspaces of $\mathcal{A}$ are not closed under the addition or the product of the algebra $\mathcal{A}$. But, as we will see later in Theorem 4.12 and Corollary 4.13, for commutative K-algebras $\mathcal{A}$, a K-subspace $V \subseteq \mathcal{A}$ with $\sqrt{V}$ algebraic over K is a Mathieu subspace of $\mathcal{A}$ iff its radical $\sqrt{V}$ is a radical ideal of $\mathcal{A}$.

Next, we give the following characterizations for ϑ-ideals and ϑ-Mathieu subspaces. Since Eqs. (2.3), (2.5) and (2.7) (below) obviously imply Eqs. (2.4), (2.6) and (2.8), respectively, the characterizations provide a different point of view to see that the notion of ϑ-Mathieu subspaces is indeed a natural generalization of the notion of ϑ-ideals.

Lemma 2.3. *Let V be a R-subspace of a R-algebra $\mathcal{A}$. For each $b \in \mathcal{A}$, we set*

$$(V : b) := \{a \in \mathcal{A} \mid ab \in V\}, \tag{2.1}$$

$$b^{-1}V := \{a \in \mathcal{A} \mid ba \in V\}, \tag{2.2}$$

where $b^{-1}V$ is an abusing notation since b might not be invertible in $\mathcal{A}$.

Then the following statements hold.

i) V is a left ideal of $\mathcal{A}$ iff for any $b \in \mathcal{A}$, we have

$$V \subseteq b^{-1}V. \tag{2.3}$$

ii) V is a left Mathieu subspace of $\mathcal{A}$ iff for any $b \in \mathcal{A}$, we have

$$\sqrt{V} \subseteq \sqrt{b^{-1}V}. \tag{2.4}$$

iii) V is a right ideal of $\mathcal{A}$ iff for any $b \in \mathcal{A}$, we have

$$V \subseteq (V : b). \tag{2.5}$$

iv) V is a right Mathieu subspace of $\mathcal{A}$ iff for any $b \in \mathcal{A}$, we have

$$\sqrt{V} \subseteq \sqrt{(V : b)}. \tag{2.6}$$

v) V is a (two-sided) ideal of $\mathcal{A}$ iff for any $b, c \in \mathcal{A}$, we have

$$V \subseteq b^{-1}(V : c). \tag{2.7}$$

vi) V is a (two-sided) Mathieu subspace of $\mathcal{A}$ iff for any $b, c \in \mathcal{A}$, we have

$$\sqrt{V} \subseteq \sqrt{b^{-1}(V : c)}. \tag{2.8}$$

Proof: The proof of the lemma is very straightforward. Here we just give a proof for the statement *vi*). The other statements can be proved similarly.

($\Leftarrow$) Let $a \in \sqrt{V}$. Then by Eq. (2.8), $a \in \sqrt{b^{-1}(V : c)}$, i.e., $a^m \in b^{-1}(V : c)$ when $m \gg 0$. Hence, by Eqs. (2.1) and (2.2), we have $ba^m c \in V$ when $m \gg 0$. It then follows from Proposition 2.1 that V is a (two-sided) Mathieu subspace.

The ($\Rightarrow$) part follows simply by reversing the arguments above. □

Next, we prove another lemma on the radicals of R-subspaces of $\mathcal{A}$, which will be needed later in Subsection 4.2.

Lemma 2.4. *Let $\mathcal{A}$ be a R-algebra (not necessarily commutative) and V a R-subspace of $\mathcal{A}$ such that $\sqrt{V} = \mathcal{A}$. Then $V = \mathcal{A}$.*

Proof: Assume otherwise and let $a \in \mathcal{A} \backslash V$. Since $a \in \mathcal{A} = \sqrt{V}$, we have $a^m \in V$ when $m \gg 0$. Since $a \notin V$, there exists $k \geq 1$ such that $a^k \notin V$ but $a^m \in V$ for all $m \geq k + 1$.

Set $b := 1 + a^k$. Since $b \in \sqrt{V}(= \mathcal{A})$, there exists $N \geq 1$ such that $b^m \in V$ for all $m \geq N$. Note that for each $m \geq N$, we also have

$$b^m = (1 + a^k)^m \equiv 1 + ma^k \mod V. \tag{2.9}$$

Therefore, $1 + ma^k \in V$ for all $m \geq N$. Consequently, we have $a^k = (1 + (N + 1)a^k) - (1 + Na^k) \in V$, which is a contradiction. □

Now let's recall the following simple but very useful property of ϑ-Mathieu subspaces, which can be easily checked (or see Proposition 4.9 in [Z4]).

Proposition 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be R-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ a R-algebra homomorphism. Then for every ϑ-Mathieu subspace M of $\mathcal{B}$, $\phi^{-1}(M)$ is a ϑ-Mathieu subspace of $\mathcal{A}$.*

One immediate consequence of the proposition above is the following corollary.

Corollary 2.6. *Let $\mathcal{B}$ be a R-algebra and $\mathcal{A}$ a R-subalgebra of $\mathcal{B}$. Then for every ϑ-Mathieu subspace M of $\mathcal{B}$, $M \cap \mathcal{A}$ is a ϑ-Mathieu subspace of $\mathcal{A}$.*

Proof: Apply Proposition 2.5 to the embedding $\iota : \mathcal{A} \to \mathcal{B}$ and note that $\iota^{-1}(M) = M \cap \mathcal{A}$. □

Proposition 2.7. *Let I be an ideal of $\mathcal{A}$ and M a R-subspace of $\mathcal{A}$. Assume that $I \subseteq M$. Then M is a ϑ-Mathieu subspace of $\mathcal{A}$ iff M/I is a ϑ-Mathieu subspace of $\mathcal{A}/I$.*

Proof: ($\Leftarrow$) Let $\pi : \mathcal{A} \to \mathcal{A}/I$ be the quotient map. Since $I \subseteq M$, we have $\pi^{-1}(M/I) = M + I = M$. Applying Proposition 2.5 to the R-algebra homomorphism π, we see that M is a ϑ-Mathieu subspace of $\mathcal{A}$.

($\Rightarrow$) Let $\bar{a}, \bar{b} \in \mathcal{A}/I$ such that $\bar{a}^m \in M/I$ for all $m \geq 1$. Let $a, b \in \mathcal{A}$ such that $\pi(a) = \bar{a}$ and $\pi(b) = \bar{b}$. Then for all $m \geq 1$, we have $a^m \in \pi^{-1}(M/I) = M$ since $\pi(a^m) = \bar{a}^m \in M/I$.

Now assume $\vartheta =$ *"left"*, i.e., M is a left Mathieu subspace of $\mathcal{A}$. Then we have $ba^m \in M$ when $m \gg 0$, whence $\bar{b}\bar{a}^m = \pi(ba^m) \in M/I$ when $m \gg 0$. Therefore, M/I is a left Mathieu subspace of $\mathcal{A}/I$. For the other specifications of ϑ, the proofs are similar. □

The following lemma is obvious but does provide a family of ϑ-Mathieu subspaces.

Lemma 2.8. *Let M be a R-subspace of $\mathcal{A}$ such that $\sqrt{M} \subseteq \operatorname{nil}(\mathcal{A})$. Then every R-subspace $V \subseteq M$ is a ϑ-Mathieu subspace of $\mathcal{A}$.*

The following lemma will be crucial for our later arguments.

Lemma 2.9. *Let a be a nonzero quasi-idempotent of $\mathcal{A}$ and V a R-subspace of $\mathcal{A}$. Then the following statements hold.*

i) a is integral over R but cannot be nilpotent. Moreover, a is invertible iff a is an invertible scalar of $\mathcal{A}$, i.e., $a \in R^\times \cdot 1_{\mathcal{A}} \subset \mathcal{A}$.
ii) $a \in \sqrt{V}$ iff $a \in V$.
iii) Assume further that $a \in V$ and V is a ϑ-Mathieu subspace of $\mathcal{A}$. Then $(a)_\vartheta \subseteq V$.

Proof: Assume $a^2 = ra$ for some $r \in R^\times$. Then it follows inductively that $a^m = r^{m-1}a$ for all $m \geq 1$, from which it is easy see that *ii*) does hold.

To show *i*), note first that a is integral over R since a is a root of the monic polynomial $t^2 - rt = 0$, and a cannot be nilpotent, for if $a^m = 0$ for some $m \geq 2$, then $a = r^{1-m}a^m = 0$, which is a contradiction. Furthermore, if $a \in \mathcal{A}^\times$, then from the equation $a(a-r) = 0$, we have $a = r \in R^\times$. Since every invertible scalar of $\mathcal{A}$ is a quasi-idempotent, we see that *i*) follows.

To show *iii*), note first that by *ii*) $a \in \sqrt{V}$. If V is a left Mathieu subspace of $\mathcal{A}$, then for each $b \in \mathcal{A}$, we have $r^{m-1}ba = ba^m \in V$ when $m \gg 0$. Since $r^{m-1} \in R^\times$ for all $m \geq 1$, we have $ba \in V$, whence $\mathcal{A}a \subseteq V$.

The *right* and *two-sided* cases can be proved similarly. The *pre-two-sided* case follows directly from the *left* and *right* cases. □

Applying Lemma 2.9, *iii*) to the identity element $1 \in \mathcal{A}$, we immediately get the following corollary, which was first noticed in [Z4].

Corollary 2.10. *For any ϑ-Mathieu subspace M of $\mathcal{A}$ with $1 \in M$, we have $M = \mathcal{A}$.*

Equivalently, any proper R-subspace $V \subset \mathcal{A}$ with $1 \in V$ cannot be a ϑ-Mathieu subspace of $\mathcal{A}$.

3. Algebraic Elements in the Radicals of Arbitrary Subspaces

In this section, we study some properties of integral or algebraic elements in the radicals of arbitrary subspaces or ϑ-Mathieu subspaces of associative algebras $\mathcal{A}$ over a commutative ring R or a field K.

Recall that for any subset $S \subseteq \mathcal{A}$, we have let $\mathfrak{r}'(S)$ to denote the subset of integral or algebraic (if the base ring R is a field) elements in the radical $\sqrt{S}$ (or $\mathfrak{r}(S)$) of S.

Lemma 3.1. *Let $\mathcal{A}$ be a R-algebra and V an arbitrary R-subspace of $\mathcal{A}$. Assume that there exists $a \in \sqrt{V}$ such that a is invertible and a^{-1} is integral over R. Then $1 \in V$.*

Proof: Note first that by replacing a by a positive power of a if necessary, we may assume that $a^m \in V$ for all $m \geq 1$.

Let $f(t)$ be a monic polynomial with coefficients in R such that $f(a^{-1}) = 0$. Write $f(t) = t^d - \sum_{k=0}^{d-1} r_k t^k$ for some $d \geq 1$ and $r_k \in R$ $(0 \leq k \leq d-1)$. Then we have

$$a^{-d} - r_{d-1}a^{1-d} - r_{d-2}a^{2-d} - \cdots - r_1 a^{-1} - r_0 = 0.$$

Multiplying a^d to the equation above, we get

$$1 = r_{d-1}a + r_{d-2}a^2 \cdots + r_1 a^{d-1} + r_0 a^d.$$

Since $a^m \in V$ for all $m \geq 1$, it follows from the equation above that $1 \in V$. $\square$

Proposition 3.2. *Let K be a field, $\mathcal{A}$ a K-algebra and V a K-subspace of $\mathcal{A}$. Then $\mathfrak{r}'(V) \cap \mathcal{A}^\times \neq \emptyset$ iff $1 \in V$.*

Proof: $(\Leftarrow)$ Since $1 \in V$, then $1 \in \sqrt{V}$. Since $1 \in \mathcal{A}$ is invertible and algebraic over K, we have $1 \in \mathfrak{r}'(V) \cap \mathcal{A}^\times$, whence $\mathfrak{r}'(V) \cap \mathcal{A}^\times \neq \emptyset$.

$(\Rightarrow)$ Let $a \in \mathfrak{r}'(V) \cap \mathcal{A}^\times$. Then $a \in \sqrt{V}$ and is invertible and algebraic over K. Since the base ring is a field K, it is easy to see that a^{-1} is also algebraic over K. Then by Lemma 3.1, we have $1 \in V$. $\square$

In order to get more results on algebraic elements in the radicals of K-subspaces of K-algebras, we need the following lemma on polynomials $f(t)$ in one variable t over a field K.

Lemma 3.3. *Let $f(t) = t^k h(t)$ for some $k \geq 0$ and $h(t) \in K[t]$ such that $h(0) \neq 0$. Then there exists a polynomial $p(t) \in K[t]$ such that the following equations hold:*

$$p(t) \equiv 0 \mod (t^k), \tag{3.1}$$

$$p^2(t) \equiv p(t) \mod (f(t)), \tag{3.2}$$

$$t^k \equiv t^k p(t) \mod (f(t)). \tag{3.3}$$

Furthermore, if $k \geq 1$ and $\deg h \geq 1$, we have

$$p(t) \not\equiv 0, 1 \mod (f(t)). \tag{3.4}$$

Proof: First, if $k = 0$, we choose $p(t) = 1$. Then it is easy to see that Eqs. (3.1)–(3.3) in the lemma hold in this case.

Assume $k \geq 1$. Since $h(0) \neq 0$, the polynomials t^k and $h(t)$ are co-prime. Therefore, there exist $u(t), v(t) \in K[t]$ such that

$$1 = t^k u(t) + h(t)v(t). \tag{3.5}$$

Let $p(t) := t^k u(t)$. Then Eq.(3.1) follows immediately. Furthermore, from Eq.(3.5) we have

$$p(t) = 1 - h(t)v(t). \tag{3.6}$$

Multiplying $p(t)$ and t^k to the both sides of the equation above, respectively, we get

$$\begin{aligned} p^2(t) &= p(t) - p(t)h(t)v(t) = p(t) - t^k u(t)h(t)v(t) \\ &= p(t) - u(t)v(t)f(t), \end{aligned} \tag{3.7}$$

$$t^k p(t) = t^k - t^k h(t)v(t) = t^k - f(t)v(t). \tag{3.8}$$

Then Eqs. (3.2) and (3.3) follow immediately from Eqs.(3.7) and (3.8), respectively.

Finally, we prove Eq. (3.4) as follows.

Assume $p(t) \equiv 0 \bmod (f(t))$. Then $f(t) \mid p(t)$, whence $h(t) \mid p(t)$. However, by Eq. (3.6), we have $h(t) \mid 1$, which contradicts the condition $\deg h \geq 1$.

Assume $p(t) \equiv 1 \bmod (f(t))$. Then we have $f(t) \mid (p(t) - 1)$, whence $t^k \mid (p(t) - 1)$. By Eq. (3.6), we also have $t^k \mid h(t)v(t)$. Hence $t^k \mid v(t)$ since $h(0) \neq 0$. Then by Eq. (3.5), we get $t^k \mid 1$, which contradicts the condition $k \geq 1$. □

Proposition 3.4. *Let V be a K-subspace of $\mathcal{A}$ and a an algebraic element of $\mathcal{A}$, which is not nilpotent nor invertible. Denote by $k(\geq 1)$ the multiplicity of $0 \in K$ as a root of the minimal polynomial of a over K. Assume further that $a^m \in V$ for all $m \geq 1$. Then there exists $p(t) \in t^k K[t]$ such that the following three statements hold:*

i) $p(a) \in V$;
ii) $p(a)$ *is a non-trivial idempotent of* $\mathcal{A}$;
iii) $a^k = a^k p(a)$.

Proof: Let $f(t)$ be the minimal polynomial of a over K and write it as $f(t) = t^k h(t)$ for some $h(t) \in K[t]$ such that $h(0) \neq 0$. Since a is not nilpotent, we have $\deg h \geq 1$. Since a is not invertible, we have $f(0) = 0$, which means $k \geq 1$ as already indicated in the theorem.

Now apply Lemma 3.3 to the polynomial $f(t)$ and let $p(t)$ be as in the same lemma. Then by Eq. (3.1), $p(t) \in t^k K[t]$, and by Eqs. (3.2)-(3.4), $p(a)$ satisfies *ii)* and *iii)*. To show *i)*, note that $p(t) \in t^k K[t]$

with $k \geq 1$. So $p(a)$ is a linear combination of some powers a^m's over K with $m \geq 1$. Since by our assumption $a^m \in V$ for all $m \geq 1$, we have $p(a) \in V$. $\square$

Theorem 3.5. *Let $\mathcal{A}$ be a K-algebra and V a K-subspace of $\mathcal{A}$. Then the following two statements are equivalent.*

1) *every element of $\mathfrak{r}'(V)$ is either nilpotent or invertible.*
2) *V contains no non-trivial idempotents.*

Proof: 1) $\Rightarrow$ 2): Assume that V contains a non-trivial idempotent e. Then by Lemma 2.9, $i)$ and $ii)$, we know that $e \in \mathfrak{r}'(V)$ and e is not nilpotent nor invertible, which contradicts 1).

2) $\Rightarrow$ 1): Assume that there exists $a \in \mathfrak{r}'(V)$ which is not nilpotent nor invertible. Note that for each $m \geq 1$, a^m is also algebraic over K and is not nilpotent nor invertible. Since $a^m \in V$ when $m \gg 0$, replacing a by a power of a if necessary, we may further assume that $a^m \in V$ for all $m \geq 1$. Then by Proposition 3.4, we get a non-trivial idempotent $p(a) \in V$, which is a contradiction. $\square$

Applying the theorem above to $V = \mathcal{A}$ and noting that $\sqrt{\mathcal{A}} = \mathcal{A}$, we immediately have the following corollary.

Corollary 3.6. *For every K-algebra $\mathcal{A}$, the following two statements are equivalent.*

1) *every algebraic element of $\mathcal{A}$ is either nilpotent or invertible.*
2) *$\mathcal{A}$ has no non-trivial idempotents.*

The following lemma and the corollary followed provide more understandings on the equivalent conditions in the corollary above.

Lemma 3.7. *Let $\mathcal{A}$ be a R-algebra. Then for the following three statements:*

1) *every element of $\mathcal{A}$ is either nilpotent or invertible;*
2) *$\mathcal{A}$ is a local R-algebra;*
3) *$\mathcal{A}$ has no non-trivial idempotent,*

we have 1) $\Rightarrow$ 2) $\Rightarrow$ 3).

Proof: 1) $\Rightarrow$ 2) is well-known, e.g., see Corollary a, p. 74 in [Pi]. To show 2) $\Rightarrow$ 3), let $\mathbf{J}(\mathcal{A})$ be the Jacobson radical of $\mathcal{A}$. Then $\mathbf{J}(\mathcal{A})$ is also the unique maximal left ideal of $\mathcal{A}$. Assume that $\mathcal{A}$ has a non-trivial idempotent e. Then it is easy to check that $1 - e$ is also a non-trivial idempotent. Furthermore, by Lemma 2.9, $i)$, both e and $1 - e$ are not invertible, whence the left ideals $e\mathcal{A}$ and $(1-e)\mathcal{A}$ are proper and hence,

both are contained in $\mathbf{J}(\mathcal{A})$. In particular, both e and $1-e$ are in $\mathbf{J}(\mathcal{A})$. But this implies $1 = e + (1-e) \in \mathbf{J}(\mathcal{A})$, which is a contradiction. $\square$

Corollary 3.8. *For every algebraic K-algebra $\mathcal{A}$, the three statements in Lemma 3.7 are equivalent to one another.*

Proof: Since $\mathcal{A}$ is algebraic, we see by Corollary 3.6 that the statements 1) and 3) in Lemma 3.7 are actually equivalent to each other. With this observation the corollary follows immediately from Lemma 3.7. $\square$

Next, we derive the following theorem on algebraic elements of the radicals of ϑ-Mathieu subspaces.

Theorem 3.9. *Let $\mathcal{A}$ be a K-algebra and M a ϑ-Mathieu subspace of $\mathcal{A}$. Let $a \in \mathcal{A}$ such that a is algebraic over K and $a^m \in M$ for all $m \geq 1$. Denote by $k \geq 0$ the multiplicity of $0 \in K$ as a root of the minimal polynomial $f(t)$ of a. Then $(a^k)_\vartheta \subseteq M$. In particular, for any $\vartheta \neq$ "pre-two-sided", the ϑ-ideal of $\mathcal{A}$ generated by a^k is contained in M.*

Proof: Assume first that $k = 0$, i.e., 0 is not a root of $f(t)$. Then a is invertible, and by Proposition 3.2, $1 \in M$. By Corollary 2.10, we have $M = \mathcal{A}$. Hence the theorem holds in this case.

Assume that $k \geq 1$. Then a is not invertible. If a is nilpotent, then $a^k = 0$, whence the theorem holds trivially in this case. So assume that a is not nilpotent nor invertible. Applying Proposition 3.4 to a with $V = M$, and letting $p(a)$ be as in the same proposition, we see that $a^k = a^k p(a)$ and $p(a)$ is a non-trivial idempotent in M.

Now, applying Lemma 2.9, *iii)* to the idempotent $p(a)$ with $V = M$, we get $(p(a))_\vartheta \subseteq M$. Furthermore, since $a^k = a^k p(a)$, we also have

$$(a^k)_\vartheta = (a^k p(a))_\vartheta \subseteq (p(a))_\vartheta \subseteq M. \tag{3.9}$$

Hence the theorem follows. $\square$

One immediate consequence of Theorem 3.9 is the following characterization of algebraic elements in the radicals of ϑ-Mathieu subspaces.

Theorem 3.10. *Let M be a ϑ-Mathieu subspace of a K-algebra $\mathcal{A}$ and a an algebraic element of $\mathcal{A}$. Then $a \in \sqrt{M}$ iff $(a^N)_\vartheta \subseteq M$ for some $N \geq 0$.*

Proof: The ($\Leftarrow$) part follows directly from the fact that for all $m \geq N$, $a^m \in (a^N)_\vartheta \subseteq M$. The ($\Rightarrow$) part can be proved as follows.

Since $a \in \sqrt{M}$, we have that $a^m \in M$ when $m \gg 0$. In particular, there exists $n \geq 1$ such that $(a^n)^m = a^{nm} \in M$ for all $m \geq 1$. Applying Theorem 3.9 to the algebraic element $a^n \in \mathcal{A}$, we have $(a^{nk})_\vartheta = \big((a^n)^k\big)_\vartheta \subseteq M$ for some $k \geq 0$, whence the theorem follows with $N = nk$. $\square$

4. **Mathieu Subspaces with Algebraic Radicals**

Throughout this section, K stands for an arbitrary field and $\mathcal{A}$ an associative algebra over K. For convenience, we denote by $\mathcal{G}(\mathcal{A})$ (resp., $\mathcal{E}_\vartheta(\mathcal{A})$) the collection of all K-subspaces (resp., ϑ-Mathieu subspaces) V of $\mathcal{A}$ such that $\sqrt{V}$ is algebraic over K.

In this section we use the results derived in the previous sections to study some properties of ϑ-Mathieu subspaces in $\mathcal{E}_\vartheta(\mathcal{A})$. Note that all the results derived in this section apply under one of the conditions in the following easy-to-check lemma.

Lemma 4.1. *Let V be a K-subspace of $\mathcal{A}$. Then $V \in \mathcal{G}(\mathcal{A})$ if one of the following four conditions holds:*

a) *$\mathcal{A}$ is algebraic over K;*
b) *V is algebraic over K;*
c) $\dim_K \mathcal{A} < \infty$.
d) $\dim_K V < \infty$.

4.1. **Characterization of $M \in \mathcal{E}_\vartheta(\mathcal{A})$ in Terms of Idempotents.**

We start with the following characterization of ϑ-Mathieu subspaces in $\mathcal{E}_\vartheta(\mathcal{A})$ in terms of idempotents of $\mathcal{A}$.

Theorem 4.2. *Let $V \in \mathcal{G}(\mathcal{A})$. Then V is a ϑ-Mathieu subspace of $\mathcal{A}$ iff for any idempotent $e \in V$, we have $(e)_\vartheta \subseteq V$.*

Proof: The ($\Rightarrow$) part follows directly from Lemma 2.9, *iii*). For the ($\Leftarrow$) part, we here just give a proof for the two-sided case. The proofs for the other three cases are similar.

Let $a, b, c \in \mathcal{A}$ such that $a^m \in V$ for all $m \geq 1$. We need to show that $ba^mc \in V$ when $m \gg 0$.

Note first that since $a \in \sqrt{V}$ and $V \in \mathcal{G}(\mathcal{A})$, a is algebraic over K. If a is nilpotent, then $ba^mc = 0 \in V$ when $m \gg 0$. If a is invertible, then $1 \in V$ by Proposition 3.2. Applying our assumption to the idempotent $1 \in V$, we have $V = \mathcal{A}$, whence $ba^mc \in V$ for all $m \geq 1$.

Finally, assume that a is not nilpotent nor invertible. Apply Proposition 3.4 to a, and let $p(a)$ and $k \geq 1$ be as in the same proposition. Then $p(a)$ is an idempotent in V, and by our assumption, the ideal

$\big(p(a)\big) \subseteq V$. Furthermore, since $a^k = a^k p(a)$ (by Proposition 3.4, *iii*)), we have $(a^k) \subseteq \big(p(a)\big) \subseteq V$. Hence, for all $m \geq k$, we have $ba^m c = ba^k(a^{m-k}c) \in (a^k) \subseteq V$. $\square$

One immediate consequence of Theorem 4.2 is the following corollary which provides a family of special ϑ-Mathieu subspaces.

Corollary 4.3. *Let $V \in \mathcal{G}(\mathcal{A})$ such that V does not contain any nonzero idempotent. Then V is a ϑ-Mathieu subspace of $\mathcal{A}$.*

Remark 4.4. *When the algebra $\mathcal{A}$ is algebraic over K, by Lemma 4.1 every K-subspace V of $\mathcal{A}$ lies in $\mathcal{G}(\mathcal{A})$. Then Theorem 4.2 gives another equivalent formulation for ϑ-Mathieu subspaces of algebraic K-algebras, which is more similar to the definition of ϑ-ideals than the one given in Definitions 1.1, 1.2 or in Proposition 2.1. For example, when $\mathcal{A}$ is algebraic over K, a K-subspace $M \subseteq \mathcal{A}$ is a left (resp., right) Mathieu subspace of $\mathcal{A}$ iff for any idempotent $a \in M$ and any $b \in \mathcal{A}$, we have $ba \in M$ (resp., $ab \in M$).*

Next, for any K-subspace V of $\mathcal{A}$ and $\vartheta \neq$ *"pre-two-sided"*, we let $I_{\vartheta,V}$ denote the ϑ-ideal of $\mathcal{A}$ which is maximum among all the ϑ-ideals of $\mathcal{A}$ contained in V. Note that by Zorn's lemma, it is easy to see that $I_{\vartheta,V}$ always exists and is unique. Actually, $I_{\vartheta,V}$ is the same as the sum of all the ϑ-ideals of $\mathcal{A}$ contained in V. For example, when V itself is a ϑ-ideal of $\mathcal{A}$, we have $I_{\vartheta,V} = V$. In particular, $I_{\vartheta,\mathcal{A}} = \mathcal{A}$.

Furthermore, for the case $\vartheta =$ *"pre-two-sided"*, we set

$$I_{\vartheta,V} := I_{left,V} + I_{right,V}.$$

In other words, $I_{\vartheta,V}$ with $\vartheta =$ *"pre-two-sided"* is the sum of the maximum left ideal contained in V and the maximum right ideal contained in V. Note that when $\mathcal{A}$ is not commutative, $I_{\vartheta,V}$ in this case is *not* necessarily a two-sided or one-sided ideal of $\mathcal{A}$.

Proposition 4.5. *Let $V \in \mathcal{G}(\mathcal{A})$ such that $I_{\vartheta,V} = 0$. Then V is a ϑ-Mathieu subspace of $\mathcal{A}$ iff V does not contain any nonzero idempotent.*

Consequently, for any $\vartheta \neq$ "pre-two-sided" and any algebraic K-algebra $\mathcal{A}$ that has no non-trivial ϑ-ideals, we have that a non-trivial K-subspace M of $\mathcal{A}$ is a ϑ-Mathieu subspace of $\mathcal{A}$ iff M does not contain any nonzero idempotent of $\mathcal{A}$.

Proof: The ($\Leftarrow$) part follows from Corollary 4.3. To show the ($\Rightarrow$) part, assume that there exists a nonzero idempotent $e \in V$. Then by Theorem 4.2, we have $(e)_\vartheta \subseteq V$, whence $(e)_\vartheta \subseteq I_{\vartheta,V}$. Since $0 \neq e \in I_{\vartheta,V}$, we have $I_{\vartheta,V} \neq 0$, which is a contradiction. $\square$

Corollary 4.6. *Let V be a K-subspace of $\mathcal{A}$ and $I_V = I_{\vartheta,V}$ with $\vartheta =$ "two-sided". Assume that $V \in \mathcal{G}(\mathcal{A})$ or $V/I_V \in \mathcal{G}(\mathcal{A}/I_V)$. Then V is a Mathieu subspace of $\mathcal{A}$ iff V/I_V does not contain any nonzero idempotent of the quotient K-algebra $\mathcal{A}/I_V$.*

Proof: First, it is easy to see that $V \in \mathcal{G}(\mathcal{A})$ implies $V/I_V \in \mathcal{G}(\mathcal{A}/I_V)$. So we may assume the latter. Second, since I_V is maximum among all the ideals of $\mathcal{A}$ that are contained in V, the quotient V/I_V does not contain any nonzero ideal of the quotient algebra $\mathcal{A}/I_V$, whence $I_{V/I_V} = 0$.

Now, applying Proposition 4.5 to the K-algebra $\mathcal{A}/I_V$ and its K-subspace V/I_V, we see that V/I_V is a Mathieu subspace of $\mathcal{A}/I_V$ iff V/I_V does not contain any nonzero idempotent of $\mathcal{A}/I_V$. On the other hand, by Proposition 2.7 we also have that V is a Mathieu subspace of $\mathcal{A}$ iff V/I_V is a Mathieu subspace of $\mathcal{A}/I_V$. Combining these two equivalences the corollary follows. □

Next we derive some consequences of Corollary 4.3 on finite dimensional ϑ-Mathieu subspaces of K-algebras.

Proposition 4.7. *Assume that $\mathcal{A}$ is purely transcendental over K, i.e., the only algebraic elements of $\mathcal{A}$ are the elements in $K \subseteq \mathcal{A}$. Then every finite dimensional K-subspace V of $\mathcal{A}$ such that $1 \notin V$ is a ϑ-Mathieu subspace of $\mathcal{A}$.*

Proof: Since $\mathcal{A}$ is purely transcendental over K and all idempotents of $\mathcal{A}$ are algebraic over K, we see that all idempotents of $\mathcal{A}$ must lie inside $K \subseteq \mathcal{A}$.

But, on the other hand, all idempotents of K are the solutions of the equation $t^2 - t = 0$ in K, which are $0, 1 \in K$. Therefore, all idempotents of $\mathcal{A}$ are trivial. Furthermore, since $1 \notin V$, we see that V does not contain any nonzero idempotent of $\mathcal{A}$. Then the proposition follows immediately from Lemma 4.1 and Corollary 4.3. □

The following characterization of one-dimensional ϑ-Mathieu subspaces of associative K-algebras will play important roles in the later Sections 5-7.

Proposition 4.8. *Let $\mathcal{A}$ be an associative K-algebra and $0 \neq a \in \mathcal{A}$. Then the one-dimensional K-subspace Ka is a ϑ-Mathieu subspace of $\mathcal{A}$ iff one of the following two statements holds:*

1) *Ka is a ϑ-ideal of $\mathcal{A}$, or equivalently, $Ka = (a)_\vartheta$.*
2) *a is not a quasi-idempotent of $\mathcal{A}$.*

It is an easy exercise to check that when $\vartheta =$ *"pre-two-sided"*, the equivalence in 1) above indeed holds, i.e., Ka is a *pre-two-sided* ideal of $\mathcal{A}$, which by definition means a (*two-sided*) ideal, iff $Ka = (a)_\vartheta = aA + Aa$.

Proof of Proposition 4.8: ($\Rightarrow$) Assume that Ka is a ϑ-Mathieu subspace of $\mathcal{A}$ but statement 2) fails, i.e., a is a nonzero quasi-idempotent of $\mathcal{A}$. Then by Lemma 2.9 *iii*), we have $(a)_\vartheta \subseteq Ka$. Since $(a)_\vartheta \supseteq Ka$, we have $(a)_\vartheta = Ka$, i.e., statement 1) holds.

($\Leftarrow$) If statement 1) holds, then Ka is a ϑ-ideal of $\mathcal{A}$ and hence, also a ϑ-Mathieu subspace of $\mathcal{A}$. Assume that statement 2) holds. Then for any $r \in K^\times$, $b := ra$ cannot be an idempotent, otherwise $a = r^{-1}b$ would be a quasi-idempotent too. Hence, Ka does not contain any nonzero idempotent of $\mathcal{A}$. Then by Lemma 4.1 and Corollary 4.3, Ka is a ϑ-Mathieu subspace of $\mathcal{A}$. $\square$

4.2. **Radicals of ϑ-Mathieu Subspaces $M \in \mathcal{E}_\vartheta(\mathcal{A})$ in Terms of Radicals of $I_{\vartheta,M}$.** Throughout this subsection, for each ϑ-Mathieu subspace M of $\mathcal{A}$, for convenience we denote by I_M the notation $I_{\vartheta,M}$ introduced in the previous subsection. In particular, when $\vartheta \neq$ *"pre-two-sided"*, I_M denotes the unique ϑ-ideal of $\mathcal{A}$ which is maximum among all the ϑ-ideals of $\mathcal{A}$ contained in M.

Lemma 4.9. *Let $\mathcal{A}$ be a K-algebra and M a ϑ-Mathieu subspace of $\mathcal{A}$. Then $\mathfrak{r}'(M) = \mathfrak{r}'(I_M)$.*

In particular, if $M \in \mathcal{E}_\vartheta(\mathcal{A})$, we have $\sqrt{M} = \sqrt{I_M}$.

Proof: Note first that since $M \supseteq I_M$, we have $\mathfrak{r}'(M) \supseteq \mathfrak{r}'(I_M)$.

To show $\mathfrak{r}'(M) \subseteq \mathfrak{r}'(I_M)$, let $a \in \mathfrak{r}'(M)$. Since a is algebraic over K, it follows from Theorem 3.10 that $(a^N)_\vartheta \subseteq M$ for some $N \geq 0$. Hence, we also have $(a^N)_\vartheta \subseteq I_M$. Consequently, $a^m \in I_M$ for all $m \geq N$, whence $a \in \mathfrak{r}'(I_M)$. $\square$

Theorem 4.10. *Let $M \in \mathcal{E}_\vartheta(\mathcal{A})$ and V a K-subspace of M such that $I_M \subseteq V$. Then V is a ϑ-Mathieu subspace of $\mathcal{A}$ and $\sqrt{V} = \sqrt{I_M}$.*

Proof: Note first that by Lemma 4.9, it suffices to show that V is a ϑ-Mathieu subspace of $\mathcal{A}$, for we obviously have $V \in \mathcal{G}(\mathcal{A})$ and $I_V = I_M$.

Let e be a nonzero idempotent in V. Hence, $e \in M$ since $V \subseteq M$. Then by Theorem 4.2, we have $(e)_\vartheta \subseteq M$, whence $(e)_\vartheta \subseteq I_M \subseteq V$. Then by Theorem 4.2 again, V is a ϑ-Mathieu subspace of $\mathcal{A}$. $\square$

Corollary 4.11. *Let $\mathcal{A}$ be a simple and algebraic K-algebra and M a proper Mathieu subspace of $\mathcal{A}$. Then $\sqrt{M} = \mathrm{nil}\,(\mathcal{A})$ and all K-subspaces $V \subseteq M$ are also Mathieu subspaces of $\mathcal{A}$.*

Proof: Since $\mathcal{A}$ is simple, we have $I_M = 0$. Since $\mathcal{A}$ is algebraic over K, by Lemmas 4.1 and 4.9 we have $\sqrt{M} = \sqrt{0} = \mathrm{nil}\,(\mathcal{A})$. Then the corollary follows from Theorem 4.10 or Lemma 2.8. $\square$

When the K-algebra $\mathcal{A}$ is commutative, we have the following characterization for the Mathieu subspaces with algebraic radicals.

Theorem 4.12. *Let $\mathcal{A}$ be a commutative K-algebra and $V \in \mathcal{G}(\mathcal{A})$. Then V is a Mathieu subspace of $\mathcal{A}$ iff $\sqrt{V}$ is an ideal of $\mathcal{A}$.*

Proof: It is well-known that the radicals of ideals of commutative algebras are (radical) ideals. Then the ($\Rightarrow$) part follows immediately from Lemma 4.9, for $\sqrt{V} = \sqrt{I_V}$ and I_V is an ideal of $\mathcal{A}$.

To show the ($\Leftarrow$) part, by Theorem 4.2 it suffices to show that for each idempotent $e \in V$, we have $(e) \subseteq V$. Equivalently, it suffices to show that the K-subspace $V_e := \{a \in \mathcal{A} \,|\, ea \in V\}$ is equals to $\mathcal{A}$ itself.

Note first that by Lemma 2.9, *ii*) we have $e \in \sqrt{V}$. Since $\sqrt{V}$ by our assumption is an ideal of $\mathcal{A}$, we have $eb \in \sqrt{V}$ for all $b \in \mathcal{A}$. Then for all $m \gg 0$, we have $eb^m = (eb)^m \in V$ or equivalently, $b^m \in V_e$. Hence, $b \in \sqrt{V_e}$ for all $b \in \mathcal{A}$, whence $\sqrt{V_e} = \mathcal{A}$. Applying Lemma 2.4 to the K-subspace V_e, we get $V_e = \mathcal{A}$. $\square$

One by-product of Theorem 4.12 is the following corollary which does not seem obvious.

Corollary 4.13. *Let $\mathcal{A}$ be a commutative K-algebra and $V \in \mathcal{G}(\mathcal{A})$. Then $\sqrt{V}$ is a radical ideal of $\mathcal{A}$ if (and only if) $\sqrt{V}$ is an ideal of $\mathcal{A}$.*

Proof: Assume that $\sqrt{V}$ is an ideal of $\mathcal{A}$. Then by Theorem 4.12, V is a Mathieu subspace of $\mathcal{A}$, and by Lemma 2.2, $\sqrt{V}$ is a radical ideal of $\mathcal{A}$. $\square$

Next, we conclude this subsection with the following two remarks.

First, as we can see from the example below, without the algebraic condition on $\sqrt{V}$ Theorem 4.12 does not always hold.

Example 4.14. *Let $\mathcal{A}$ be the Laurent polynomial algebra $\mathbb{C}[t^{-1}, t]$ in one variable t over $\mathbb{C}$ and V the subspace of all Laurent polynomials in $\mathcal{A}$ without constant terms. Then by the Duistermaat-van der Kallen theorem* [DK]*, V is a Mathieu subspace of $\mathcal{A}$ and $\sqrt{V} = t\mathbb{C}[t] \bigcup t^{-1}\mathbb{C}[t^{-1}]$, which is not even a $\mathbb{C}$-subspace and hence, not an ideal of $\mathcal{A}$.*

Second, even though the univariate polynomial algebra $K[t]$ is purely transcendental over K, by using Theorems 4.10 and 4.12, it has been shown recently in [EZ] that the following theorem actually also holds.

Theorem 4.15. ([EZ]) *Let V be a K-subspace of the univariate polynomial algebra $K[t]$. Then V is a Mathieu subspace of $K[t]$ iff $\sqrt{V} = \sqrt{I_V}$.*

But, it has also been shown in [EZ] that the theorem above fails for multi-variable polynomial algebras.

4.3. **Unions and Intersections of Mathieu Subspaces with Algebraic Radicals.** First, let's prove the following proposition on the intersections of ϑ-Mathieu subspaces.

Proposition 4.16. *Let M_i $(i \in I)$ be a family of ϑ-Mathieu subspaces of a K-algebra $\mathcal{A}$. Assume that $M_i \in \mathcal{G}(\mathcal{A})$ for some $i \in I$, or the intersection $\bigcap_{i\in I} M_i \in \mathcal{G}(\mathcal{A})$. Then $\bigcap_{i\in I} M_i$ is also a ϑ-Mathieu subspace of $\mathcal{A}$.*

Proof: Note first that the condition that $M_i \in \mathcal{G}(\mathcal{A})$ for some $i \in I$ obviously implies the condition $\bigcap_{i\in I} M_i \in \mathcal{G}(\mathcal{A})$. So we may assume the latter.

Let e be any idempotent in $\bigcap_{k\geq 1} M_k$. Then for any $i \in I$, we have $e \in M_i$ and by Lemma 2.9 *iii*), $(e)_\vartheta \subseteq M_i$, whence $(e)_\vartheta \subseteq \bigcap_{i\in I} M_i$. Applying Theorem 4.2 to $\bigcap_{i\in I} M_i$, the proposition follows. $\square$

It is worthy to point out that it is easy to check (or see Proposition 4.9 in [Z4]) that in general the intersection of any finitely many ϑ-Mathieu subspaces is always a ϑ-Mathieu subspace. However, when $|I| = \infty$, Proposition 4.16 without the algebraic conditions does not always hold.

Example 4.17. *Let $\mathcal{A}$ be the polynomial algebra $K[t]$ in one variable t over K and M_i $(i \geq 0)$ the K-subspace of $\mathcal{A}$ spanned by the monomials t^k with $k \geq 1$ but $k \neq 2j+1$ for all $0 \leq j \leq i$. Then it is easy to check that for each $i \geq 0$, $\sqrt{M_i} = tK[t]$ and M_i is a Mathieu subspace of $K[t]$.*

On the other hand, we also have $M := \bigcap_{i\geq 0} M_i = t^2K[t^2]$. Note that $t^2 \in \sqrt{M}$. But, for each $m \geq 1$, $t(t^2)^m = t^{2m+1} \notin M$. Hence, the intersection M of M_i $(i \geq 0)$ is not a Mathieu subspace of $\mathcal{A}$.

Next we consider unions of ascending sequences of ϑ-Mathieu subspaces under certain conditions.

Proposition 4.18. *Let M_i $(i \geq 1)$ be a sequence of non-trivial ϑ-Mathieu subspaces of $\mathcal{A}$ such that $M_i \subseteq M_{i+1}$ for all $i \geq 1$. Assume that $\bigcup_{i\geq 1} M_i \in \mathcal{G}(\mathcal{A})$. Then $\bigcup_{i\geq 1} M_i$ is also a non-trivial ϑ-Mathieu subspace of $\mathcal{A}$.*

Proof: First, since M_i is non-trivial for each $i \geq 1$, by Lemma 2.10 we have $1 \notin M_i$, whence $1 \notin \bigcup_{i\geq 1} M_i$. So the union $\bigcup_{i\geq 1} M_i$ is also non-trivial.

Second, let e be an idempotent in $\bigcup_{i\geq 1} M_i$. Then $e \in M_k$ for some $k \geq 1$, and by Lemma 2.9 *iii*), $(e)_\vartheta \subseteq M_k$. Hence we have $(e)_\vartheta \subseteq \bigcup_{i\geq 1} M_i$. Then by Theorem 4.2, $\bigcup_{i\geq 1} M_i$ is a ϑ-Mathieu subspace of $\mathcal{A}$. $\square$

One remark on Proposition 4.18 is that without the algebraic condition on the radical of the union, the proposition does not necessarily hold.

Example 4.19. *Let $\mathcal{A} = K[t]$ as in Example 4.17 and V_i $(i \geq 1)$ the K-subspace of $\mathcal{A}$ spanned by the monomials t^{2j} $(1 \leq j \leq i)$. Note that for any $i \geq 1$, we have $1 \notin V_i$ and $\dim_K V_i < \infty$. Then it follows from Proposition 4.7 that for any $i \geq 1$, V_i is a Mathieu subspace of $\mathcal{A}$.*

But, on the other hand, we have $\bigcup_{i\geq 1} V_i = t^2K[t^2]$, which as shown in Example 4.17, is not a Mathieu subspace of $\mathcal{A}$.

Next, we use Zorn's lemma and Propositions 4.16 and 4.18 to derive existences of certain *maximal* (resp., *minimal*) non-trivial ϑ-Mathieu subspaces for algebraic K-algebras $\mathcal{A}$.

First, note that if $\mathcal{A}$ is algebraic over K, then by Lemma 4.1 the algebraic conditions in Propositions 4.16 and 4.18 are automatically satisfied. With this observation and by Zorn's lemma, we immediately have the following proposition.

Proposition 4.20. *Let $\mathcal{A}$ be an algebraic K-algebra and V a K-subspace of $\mathcal{A}$. Then the following statements hold.*

i) There exists at least one ϑ-Mathieu subspace of $\mathcal{A}$ which is maximal among all the ϑ-Mathieu subspaces of $\mathcal{A}$ contained in V.

ii) There exists a unique ϑ-Mathieu subspace M of $\mathcal{A}$ which is minimum among all the ϑ-Mathieu subspaces W of $\mathcal{A}$ with $V \subseteq W$. Actually, M is given by the intersection of all ϑ-Mathieu subspaces that contain V.

iii) Any non-empty collection of proper ϑ-Mathieu subspaces M of $\mathcal{A}$ with $V \subseteq M$ has at least one maximal element and a (unique) minimum element.

Theorem 4.21. *Assume that $\mathcal{A}$ is algebraic over K but $\mathcal{A} \neq K$. Then for any proper ϑ-Mathieu subspace M of $\mathcal{A}$, there exists a maximal non-trivial ϑ-Mathieu subspace of $\mathcal{A}$ which contains M.*

In particular, (by taking $M = 0$), $\mathcal{A}$ has at least one maximal non-trivial ϑ-Mathieu subspace.

Proof: Let $\mathcal{F}$ be the collection of the non-trivial ϑ-Mathieu subspaces J of $\mathcal{A}$ such that $M \subseteq J$. If $M \neq 0$, then $M \in \mathcal{F}$. If $M = 0$, then by Lemma 6.4 in the later Section 6, $\mathcal{A}$ has at least one non-trivial ϑ-Mathieu subspace J, which obviously lies in $\mathcal{F}$. Therefore, in any case $\mathcal{F} \neq \emptyset$. Then the theorem follows directly from Proposition 4.20 *iii*). $\square$

Corollary 4.22. *Let V be a K-subspace of an algebraic K-algebra $\mathcal{A}$ such that the ϑ-ideal generated by elements of V is non-trivial . Then there exists a maximal non-trivial ϑ-Mathieu subspace M of $\mathcal{A}$ such that $V \subseteq M$.*

Proof: Since any ϑ-ideal is a ϑ-Mathieu subspace, the corollary follows immediately from Theorem 4.21 by taking M to be the ϑ-ideal generated by elements of V. $\square$

5. Co-dimension One Mathieu Subspaces and the Minimal Non-trivial Mathieu Subspaces of Matrix Algebras over Fields

Let K be an arbitrary field and $n \geq 1$. In this section we classify the co-dimension one ϑ-Mathieu subspaces and the minimal non-trivial ϑ-Mathieu subspaces for the matrix algebra $M_n(K)$ of $n \times n$ matrices with entries in K.

First, let's fix the following notations that will be used throughout this section.

We denote by I_n the *identity* matrix in $M_n(K)$. For each $X \in M_n(K)$, we denote by $\operatorname{Tr} X$ the *trace* of the matrix X and set

$$H_X := \{A \in M_n(K) \mid \operatorname{Tr}(AX) = 0\}. \tag{5.1}$$

When $X = I_n$, H_{I_n} will also be denoted by H, i.e.,

$$H := \{A \in M_n(K) \mid \operatorname{Tr} A = 0\}. \tag{5.2}$$

For any $X, Y \in M_n(K)$, we denote by $X \sim Y$ if $X = sY$ for some $s \in K^\times$. Note that by Lemma 5.2 below, we have

$$H_X = H_Y \Leftrightarrow X \sim Y. \tag{5.3}$$

In particular, we have

$$H_X = H \Leftrightarrow X \sim I_n. \tag{5.4}$$

With the notations fixed above, the first main result of this section can be stated as follows.

Theorem 5.1. *Let K be a field and $n \geq 1$. Then the following two statements hold:*

i) if $\mathrm{char.}\, K = 0$ or $\mathrm{char.}\, K = p > n$, then H is the only co-dimension one ϑ-Mathieu subspace of $M_n(K)$;

ii) if $\mathrm{char.}\, K = p > 0$ and $p \leq n$, then $M_n(K)$ has no co-dimension one ϑ-Mathieu subspaces.

In order to prove the theorem, we first need to prove the following two lemmas.

Lemma 5.2. *For every co-dimension one K-subspace V of $M_n(K)$, there exists $0 \neq X \in M_n(K)$ such that $V = H_X$. Furthermore, X is unique up to nonzero scalar multiplications.*

Proof: First, let's consider the following K-bilinear form of $M_n(K)$:

$$\begin{aligned} (\cdot,\cdot) : M_n(K) \times M_n(K) &\to K \\ (A, \quad B) &\to \mathrm{Tr}\,(AB). \end{aligned} \tag{5.5}$$

It is well-known and also easy to check that the bilinear form above is non-singular. Hence, it induces a K-linear isomorphism

$$\begin{aligned} \phi : M_n(K) &\xrightarrow{\sim} \mathrm{Hom}_K(M_n(K), K) \\ B \quad &\to \qquad \phi_B, \end{aligned} \tag{5.6}$$

where $\phi_B : M_n(K) \to K$ is the linear functional of $M_n(K)$ defined by setting for all $A \in M_n(K)$,

$$\phi_B(A) := \mathrm{Tr}\,(AB). \tag{5.7}$$

Note that any co-dimension one K-subspace of $M_n(K)$ is the kernel of a nonzero linear functional of $M_n(K)$, which is unique up to nonzero scalar multiplications. Then by the K-linear isomorphism in Eq. (5.6), we see that for the co-dimension one subspace V in the lemma, there exists $0 \neq X \in M_n(K)$, which is unique up to nonzero scalar multiplications, such that $V = \mathrm{Ker}\ \phi_X$. Furthermore, by Eqs. (5.7) and (5.1), we also have $\mathrm{Ker}\ \phi_X = H_X$, whence the lemma follows. □

Lemma 5.3. *Let $n \geq 2$ and $0 \neq X \in M_n(K)$ such that $X \not\sim I_n$. Then there exist non-trivial idempotents $A, B \in M_n(K)$ such that*

$$AX \neq 0; \tag{5.8}$$

$$XB \neq 0; \tag{5.9}$$

$$\mathrm{Tr}\,(AX) = \mathrm{Tr}\,(XB) = 0. \tag{5.10}$$

Proof: First, it is easy to check that the existence of the idempotent B for X follows from that of the idempotent A for X^τ by letting $B = A^\tau$, where X^τ and A^τ are the transposes of X and A, respectively. So it suffices to show the existence of the non-trivial idempotent A.

We first consider the case $n = 2$. Write $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $a, b, c, d \in K$. We divide the proof into the following three different cases.

<u>*Case 1*</u>: If $b \neq 0$, let $A := \begin{pmatrix} 1 & 0 \\ -ab^{-1} & 0 \end{pmatrix}$. Then we have

$$AX = \begin{pmatrix} a & b \\ -a^2b^{-1} & -a \end{pmatrix} \neq 0 \tag{5.11}$$

$$\mathrm{Tr}\,(AX) = 0 \quad \text{and} \quad A^2 = A. \tag{5.12}$$

<u>*Case 2*</u>: If $b = 0$ but $c \neq 0$, let $A := \begin{pmatrix} 0 & -c^{-1}d \\ 0 & 1 \end{pmatrix}$. Then we have

$$AX = \begin{pmatrix} -d & -c^{-1}d^2 \\ c & d \end{pmatrix} \neq 0 \tag{5.13}$$

$$\mathrm{Tr}\,(AX) = 0 \quad \text{and} \quad A^2 = A. \tag{5.14}$$

<u>*Case 3*</u>: If $b = c = 0$, then $a \neq d$ since by our assumption $X \not\sim I_2$. In particular, a and d cannot be both zero.

Let $A := \frac{1}{d-a} \begin{pmatrix} d & d \\ -a & -a \end{pmatrix}$. Then we have

$$AX = \frac{1}{d-a} \begin{pmatrix} ad & d^2 \\ -a^2 & -ad \end{pmatrix} \neq 0 \tag{5.15}$$

$$\mathrm{Tr}\,(AX) = 0 \quad \text{and} \quad A^2 = A. \tag{5.16}$$

It is straightforward to check that all the equations (5.11)–(5.16) do hold and that the idempotent A in each case is non-trivial. So we omit the details here.

Next, we consider the case $n \geq 3$. Since $X \not\sim I_n$, it is easy to see that there exist $1 \leq m < k \leq n$ such that the 2×2-minor of X on m^{th}, k^{th} rows and m^{th}, k^{th} columns is not a multiple of I_2.

Since in general idempotents and also traces of matrices are preserved by conjugations, by applying some conjugations by permutation matrices to X if it is necessary, we may further assume $m = 1$ and $k = 2$. We denote by X' this 2×2 minor of X.

By the lemma for the case $n = 2$, there exists a non-trivial idempotent $A' \in M_2(K)$ such that $A'X' \neq 0$ and $\mathrm{Tr}\,(A'X') = 0$. Let $A := \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix} \in M_n(K)$. Then it is easy to check that A is a non-trivial idempotent of $M_n(K)$ which satisfies $AX \neq 0$ and $\mathrm{Tr}\,(AX) = 0$. Hence, the lemma also holds for the case $n \geq 3$. $\square$

Note that one bi-product of Lemmas 5.2 and 5.3 above is the following corollary.

Corollary 5.4. *Let $n \geq 2$ and V be a co-dimension one K-subspace of $M_n(K)$ such that $V \neq H$. Then V contains at least one non-trivial idempotent of $M_n(K)$.*

Proof: By Lemma 5.2, we know that $V = H_X$ for some $0 \neq X \in M_n(K)$. Since $V \neq H$, we have $X \not\sim I_n$. Then by Lemma 5.3, there exists a non-trivial idempotent A of $M_n(K)$, which satisfis Eq. (5.10). Hence by Eq. (5.1), we have $A, B \in H_X = V$. $\square$

Now we can prove the first main result of this section as follows.

Proof of Theorem 5.1: Note first that if $n = 1$, then the theorem obviously holds. So we assume $n \geq 2$.

Let V be a co-dimension one K-subspace of $M_n(K)$ such that $V \neq H$. Then by Lemma 5.2, $V = H_X$ for some $0 \neq X \in M_n(X)$. Note that by Eq. (5.4), $X \not\sim I_n$ since $V \neq H$. Next, we show that V cannot be a left or right Mathieu subspace of $M_n(X)$.

Assume that V is a left Mathieu subspace of $M_n(K)$. Let A be the non-trivial idempotent as in Lemma 5.3. Then by Eqs. (5.1) and (5.10), we have $A \in H_X = V$. Furthermore, by Lemma 2.9 *iii*), we have $CA \in V = H_X$ for all $C \in M_n(K)$. More precisely, we have

$$0 = \mathrm{Tr}\,\big((CA)X\big) = \mathrm{Tr}\,\big(C(AX)\big) \tag{5.17}$$

for all $C \in M_n(K)$.

Since the K-bilinear form in Eq. (5.5) is non-singular, we have $AX = 0$. But this contradicts Eq. (5.8) in Lemma 5.3. Therefore, V cannot be a left Mathieu subspace of $M_n(K)$.

Assume that V is a right Mathieu subspace of $M_n(K)$. Let B be the non-trivial idempotent as in Lemma 5.3. Then by Eqs. (5.1) and (5.10)

we have $B \in H_X = V$, and by Lemma 2.9 *iii*), $BC \in V = H_X$ for all $C \in M_n(K)$. More precisely, we have

$$0 = \mathrm{Tr}\left((BC)X\right) = \mathrm{Tr}\left(B(CX)\right) = \mathrm{Tr}\left((CX)B\right) = \mathrm{Tr}\left(C(XB)\right)$$

for all $C \in M_n(K)$.

Then by the non-singularity of the K-bilinear form in Eq. (5.5) again, we have $XB = 0$, which contradicts Eq. (5.9) in Lemma 5.3. Therefore, V cannot be a right Mathieu subspace of $M_n(K)$ either.

Therefore, for any specification of ϑ, the only possible co-dimension one ϑ-Mathieu subspace of $M_n(K)$ is the K-subspace H of the trace-zero matrices in $M_n(K)$, which we will consider next.

Assume first *char.* $K = p \leq n$. Let $e_p := \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} \in M_n(K)$. Note that e_p is a nonzero idempotent lying in H, and $(e_p)_\vartheta$ clearly contains the subalgebra $\begin{pmatrix} M_p(K) & 0 \\ 0 & 0 \end{pmatrix} \subseteq M_n(K)$, which certainly cannot be entirely contained in H. Therefore, we have $(e_p)_\vartheta \nsubseteq H$. Then by Lemma 2.9, *iii*) or Theorem 4.2, H in this case cannot be a ϑ-Mathieu subspace of $M_n(K)$, whence the statement *ii*) of the theorem follows.

Now, assume *char.* $K = 0$ or *char.* $K = p > n$. Then it is well-known in linear algebra that for any $A \in M_n(K)$, A is nilpotent iff for all $m \geq 1$, $\mathrm{Tr}\,(A^m) = 0$, i.e., $A^m \in H$. Hence, we have $\sqrt{H} = \mathrm{nil}\,(M_n(K))$. Then by Lemma 2.8 the statement *i*) of the theorem also follows. □

Next we give a classification for the minimal non-trivial ϑ-Mathieu subspaces of the matrix algebras $M_n(K)$ $(n \geq 2)$.

Proposition 5.5. *A K-subspace $V \subset M_n(K)$ $(n \geq 2)$ is a minimal non-trivial ϑ-Mathieu subspace of $M_n(K)$ iff $V = KA$ for some nonzero $A \in M_n(K)$ which is not a quasi-idempotent.*

To prove the proposition, we need first to show the following lemma.

Lemma 5.6. *For any $n \geq 2$ and $0 \neq A \in M_n(K)$, we have*

i) $(A)_\vartheta \neq KA$;
ii) $(A)_\vartheta$ *contains at least one element which is not a quasi-idempotent.*

Note that from the well-known fact that $M_n(K)$ is a simple K-algebra (e.g., see the lemma on p. 9 in [Pi]), it follows immediately that the lemma holds for the two-sided case, since in this case $(A)_\vartheta = (A) = M_n(K)$. But, for the other cases, we need a different argument given below, which actually works for all the cases.

Proof of Lemma 5.6: Note first that for any ϑ, $(A)_\vartheta$ contains either the left ideal generated by A or the right ideal generated by A. Therefore, it suffices to show the proposition for the two cases: $\vartheta =$ *"left"* and $\vartheta =$ *"right"*.

We here just give a proof for the former case. The latter case follows from the former one for the transpose A^τ of A, or by applying the similar arguments. So for the rest of the proof, we set $\vartheta =$ *"left"*.

i) Assume otherwise, i.e., $(A)_\vartheta = KA$. Then for any $X \in M_n(K)$, we have $XA = rA$ for some $r \in K$. Consequently, each column of A is a common eigenvector of all matrices $X \in M_n(K)$, which is clearly impossible unless the column is equal to zero. Therefore, we have $A = 0$, which is a contradiction.

ii) Note first that since quasi-idempotents are preserved by taking conjugations, we may replace A by any conjugation of A.

Write $A = X \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} Y$ for some $1 \leq k \leq n$ and invertible $X, Y \in M_n(K)$. Replacing A by YAY^{-1}, we have $A = YX \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$. Since YX is invertible, the left ideal $(A)_\vartheta$ generated by A is the same as the left ideal generated by $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$. Hence, we may assume $A = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ for some $1 \leq k \leq n$.

Now, let $B = (b_{ij}) \in M_n(K)$ such that $b_{ij} = 1$ if $i = 2$ and $j = 1$; and 0 otherwise. Then we have $B = BA \in (A)_\vartheta$. Since B is nonzero and nilpotent, it follows from Lemma 2.9 *i*) that B cannot be a quasi-idempotent, whence the statement follows. □

Proof of Proposition 5.5: The ($\Leftarrow$) part follows directly from Proposition 4.8. To show the ($\Rightarrow$) part, we first show $\dim_K V = 1$.

Assume otherwise. Then for any nonzero $A \in V$, the line $KA \neq V$ and hence, cannot be a ϑ-Mathieu subspace of $M_n(K)$, for V is minimal. Applying Proposition 4.8 to A, we see that A must be a quasi-idempotent. Therefore, all elements of V must be quasi-idempotents. Moreover, by Lemma 2.9 *iii*), for any nonzero $A \in V$, we have $(A)_\vartheta \subset V$, whence all elements of $(A)_\vartheta$ are also quasi-idempotents. But this contradicts Lemma 5.6 *ii*).

Now, write $V = KA$ for some $0 \neq A \in M_n(K)$. Then from Proposition 4.8 and Lemma 5.6 *i*), it follows that A cannot be a quasi-idempotent. □

Finally, we conclude this section with the following remarks on the maximal non-trivial ϑ-Mathieu subspaces of $M_n(K)$.

In contrast to the minimal non-trivial case, the situation for the maximal non-trivial ϑ-Mathieu subspaces of $M_n(K)$ becomes much more complicated. Even though Theorem 5.1 classifies the co-dimension one maximal ϑ-Mathieu subspace of $M_n(K)$, there are also many others (with different co-dimensions).

For example, pick up any $A \in M_n(K)\backslash H$ (i.e., $\operatorname{Tr} A \neq 0$) such that A is not a quasi-idempotent. Then by Proposition 4.8, the line KA is a ϑ-Mathieu subspace of $\mathcal{A}$. If $n \geq 2$, then by Proposition 4.21 or by counting dimensions, KA is contained in at least one maximal non-trivial ϑ-Mathieu subspace W of $M_n(K)$. But, since $A \in W$ and $A \notin H$, we have $W \neq H$.

The situation for the two-sided case can be slightly improved by the following proposition.

Proposition 5.7. *Let V be a proper K-subspace of $M_n(K)$. Then V is a Mathieu subspace of $M_n(K)$ iff V does not contain any nonzero idempotent of $M_n(K)$.*

Proof: The ($\Leftarrow$) part follows immediately from Lemma 4.1 and Corollary 4.3 since $\dim_K M_n(K) < \infty$.

To show ($\Rightarrow$) part, assume otherwise, i.e., there exists a nonzero idempotent $A \in V$. Then by Lemma 2.9 *iii*), the ideal (A) of $M_n(K)$ generated by A is also contained in V. But, on the other hand, it is well-known that $M_n(K)$ is a simple K-algebra (e.g., see the lemma on p. 9 in [Pi]). Hence $(A) = M_n(K)$, whence $V = M_n(K)$. But this contradicts the assumption that V is proper. $\square$

It will be interesting if one can get a more explicit classification (other than the ones given by Theorem 4.2 and Proposition 5.7 plus the maximality) of all maximal non-trivial ϑ-Mathieu subspaces for matrix algebras $M_n(K)$ ($n \geq 2$), or even more generally, for all finite dimensional or algebraic K-algebras.

6. **Strongly Simple Algebras**

As we have mentioned earlier, the notion of Mathieu subspaces can be viewed as a natural generalization of the notion of ideals. Note that one of the most important families of (associative) algebras are simple algebras, i.e., the algebras that have no non-trivial ideals. Then parallel to simple algebras, we have the following family of special algebras.

Definition 6.1. *Let R be a commutative ring and $\mathcal{A}$ a R-algebra. We say that $\mathcal{A}$ is a strongly simple R-algebra if $\mathcal{A}$ has no non-trivial* (*two-sided*) *Mathieu subspaces.*

Formally, one may also consider *left* (resp., *right*, *pre-two-sided*) *strongly simple algebras*, i.e., the algebras that have no non-trivial *left* (resp., *right*, *pre-two-sided*) Mathieu subspaces. But, as we will show in Theorem 6.2 below, every (two-sided) strongly simple algebras is commutative. From this fact, it is easy to see that the notion of *left*, *right* or *pre-two-sided*) strongly simple algebras is actually equivalent to the notion of (two-sided) strongly simple algebras. In other words, an algebra is *left*, *right* or *pre-two-sided* strongly simple iff it is (two-sided) strongly simple.

In this section, we give a characterization for strongly simple algebras $\mathcal{A}$ over arbitrary commutative rings R. For convenience, throughout the rest of this section except in Corollary 6.8, we assume that the base ring R is contained in the R-algebra $\mathcal{A}$. Note that by replacing R by $R \cdot 1_{\mathcal{A}} \subseteq \mathcal{A}$, this condition will be satisfied. Furthermore, when R is an integral domain, we denote by K_R the field of fractions of R. Note that by *"integral domains"* we always mean *commutative* domains.

Under the assumption and notation above, the first main result of this section can be stated as follows.

Theorem 6.2. *Let R be a commutative ring and $\mathcal{A}$ a R-algebra. Then $\mathcal{A}$ is a strongly simple R-algebra (if and) only if the following three statements hold:*

i) R is an integral domain;
ii) $\mathcal{A} \simeq K_R$ as R-algebras;
iii) K_R as a R-algebra is strongly simple.

One immediate consequence of the theorem above is the following corollary.

Corollary 6.3. *Let R be a commutative ring and $\mathcal{A}$ a R-algebra. Assume that either R is not an integral domain, or $\mathcal{A}$ is not commutative, or $\mathcal{A}$ is commutative but not a field. Then $\mathcal{A}$ has at least one non-trivial Mathieu subspace.*

In order to prove Theorem 6.2, we first need to show the following lemma which is the special case of the theorem when the base ring R is a field K.

Lemma 6.4. *Let K be a field and $\mathcal{A}$ a K-algebra. Then $\mathcal{A}$ is strongly simple (if and) only if $\mathcal{A} = K$.*

Proof: Assume otherwise, i.e., $\mathcal{A} \neq K$. Then there exists $a \in \mathcal{A}$ such that a is linearly independent with $1 \in \mathcal{A}$ over K. Throughout the rest of the proof, we fix such an element a and derive a contradiction as follows.

First, since every non-trivial ideal of $\mathcal{A}$ is a non-trivial Mathieu subspace of $\mathcal{A}$, we see that $\mathcal{A}$ cannot have any non-trivial ideals, which means that $\mathcal{A}$ must be a simple K-algebra.

Second, for any nonzero $b \in \mathcal{A}$, the one-dimensional K-subspace $Kb \subset \mathcal{A}$ is non-trivial but cannot be a Mathieu subspace of $\mathcal{A}$. Then by Proposition 4.8, b must be a quasi-idempotent of $\mathcal{A}$. Therefore, all nonzero elements of $\mathcal{A}$ are quasi-idempotents.

In particular, the element $a \in \mathcal{A}$ fixed at the beginning is a quasi-idempotent. Replacing a by a scalar multiple of a, we further assume from now on that a is an idempotent which is linearly independent with $1 \in \mathcal{A}$.

Next, with the two observations above in mind we consider the following two different cases.

<u>*Case 1*</u>: Assume $K \simeq \mathbb{Z}_2$. Then in this case all elements of $\mathcal{A}$ are actually idempotents instead of just being quasi-idempotents (since the only nonzero element of the base field K is $1 \in K$). It is well-known or from the simple argument below that $\mathcal{A}$ in this case is actually a commutative algebra. Since $\mathcal{A}$ is also simple, we see that $\mathcal{A}$ in this case is actually a field extension of $\mathbb{Z}_2$.

Let $b, c \in \mathcal{A}$. Then b, c and $b + c$ are all idempotents. From the equations $(b + c)^2 = b + c$; $b^2 = b$ and $c^2 = c$, it is easy to see that $bc = -cb = cb$.

Now, let $a \in \mathcal{A}$ be the idempotent fixed above. Since $a \neq 0$ and $\mathcal{A}$ is a field, a is invertible. Then by Lemma 2.9 i), we have $a \in K^\times$. But this contradicts our assumption that a and 1 are linearly independent over K.

<u>*Case 2*</u>: Assume $K \not\simeq \mathbb{Z}_2$. Then there exists $r \in K^\times$ such that $r \neq -1$. Set $b := 1 + ra$, where a is as fixed before. Note that $b \neq 0$ since 1 and a are linearly independent over K. Then we have $b^2 = sb$ for some $s \in K^\times$. More precisely, we have

$$\begin{aligned} s(1 + ra) = (1 + ra)^2 &= 1 + 2ra + r^2a^2 \\ &= 1 + 2ra + r^2a = 1 + (2 + r)ra. \end{aligned}$$

By comparing the coefficients of 1 and a in the equation above, we get

$$\begin{cases} s = 1, \\ sr = (2 + r)r. \end{cases}$$

Solving the equation above, we get $r = -1$, which is a contradiction again. Therefore, the lemma holds. $\square$

The following lemma will also be important to us.

Lemma 6.5. *Let S be a subring of a R-algebra $\mathcal{A}$ such that $R \subseteq S \subseteq Z(\mathcal{A})$, where $Z(\mathcal{A})$ denotes the center of $\mathcal{A}$. Assume that $\mathcal{A}$ as a R-algebra is strongly simple. Then $\mathcal{A}$ as a S-algebra is also strongly simple.*

Proof: Since $S \subseteq Z(\mathcal{A})$, $\mathcal{A}$ can also be viewed as a S-algebra (in the obvious way). Moreover, since $R \subseteq S$, every S-subspace of $\mathcal{A}$ is also a R-subspace of $\mathcal{A}$. With these observations, the lemma follows immediately from the definition of Mathieu subspaces (see Definitions 1.1 and 1.2) and that of strongly simple algebras (see Definition 6.1). □

Now we can prove Theorem 6.2 as follows.

Proof of Theorem 6.2: First, let $0 \neq r \in R \subseteq \mathcal{A}$. Since r commutes with all elements of $\mathcal{A}$, $\mathcal{A}r$ is a nonzero (two-sided) ideal and hence, also a nonzero Mathieu subspace of $\mathcal{A}$. Since $\mathcal{A}$ is a strongly simple R-algebra, we have $\mathcal{A}r = \mathcal{A}$. In particular, $1 \in \mathcal{A}r$ and r is invertible in $\mathcal{A}$. Therefore, all nonzero elements of R are invertible in $\mathcal{A}$, whence R must be an integral domain, i.e., the statement $i)$ in the theorem holds.

Furthermore, since $R \subseteq \mathcal{A}$, we may also assume that $\mathcal{A}$ contains the field of fractions K_R of R. Since all elements of R are central elements of $\mathcal{A}$, it is easy to check that so are all elements of $K_R \subseteq \mathcal{A}$. Therefore, $\mathcal{A}$ can also be viewed as a K_R-algebra.

Now, since $\mathcal{A}$ is strongly simple as a R-algebra, by Lemma 6.5 with $S = K_R$, it is also strongly simple as a K_R-algebra. Then by Lemma 6.4, we have $\mathcal{A} = K_R$. Therefore, the statement $ii)$ in the theorem holds. The statement $iii)$ follows from the statement $ii)$ and our assumption on the R-algebra $\mathcal{A}$. □

From Theorem 6.2, we see that in order to classify all strongly simple algebras, it suffices to classify all the integral domains R whose field of fractions $K_R \neq R$ and as a R-algebra is strongly simple.

We have not succeeded in classifying this special family of integral domains. Instead, we show next that no Noetherian domain or Krull domain belongs to this family. To do so, we first need to prove the following lemma.

Lemma 6.6. *Let R be an integral domain with $R \neq K_R$. Assume that there exists a non-trivial real-valued additive valuation ν of K_R such that $\nu(r) \geq 0$ for all $r \in R$. Then K_R as a R-algebra is not strongly simple.*

Proof: Since ν is non-trivial, i.e., $\nu(a) \neq 0$ for some $0 \neq a \in K_R$, there exists a positive $\beta \in \mathbb{R}$ such that $M_\beta := \{a \in K_R \,|\, \nu(a) \geq \beta\} \neq 0$. Note that $M_\beta \neq K_R$ either since for each $a \in M_\beta$, we have $\nu(a^{-1}) = -\nu(a) < 0$, whence $a^{-1} \notin M_\beta$.

Furthermore, by our assumption that $\nu(r) \geq 0$ for all $r \in R$, it is easy to check that M_β is a Mathieu subspace of K_R. Therefore, M_β is a non-trivial Mathieu subspace of K_R, whence K_R is not a strongly simple R-algebra. $\square$

For general discussions on valuations, and also on Krull domains needed below, see [Sc], [R], [AM], [ZS], [Bou] and [Fo].

Proposition 6.7. *Let R be a Krull domain or a Noetherian domain such that $R \neq K_R$, i.e., R is not a field. Then no R-algebra is strongly simple. Equivalently, every R-algebra $\mathcal{A}$ has at least one non-trivial Mathieu subspace.*

Proof: Note that by Theorem 6.2, it suffices to show that K_R as a R-algebra is not strongly simple.

Assume first that R is a Krull domain. Since R is not a field, by the very definition of Krull domains (e.g., see p. 480 in [Bou]), we see that R satisfies the hypothesis in Lemma 6.6. Hence, by Lemma 6.6 K_R cannot be a strongly simple R-algebra.

Now, assume that R is a Noetherian domain. Let $\bar{R}$ be the integral closure of R in K_R. Then by the Mori-Nagata integral closure theorem (see Theorem 4.3, p. 18 in [Fo] or Corollary 2.3, p. 161 in [H]), $\bar{R}$ is a Krull domain. Note that since R is not a field, it is well-known (e.g., see Proposition 5.7, p. 61 in [AM]) that $\bar{R}$ is not a field either.

Furthermore, since the field of fractions $K_{\bar{R}}$ of $\bar{R}$ is the same as K_R, by the Krull domain case that we just proved above, K_R is not strongly simple as a $\bar{R}$-algebra, and by Lemma 6.5 with $S = \bar{R}$, K_R is not strongly simple as a R-algebra either. $\square$

Since $\mathbb{Z}$ and all its quotient rings are obviously Noetherian, from Theorem 6.2 and Proposition 6.7 we immediately have the following classification for *strongly simple rings* $\mathcal{A}$, i.e., strongly simple algebras $\mathcal{A}$ over $\mathbb{Z}$ (without the convenient assumption $\mathbb{Z} \subseteq \mathcal{A}$).

Corollary 6.8. *Let $\mathcal{A}$ be an arbitrary commutative or noncommutative ring. Then $\mathcal{A}$ as a $\mathbb{Z}$-algebra is strongly simple iff $\mathcal{A} \simeq \mathbb{Z}_p$ for some prime $p > 0$. In other words, all rings (as $\mathbb{Z}$-algebras) except the finite fields $\mathbb{Z}_p$'s have non-trivial Mathieu subspaces.*

Next, we conclude this section with the following remarks.

Remark 6.9. *i) By Lemma 6.5, we see that Proposition 6.7 also holds if there exists a Noetherian or Krull domain S of K_R such that S is not a field and S contains R.*

ii) After an earlier version of this paper was circulated, M. de Bondt [Bon] *has recently found some examples of integral domains R such that R is not a field and K_R is strongly simple as a R-algebra. He also showed that for any integral domain R that has at least one prime ideal of height one, the field of fractions K_R as a R-algebra is not strongly simple. Therefore, by Theorem 6.2 we see that Proposition 6.7 actually holds for all the integral domains with prime ideals of height one.*

7. Quasi-Stable Algebras

First, let's introduce the following notions for associative algebras.

Definition 7.1. *Let $\mathcal{A}$ be an associative R-algebra. We say that $\mathcal{A}$ is ϑ-quasi-stable (resp., ϑ-stable) if every R-subspace V of $\mathcal{A}$ with $1 \notin V$ is a ϑ-Mathieu subspace (resp., ϑ-ideal) of $\mathcal{A}$.*

For the justifications of the terminologies in the definition above, see Section 3 in [Z6].

In contrast to *strongly simple algebras* studied in the previous section, which have as less ϑ-Mathieu subspaces as possible, *ϑ-quasi-stable algebras* by Corollary 2.10 are the algebras that have as many ϑ-Mathieu subspaces as possible. One of the motivations for the study of ϑ-quasi-stable algebras comes from the following proposition and the corollary followed.

Proposition 7.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be R-algebras and $\phi : \mathcal{B} \to \mathcal{A}$ a R-algebra homomorphism. Assume that $\mathcal{A}$ is ϑ-quasi-stable. Then for every R-subspace V of $\mathcal{A}$ such that $1_{\mathcal{A}} \notin V$, the pre-image $\phi^{-1}(V)$ is a ϑ-Mathieu subspace of $\mathcal{B}$.*

Proof: Since $1_{\mathcal{A}} \notin V$ and $\mathcal{A}$ is a ϑ-quasi-stable R-algebra, we have that V is a ϑ-Mathieu subspace of $\mathcal{A}$. Then by Proposition 2.5, $\phi^{-1}(V)$ is a ϑ-Mathieu subspace of $\mathcal{B}$. $\square$

Corollary 7.3. *Let $\mathcal{B}$ be a R-algebra and I an ideal of $\mathcal{B}$ such that $\mathcal{B}/I$ is a ϑ-quasi-stable R-algebra. Then every R-subspace M of $\mathcal{B}$ with $I \subseteq M$ and $1 \notin M$ is a ϑ-Mathieu subspace of $\mathcal{B}$.*

Proof: If $I = \mathcal{B}$, the corollary holds vacuously. So we assume $I \neq \mathcal{B}$. Let $\mathcal{A} := \mathcal{B}/I$ and $\pi : \mathcal{B} \to \mathcal{A}$ the quotient R-algebra homomorphism. Set $V := \pi(M)$. Then by the assumptions $1_{\mathcal{B}} \notin M$ and $I \subseteq M$, it is easy to check that $1_{\mathcal{A}} \notin V$ and $M = \pi^{-1}(V)$. Applying Proposition

7.2 to the R-subspace $V \subset \mathcal{A}$ with $\phi = \pi$, we see that the corollary follows. $\square$

One family of quasi-stable R-algebras is given by the following proposition.

Proposition 7.4. *Let $\mathcal{A}$ be a R-algebra such that $\mathcal{A}$ is integral over R and every element of $\mathcal{A}$ is either invertible or nilpotent. Then $\mathcal{A}$ is a ϑ-quasi-stable R-algebra.*

Proof: Let V be a R-subspace of $\mathcal{A}$ such that $1 \notin V$. Since $\mathcal{A}$ is integral over R, by Lemma 3.1 the radical $\sqrt{V}$ of V does not contain any invertible element of $\mathcal{A}$. Hence by our assumption on $\mathcal{A}$, we have $\sqrt{V} \subseteq \mathrm{nil}\,(\mathcal{A})$. Then by Lemma 2.8, V is a ϑ-Mathieu subspace of $\mathcal{A}$. Hence the proposition follows. $\square$

Corollary 7.5. *Every left or right Artinian local R-algebra $\mathcal{A}$ that is integral over R is ϑ-quasi-stable. In particular, every commutative Artinian local ring as a $\mathbb{Z}$-algebra is quasi-stable if it is integral over $\mathbb{Z}$.*

Proof: Since $\mathcal{A}$ is local, it's Jacobson radical $\mathbf{J}(\mathcal{A})$ is also the unique maximal left ideal of $\mathcal{A}$. Hence, all non-invertible elements of $\mathcal{A}$ are contained in $\mathbf{J}(\mathcal{A})$. Since $\mathcal{A}$ is left or right Artinian, it is well-known (e.g., see the proposition on p. 61 in [Pi]) that the Jacobson radical $\mathbf{J}(\mathcal{A})$ is nilpotent, i.e., $\mathbf{J}(\mathcal{A})^k = 0$ for some $k \geq 1$. Consequently, all the elements in $\mathbf{J}(\mathcal{A})$ are nilpotent. Therefore, all elements of $\mathcal{A}$ are either invertible or nilpotent, and by Proposition 7.4, $\mathcal{A}$ is ϑ-quasi-stable. $\square$

Next, we give the following classification for ϑ-quasi-stable algebras $\mathcal{A}$ over arbitrary fields K.

Theorem 7.6. *Let K be a field and $\mathcal{A}$ a K-algebra. Then $\mathcal{A}$ is ϑ-quasi-stable iff either $\mathcal{A} \simeq K \dot{+} K$ or $\mathcal{A}$ is an algebraic local K-algebra.*

Two remarks on the theorem above are as follows.

First, by Corollary 3.8 we see that for any algebraic K-algebra $\mathcal{A}$, $\mathcal{A}$ is local iff every element of $\mathcal{A}$ is either nilpotent or invertible. Therefore, by Theorem 7.6 we see that Proposition 7.4 with $R = K$ actually has covered most of the ϑ-quasi-stable algebras over K.

Second, from Theorem 7.6, Corollary 3.8, Lemma 4.1, Corollary 4.3, or from the proof of Theorem 7.6 given below, it follows that the ϑ-quasi-stableness for algebras over fields actually does not depend on the specifications of ϑ. More precisely, we have the following corollary.

Corollary 7.7. *Let K be a field and $\mathcal{A}$ a K-algebra. Then $\mathcal{A}$ is ϑ-quasi-stable for one specification of ϑ iff $\mathcal{A}$ is ϑ-quasi-stable for all specifications of ϑ iff $\mathcal{A}$ is (two-sided) quasi-stable.*

To prove Theorem 7.6, we start with the following lemma.

Lemma 7.8. *Every ϑ-quasi-stable K-algebra $\mathcal{A}$ is algebraic over K.*

Proof: Assume otherwise and let a be a (nonzero) element of $\mathcal{A}$ which is transcendental over K. Denote by V the K-subspace of $\mathcal{A}$ spanned by a^{2k} $(k \geq 1)$ over K. Then we have $1 \notin V$, otherwise a would be algebraic over K. So V is a ϑ-Mathieu subspace of $\mathcal{A}$, for $\mathcal{A}$ is ϑ-quasi-stable.

Since $(a^2)^m = a^{2m} \in V$ for all $m \geq 1$, there exists a large enough $N \geq 1$ such that $a^{2N+1} = (a^2)^N a \in V$. But this means that the odd power a^{2N+1} can be written as a linear combination of some even powers of a, whence a is algebraic over K. Hence, we get a contradiction. $\square$

<u>*Proof of Theorem 7.6:*</u> ($\Leftarrow$) Assume first that $\mathcal{A} \simeq K\dot{+}K$, then it is easy to check that the only non-trivial idempotents of $\mathcal{A}$ are $a := (1,0)$ and $b := (0,1)$. Note that the lines Ka and Kb are obviously ideals of $\mathcal{A}$ and hence, also Mathieu subspaces of $\mathcal{A}$. Then by Proposition 4.8, it is easy to see that every non-trivial subspace V of $\mathcal{A}$ (which is necessarily a line of $\mathcal{A}$) with $1_{\mathcal{A}} = (1,1) \notin V$ is a Mathieu subspace of $\mathcal{A}$. Therefore, $\mathcal{A}$ is quasi-stable and hence, also ϑ-quasi-stable for all possible ϑ.

Now assume that $\mathcal{A}$ is an algebraic local K-algebra. Then by Corollary 3.8, $\mathcal{A}$ has no non-trivial idempotent. Let V be a R-subspace of $\mathcal{A}$ such that $1 \notin V$. Then V contains no nonzero idempotent of $\mathcal{A}$. By Lemma 4.1 and Corollary 4.3, V is a ϑ-Mathieu subspace of $\mathcal{A}$. Therefore, $\mathcal{A}$ is ϑ-quasi-stable.

($\Rightarrow$) Assume that $\mathcal{A}$ is not an algebraic local K-algebra. Then by Corollary 3.8, $\mathcal{A}$ has at least one non-trivial idempotent, say, $e \in \mathcal{A}$. Note that e is linearly independent with $1 \in \mathcal{A}$ over K since the only idempotents of K are $0, 1 \in K$.

Let $\mathcal{B}$ be the two-dimensional K-subspace of $\mathcal{A}$ spanned by $1, e \in \mathcal{A}$ over K. Then it is easy to check that $\mathcal{B}$ is actually a K-subalgebra of $\mathcal{A}$ which is isomorphic to the K-algebra $K\dot{+}K$ via the following K-algebra isomorphism:

$$\begin{array}{rcl} \phi : K\dot{+}K & \longrightarrow & \mathcal{B} \\ (r,\ s) & \longrightarrow & r(1-e) + se. \end{array}$$

Next we show $\mathcal{B} = \mathcal{A}$, from which the theorem will follow.

First, by the fact that 1 and e are linearly independent over K, we have $1 \notin K(1-e)$ and $1 \notin Ke$. Second, since $\mathcal{A}$ is ϑ-quasi-stable, both Ke and $K(1-e)$ are ϑ-Mathieu subspaces of $\mathcal{A}$. But, on the other hand, since e and $(1-e)$ are non-trivial idempotents, it follows from Proposition 4.8 that Ke and $K(1-e)$ are actually ϑ-ideals of $\mathcal{A}$.

Assume $\vartheta =$ *"left"*, *"pre-two-sided"* or *"two-sided"*. Then for each $a \in \mathcal{A}$, we have

$$ae = re, \tag{7.1}$$

$$a(1-e) = s(1-e), \tag{7.2}$$

for some $r, s \in K$.

Taking the sum of the two equations above, we get $a = re + s(1-e)$, whence $a \in \mathcal{B}$. Therefore, we do have $\mathcal{B} = \mathcal{A}$ when $\vartheta \neq$ *"right"*. The case $\vartheta =$ *"right"* can be proved similarly. Therefore, the theorem holds. □

From Theorem 7.6, we immediately have the following examples of quasi-stable K-algebras.

Example 7.9. 1) *every algebraic field extension of K or more generally, every algebraic division algebra over K is a quasi-stable K-algebra.*

2) *Let p be a prime and $\mathcal{A} := \mathbb{Z}/(p^k)$ for some $k \geq 1$. Then $\mathcal{A}$ as a $\mathbb{Z}$-algebra is algebraic and local and hence, a quasi-stable $\mathbb{Z}$-algebra. Actually, $\mathcal{A}$ is also a stable $\mathbb{Z}$-algebra since every $\mathbb{Z}$-subspace of $\mathcal{A}$ is an ideal of $\mathcal{A}$.*

3) *Let K be a field and t a free variable. For every $k \geq 1$ and irreducible $f(t) \in K[t]$, the quotient algebra $\mathcal{A} := K[t]/(f^k)$ is an algebraic and local K-algebra. Therefore, $\mathcal{A}$ is a quasi-stable K-algebra.*

Note that all the quasi-stable algebras in the example above are Artinian. However, this is not always the case.

Example 7.10. *Let $\mathcal{B} = K[x_i \,|\, i \geq 1]$ be the polynomial algebra over K in the infinitely many commutative free variables x_i $(i \geq 1)$, and I the ideal of $\mathcal{B}$ generated by x_i^{i+1} $(i \geq 1)$. Set $\mathcal{A} := \mathcal{B}/I$. Then it is easy to see that $\mathcal{A}$ is an algebraic local K-algebra whose maximal ideal $\mathfrak{m}$ is the ideal generated by the images of x_i $(i \geq 1)$ in $\mathcal{A}$. Hence, $\mathcal{A}$ by Theorem 7.6 is a quasi-stable K-algebra.*

On the other hand, since the maximal ideal $\mathfrak{m}$ of $\mathcal{A}$ is obviously not finitely generated, $\mathcal{A}$ is not Noetherian and hence, not Artinian either.

The following proposition generalizes the construction in Example 7.9, 2) and 3) for quasi-stable algebras.

Proposition 7.11. *Let $\mathcal{A}$ be a commutative K-algebra and $\mathfrak{m}$ a maximal ideal of $\mathcal{A}$ such that $\mathcal{A}/\mathfrak{m}$ is an algebraic field extension of K. Then for every $k \geq 1$, $\mathcal{A}/\mathfrak{m}^k$ is a quasi-stable K-algebra.*

Proof: It is easy to see that $\mathcal{A}/\mathfrak{m}^k$ is a local K-algebra with the maximal ideal $\mathfrak{m}/\mathfrak{m}^k$. Then by Theorem 7.6, we only need to show that $\mathcal{A}/\mathfrak{m}^k$ is algebraic over K.

Let $a \in \mathcal{A}$. Since $\mathcal{A}/\mathfrak{m}$ is algebraic over K, it is easy to see that there exists a nonzero polynomial $f(t) \in K[t]$ such that $f(a) \in \mathfrak{m}$. Then we have $f^k(a) \in \mathfrak{m}^k$ and $f^k(\bar{a}) = 0$, where $\bar{a}$ denotes the image of a in $\mathcal{A}/\mathfrak{m}^k$. Therefore $\bar{a}$ is algebraic over K for all $a \in \mathcal{A}$, whence $\mathcal{A}/\mathfrak{m}^k$ is algebraic over K. $\square$

From Proposition 7.11 and Corollary 7.3, we immediately have the following corollary.

Corollary 7.12. *Let $\mathcal{A}$ and $\mathfrak{m}$ be as Proposition 7.11 and V a K-subspace of $\mathcal{A}$. Assume that $1 \notin V$ and $\mathfrak{m}^k \subseteq V$ for some $k \geq 1$. Then V is a Mathieu subspace of $\mathcal{A}$.*

In contrast to ϑ-quasi-stable K-algebras, ϑ-stable K-algebras do not seem very interesting. But, for the completeness and also for the purpose of comparison with ϑ-quasi-stable algebras, here we conclude this paper with the following classification of ϑ-stable K-algebras.

Proposition 7.13. *Let K be a field and $\mathcal{A}$ a K-algebra. Then $\mathcal{A}$ is ϑ-stable iff one of the following two statements holds:*

1) $\mathcal{A} = K$;
2) $K \simeq \mathbb{Z}_2$ *and* $\mathcal{A} \simeq \mathbb{Z}_2 \dot{+} \mathbb{Z}_2$.

Proof: The ($\Leftarrow$) part of the proposition can be easily checked. To show the ($\Rightarrow$) part, we assume $\mathcal{A} \neq K$, and claim first that the following equation holds:

$$\mathcal{A}^\times = K^\times. \tag{7.3}$$

Assume otherwise and let $a \in \mathcal{A}^\times \backslash K$. Then $1 \notin Ka$. Since $\mathcal{A}$ is ϑ-stable, Ka is a ϑ-ideal of $\mathcal{A}$. But for any ϑ, this implies $1 = a^{-1}a = aa^{-1} \in Ka$, which is a contradiction. Therefore, Eq. (7.3) does hold.

On the other hand, since every ϑ-stable algebra is obviously ϑ-quasi-stable, hence $\mathcal{A}$ by our hypothesis is also ϑ-quasi-stable. Then by Theorem 7.6, we have that either $\mathcal{A} \simeq K \dot{+} K$ or $\mathcal{A}$ is an algebraic local K-algebra.

In the latter case, it follows from Corollary 3.8 that all elements of $\mathcal{A}$ are either nilpotent or invertible. Then by Eq. (7.3), all elements in $\mathcal{A} \backslash K$ are nilpotent. But this is impossible by the argument below.

Let $a \in \mathcal{A}\backslash K$ and set $b := 1 - a$. Then $b \notin K$. Hence, both a and b are nilpotent. But, on the other hand, since a is nilpotent, b has inverse $\sum_{i\geq 0} a^i$ in $\mathcal{A}$. Therefore, we have $b \in \mathcal{A}^\times$ (and $b \notin K$), which contradicts Eq. (7.3).

Therefore, we must have $\mathcal{A} \simeq K \dot{+} K$. If $K \not\simeq \mathbb{Z}_2$, then there exist $r, s \in K^\times$ such that $r \neq s$. Set $a := (r, s)$. Then $a \in \mathcal{A}^\times$ and a does not lie in the base field $K \simeq K \cdot 1_{\mathcal{A}} \subset \mathcal{A}$, since $1_{\mathcal{A}} = (1, 1)$. But this contradicts Eq. (7.3) again. Hence, the theorem follows. □

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