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Bianchi type VI cosmological models: A Scale-Covariant study

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Abstract A model for an anisotropic Bianchi type VI universe in a Scale Covariant theory of gravitation (Canuto et al. 1977) is analyzed. Exact solutions to the corresponding field equations are found under some specific assumptions. A finite singularity is found in the model at the initial time $t = 0$. All the physical parameters are studied and thoroughly discussed. The model behaves like a big bang singular model o f the universe.

Key words: Cosmology. Bianchi type VI model. Scale Covariant theory.

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1 Introduction

Canuto et al. (1977) have formulated a Scale-Covariant theory of gravitation by associating the mathematical operation of scale transformation with the physics of using different dynamical systems to measure space-time distances. A Scale-Covariant theory provides the necessary theoretical framework to sensibly discuss the possible variation of the gravitational constant G without compromising the validity of general relativity. In this theory, we measure physical quantities in atomic units whereas Einstein's field equations in gravitational units. If we consider $d\bar{s}^2 = \bar{g}_{ij}dx^i dx^j$, the line element in Einstein units, the corresponding line element in any other units (in atomic units) will be written as

$$ds = \phi^{-1}(x)d\bar{s}. \tag{1}$$

The metric tensor in the two systems of units are related by a conformal transformation

$$\bar{g}_{ij} = \phi^2 g_{ij}, \tag{2}$$

where the metric $\bar{g}_{ij}$ giving macroscopic metric properties and g_{ij} giving microscopic metric properties. Here we consider the gauge function ϕ as a function of time.

Friedmann-Robertson Walker(FRW) space-time models are widely acceptable as a good approximation of the present stage of the evolution of the universe although it is spatially homogeneous and isotropic in nature. However, the large scale matter distribution in the observable universe, largely manifested in the form of discrete structures, does not exhibit a high degree of homogeneity. Also the recent space investigations detect anisotropy in the cosmic microwave background. So the recent experimental data support the existence of an anisotropic phase that approaches an isotropic phase. These theoretical arguments (Saha 2004) lead one to consider models with an anisotropic background. Bianchi type space-times play a vital role in understanding and description of the early stages of evolution of the universe. Bianchi type VI (Saha 2004) space-time is inhomogeneous and anisotropic.

Scale-Covariant theory in different Bianchi space-times has been studied so far by several authors. Shri Ram et al. (2009) have studied a spatially homogeneous Bianchi type V cosmological model in Scale-Covariant theory of gravitation. Reddy et al. (2007) have developed a cosmological model with negative constant deceleration parameter in Scale-Covariant theory of

gravitation. Beesham (1986) has obtained a solution for Bianchi type I cosmological model in the Scale-Covariant theory. Higher dimensional string cosmologies in Scale-Covariant theory of gravitation have been investigated by Venkateswarlu and Kumar (2004). Reddy et al. (1993) have presented the exact Bianchi type II, VIII and IX cosmological models in Scale-Covariant theory of gravitation. In this paper, we obtain exact solution to the field equations of Scale-Covariant theory for Bianchi type VI space-time metric.

2 Field Equations, Metric and General Expressions

Canuto et al. (1977) transformed the general Einstein's field equations by using the conformal transformations equations (1) and (2) as follows:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + f_{\mu\nu}(\phi) = -8\pi G T_{\mu\nu} + \Lambda(\phi)g_{\mu\nu}, \tag{3}$$

where

$$\phi^2 f_{\mu\nu} = 2\phi\phi_{\mu;\nu} - 4\phi_\mu\phi_\nu - g_{\mu\nu}(\phi\phi^{\lambda}_{;\lambda} - \phi^\lambda\phi_\lambda), \tag{4}$$

for any scalar ϕ, $\quad \phi_\mu = \phi_{,\mu}$. Here comma denotes ordinary partial differentiation whereas a semi-colon denotes a covariant differentiation.

The Bianchi VI space-time metric is given as

$$ds^2 = dt^2 - A^2dx^2 - e^{-2mx}B^2dy^2 - e^{2nx}C^2dz^2, \tag{5}$$

with the scale factors A, B, C being functions of time only. Here m ,n are some arbitrary constants. Here the source of gravitational field is considered as a perfect fluid. So for a perfect fluid, the energy momentum tensor is given by

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p g_{\mu\nu}, \tag{6}$$

where ρ is the energy-density, p the pressure and u^μ is the four velocity vector of the fluid following $u^\mu u_\mu = 1$.

The general formulas of certain physical parameters for the metric equation (5) are given as follows:

The expansion scalar is given by

$$\theta = u^{\mu}_{;\mu} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}, \tag{7}$$

where a dot(.) denotes differentiation with respect to time t. The shear scalar has the form

$$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{1}{2}\left[\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{B}}{B}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2\right] - \frac{\theta^2}{6}. \tag{8}$$

We also introduce generalized Hubble parameter H:

$$H = \frac{1}{3}\left(H_1 + H_2 + H_3\right), \tag{9}$$

with $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$ and $H_3 = \frac{\dot{C}}{C}$ are the directional Hubble parameters in the directions of x, y and z respectively. Let us introduce the function V and average scale factor a:

$$V = ABC, \tag{10}$$

$$a = (ABC)^{1/3}. \tag{11}$$

It should be noted that the parameters H, V and a are connected by the following relation

$$H = \frac{1}{3}\frac{\dot{V}}{V} = \frac{\dot{a}}{a}. \tag{12}$$

The field equations (3) and (4) to the metric equation (5) for perfect fluid equation (6), are given as following set of equations

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{mn}{A^2} - 2\frac{\dot{A}}{A}\frac{\dot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{V}}{V}\right) + \frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}^2}{\phi^2} = -8\pi G p, \tag{13}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} - \frac{n^2}{A^2} - 2\frac{\dot{B}}{B}\frac{\dot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{V}}{V}\right) + \frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}^2}{\phi^2} = -8\pi G p, \tag{14}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} - \frac{m^2}{A^2} - 2\frac{\dot{C}}{C}\frac{\dot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{V}}{V}\right) + \frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}^2}{\phi^2} = -8\pi G p, \tag{15}$$

$$\frac{\dot{A}}{A}\frac{\dot{B}}{B}+\frac{\dot{A}}{A}\frac{\dot{C}}{C}+\frac{\dot{B}}{B}\frac{\dot{C}}{C}-\frac{m^2-mn+n^2}{A^2}+\frac{\dot{\phi}}{\phi}\left(\frac{\dot{V}}{V}\right)-\frac{\ddot{\phi}}{\phi}+3\frac{\dot{\phi}^2}{\phi^2}=8\pi G\rho, \quad (16)$$

$$m\frac{\dot{B}}{B}-n\frac{\dot{C}}{C}-(m-n)\frac{\dot{A}}{A}=0. \quad (17)$$

Here we have used definition (10). The Bianchi identity reads

$$\dot{\rho}+(\rho+p)\frac{\dot{V}}{V}+\rho\frac{\dot{\phi}}{\phi}+3p\frac{\dot{\phi}}{\phi}=0. \quad (18)$$

From equation (17), we find the following relation between the metric functions A, B, C as

$$\left(\frac{B}{A}\right)^m=k\left(\frac{C}{A}\right)^n, \quad (19)$$

with the integration constant k. Taking into account the definition (10), from equation (19), we can write the scale factors B and C in terms of A and V, such that

$$B=\left(kV^nA^{(m-2n)}\right)^{\frac{1}{m+n}}, \quad (20)$$

$$C=\left(\frac{1}{k}V^mA^{(n-2m)}\right)^{\frac{1}{m+n}}. \quad (21)$$

Summing equations (13), (14), (15) and 3 times equation (16), in view of the equation (10) for volume scalar, we obtain a non-linear differential equation as

$$\frac{\ddot{V}}{V}+\frac{2}{A^2}(mn-n^2-m^2)+2\frac{\dot{\phi}}{\phi}\frac{\dot{V}}{V}+3\frac{\dot{\phi}^2}{\phi^2}=12\pi G(-p+\rho). \quad (22)$$

Taking into account that the perfect fluid obeys the equation of state $p=\gamma\rho, (0<\gamma<1)$, the equation (18) becomes

$$\rho V^{1+\gamma}\phi^{1+3\gamma} = a_0 \tag{23}$$

where a_0 is an integration constant.

We consider the gauge function ϕ (Canuto et al. (1977) and Shri Ram et al. (2009)) as

$$\phi = \phi_0 a^{\alpha} = \phi_0 V^{\alpha/3}, \tag{24}$$

where α and ϕ_0 are arbitrary constants. Now in view of equations (23), (24) equation (22) reduces to

$$\frac{\ddot{V}}{V} + \frac{\alpha(\alpha+2)}{3}\left(\frac{\dot{V}}{V}\right)^2 + \frac{2}{A^2}(mn - n^2 - m^2) = \frac{12\pi G(1-\gamma)\rho_0}{V^{1+\gamma+\alpha/3+\alpha\gamma}} \tag{25}$$

where ρ_0 is an arbitrary constant. As we can see, there are two unknown functions A and V in the above equation (25). Let us demand an additional assumption relating to these two variables. So we consider here that the scale factor A is related to the volume scalar V with the relation $A = \sqrt{V}$ (Saha 2004). This assumption provide us the exact solutions to the field equations at the same time leaving the spacetime anisotropic.

Note that such an assumption imposes restrictions on the metric functions. Now, in what follows, we try to find an exact solution of the field equations in Scale-Covariant theory with the help of the equation (25).

3 Exact Solutions

Under the assumption $A = \sqrt{V}$, we obtain the following equation for V, by solving the differential equation (25) as

$$V = \frac{3v_0}{2(\alpha^2+2\alpha+3)}t^2 + \frac{3v_1}{3-\alpha^2-2\alpha}t^{\frac{3-\alpha^2-2\alpha}{3}} + v_2, \tag{26}$$

where $v_0 = 12\pi G(1-\gamma)\rho_0 + 2(m^2+n^2-mn) > 0$, v_1 and v_2 are integration constants. It should be noted that in case of a non-zero v_2, V is non-trivial even at $t = 0$, which imposes that v_2 is essentially positive. For $v_2 = 0$ we have the model, when V becomes zero at the initial time, i.e., $V\big|_{v_2=0,t=0} = 0$.

We also have a relationship between γ and α as $\alpha = -\frac{3\gamma}{3\gamma+1}$, $\alpha \in (-3/4, 0)$.

The equation (26), in view of (10), gives the following expressions of the scale factors A, B and C as follows:

$$A = \left[\frac{3v_0}{2(\alpha^2+2\alpha+3)}t^2 + \frac{3v_1}{3-\alpha^2-2\alpha}t^{\frac{3-\alpha^2-2\alpha}{3}} + v_2\right]^{1/2}, \tag{27}$$

$$B = B_0\left[\frac{3v_0}{2(\alpha^2+2\alpha+3)}t^2 + \frac{3v_1}{3-\alpha^2-2\alpha}t^{\frac{3-\alpha^2-2\alpha}{3}} + v_2\right]^{\frac{m}{2(m+n)}}, \tag{28}$$

and

$$C = C_0\left[\frac{3v_0}{2(\alpha^2+2\alpha+3)}t^2 + \frac{3v_1}{3-\alpha^2-2\alpha}t^{\frac{3-\alpha^2-2\alpha}{3}} + v_2\right]^{\frac{n}{2(m+n)}}, \tag{29}$$

where $B_0 = k^{1/(m+n)}$ and $C_0 = (1/k)^{1/(m+n)}$.

The expressions for the gauge function ϕ and the average scale factor a are given by

$$\phi = \phi_0\left[\frac{3v_0}{2(\alpha^2+2\alpha+3)}t^2 + \frac{3v_1}{3-\alpha^2-2\alpha}t^{\frac{3-\alpha^2-2\alpha}{3}} + v_2\right]^{\alpha/3}, \tag{30}$$

and

$$a = \left[\frac{3v_0}{2(\alpha^2+2\alpha+3)}t^2 + \frac{3v_1}{3-\alpha^2-2\alpha}t^{\frac{3-\alpha^2-2\alpha}{3}} + v_2\right]^{1/3}. \tag{31}$$

Using the above expressions in equations (7)-(9), the expansion scalar θ, shear scalar σ^2 and the Hubble parameter H are written as,

$$\theta = \frac{\frac{3v_0}{(\alpha^2+2\alpha+3)}t^{\frac{3+\alpha^2+2\alpha}{3}} + v_1}{\frac{3v_0}{2(\alpha^2+2\alpha+3)}t^{\frac{6+\alpha^2+2\alpha}{3}} + \frac{3v_1}{3-\alpha^2-2\alpha}t + v_2 t^{\frac{\alpha^2+2\alpha}{3}}}, \tag{32}$$

$$\sigma^2 = \frac{(m^2+n^2-mn)}{12(m+n)^2}\left[\frac{\frac{3v_0}{(\alpha^2+2\alpha+3)}t^{\frac{3+\alpha^2+2\alpha}{3}} + v_1}{\frac{3v_0}{2(\alpha^2+2\alpha+3)}t^{\frac{6+\alpha^2+2\alpha}{3}} + \frac{3v_1}{3-\alpha^2-2\alpha}t + v_2 t^{\frac{\alpha^2+2\alpha}{3}}}\right]^2, \tag{33}$$

and

$$H=\frac{1}{3}\left[\frac{\frac{3v_0}{(\alpha^2+2\alpha+3)}t^{\frac{3+\alpha^2+2\alpha}{3}}+v_1}{\frac{3v_0}{2(\alpha^2+2\alpha+3)}t^{\frac{6+\alpha^2+2\alpha}{3}}+\frac{3v_1}{3-\alpha^2-2\alpha}t+v_2t^{\frac{\alpha^2+2\alpha}{3}}}\right]. \tag{34}$$

The directional Hubble parameters can be obtained as

$$H_1=\frac{\frac{3v_0}{(\alpha^2+2\alpha+3)}t^{\frac{3+\alpha^2+2\alpha}{3}}+v_1}{\frac{3v_0}{(\alpha^2+2\alpha+3)}t^{\frac{6+\alpha^2+2\alpha}{3}}+\frac{6v_1}{3-\alpha^2-2\alpha}t+2v_2t^{\frac{\alpha^2+2\alpha}{3}}}, \tag{35}$$

$$H_2=\frac{\frac{3mv_0}{(m+n)(\alpha^2+2\alpha+3)}t^{\frac{3+\alpha^2+2\alpha}{3}}+\frac{mv_1}{(m+n)}}{\frac{3v_0}{(\alpha^2+2\alpha+3)}t^{\frac{6+\alpha^2+2\alpha}{3}}+\frac{6v_1}{3-\alpha^2-2\alpha}t+2v_2t^{\frac{\alpha^2+2\alpha}{3}}}, \tag{36}$$

and

$$H_3=\frac{\frac{3nv_0}{(m+n)(\alpha^2+2\alpha+3)}t^{\frac{3+\alpha^2+2\alpha}{3}}+\frac{nv_1}{(m+n)}}{\frac{3v_0}{(\alpha^2+2\alpha+3)}t^{\frac{6+\alpha^2+2\alpha}{3}}+\frac{6v_1}{3-\alpha^2-2\alpha}t+2v_2t^{\frac{\alpha^2+2\alpha}{3}}}. \tag{37}$$

Now the value of the energy-momentum tensor ρ and the pressure p can be found as follows:

$$\rho=\frac{\rho_0}{\frac{3v_0}{2(\alpha^2+2\alpha+3)}t^2+\frac{3v_1}{3-\alpha^2-2\alpha}t^{\frac{3-\alpha^2-2\alpha}{3}}+v_2}, \tag{38}$$

and

$$p=\frac{\gamma\rho_0}{\frac{3v_0}{2(\alpha^2+2\alpha+3)}t^2+\frac{3v_1}{3-\alpha^2-2\alpha}t^{\frac{3-\alpha^2-2\alpha}{3}}+v_2}. \tag{39}$$

We now investigate the behavior of the above cosmological model by analyzing the different physical parameters. The above set of exact solutions shows that the expansion scalar θ, shear scalar σ^2 and the Hubble parameter H are infinite at the time $t=0$. At the same time $t=0$, all the directional Hubble's parameters are also infinite. The pressure and density both will be infinite at this epoch at $t=0$ iff $v_2=0$. These characteristics of different

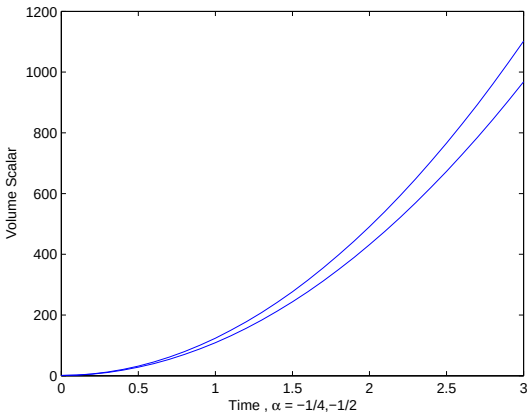

Figure 1: Variation of Volume scalar V with time t.

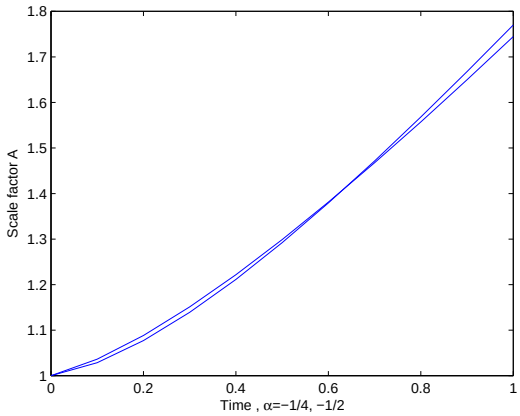

Figure 2: Variation of the Scale factor A with time t.

physical parameters identify the existence of singularity in the model at the initial time $t = 0$. Now one can also observe that all these parameters θ, σ^2, H, H_1, H_2, H_3, ρ and p are become zero at the large time $t \to \infty$, even for $v_2 \neq 0$. That is all these physical parameters are decreasing functions of time. Therefore this model describes a continuously expanding and shearing universe with the singularity at $t = 0$. This model gives an empty space for large time.

Let us now study the behavior of the volume scalar and the scale factors A, B, C in this model. From Figure 1 , it can be seen that the volume scalar $V(\alpha = -1/2, -1/4)$ is the increasing function of time. That is V is zero at $t = 0$ and it takes infinite value at $t \to \infty$. As A is a function of V, namely $A = \sqrt{V}$ the behavior of A is almost the same as that of V. As far as B and C are concerned, depending on the values of m and n they either expands rapidly or slowly. These variations can be observed through the Figures 2, 3, 4, 5 and 6 respectively.

4 Conclusion

In this paper, we have obtained an exact solution for the field equations of Scale-Covariant theory of gravitation in Bianchi type VI line element of the universe. Under some specific assumptions, exact solutions to the corresponding field equations are found. It is found that one of the metric functions (A) is an expanding one with acceleration whereas depending on the choice of the parameters two other metric functions B and C expand

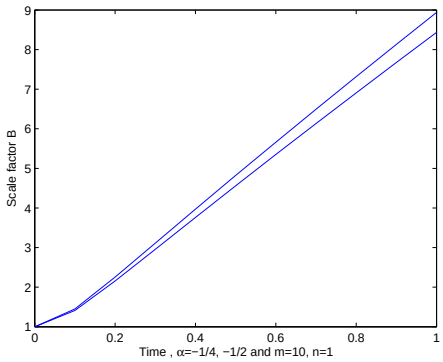

Figure 3: Variation of the Scale factor B with time t .

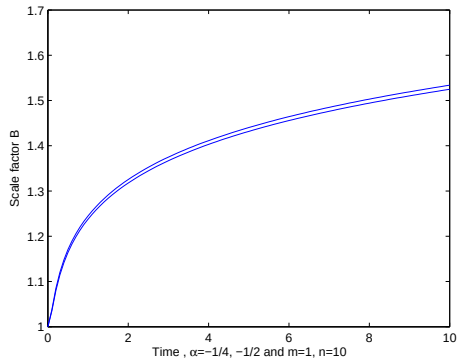

Figure 4: Variation of the Scale factor B with time t.

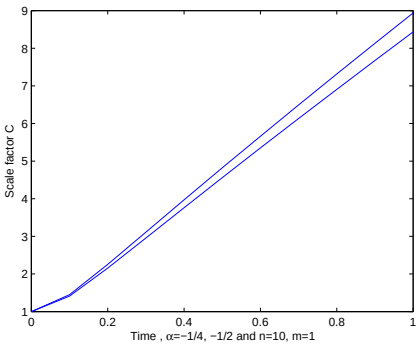

Figure 5: Variation of the Scale factor C with time t.

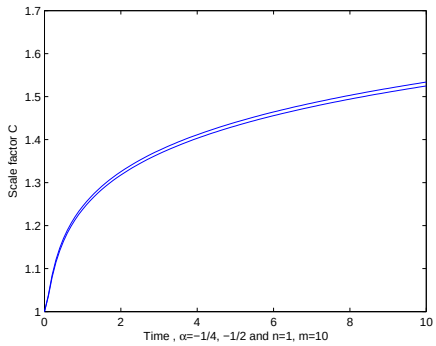

Figure 6: Variation of the Scale factor C with the time t.

either with acceleration or deceleration. The model in question does not allow isotropization of the initial anisotropic space-time. All the physical and kinematical parameters have been thoroughly discussed. The solution so obtained, represents a continuously expanding and shearing model of the universe with the singularity at the initial time $t = 0$. This model gives an empty space for large time.

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