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Nonlinear Spinor Fields in Bianchi type-I spacetime: Problems and Possibilities

Bijan Saha

Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980 Dubna, Moscow region, Russia[*]

Within the scope of Bianchi type-I cosmological model we study the role of spinor field in the evolution of the Universe. It is found that due to the spinor affine connections the energy momentum tensor of the spinor becomes non-diagonal, whereas the Einstein tensor is diagonal. This non-triviality of non-diagonal components of the energy-momentum tensor imposes some severe restrictions either on the spinor field or on the metric functions or on both of them. In case if the restrictions are imposed on the components of spinor field only, we come to a situation when spinor field becomes massless and invariants constructed from bilinear spinor forms also become trivial. Imposing restriction wholly on metric functions we obtain FRW model, while if the restrictions are imposed both on metric functions and spinor field components, we come to LRS BI model. In both cases the system is solved completely. It was found that if the relation between the pressure and energy density obeys a barotropic equation of state, only a non-trivial spinor mass can give rise to a dynamic EoS parameter.

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[*] bijan@jinr.ru; http://bijansaha.narod.ru

I. INTRODUCTION

The journey of Einstein's General Theory of Relativity in cosmological area was never a smooth one. The introduction of the cosmological constant and its' further omission opened the thorny road from the very beginning. But with many fundamental questions remaining unanswered and further development and new findings of observational cosmology lead to the conclusion that Einstein's General Relativity is not the final theory of gravitational interactions. These issues come from cosmology and quantum field theory. The presence of Big Bang singularity, flatness and horizontal problems [1] lead to the fact that the standard cosmological model [2] based on GR and the standard model of particle physics are inadequate to describe the Universe at extreme regime. The absence of the genuine quantum gravity theory leads to develop alternative theory of gravity, where, at least, in semi-classical limits, GR and its positive results could be recovered.

A fruitful approach in this search is the extended theories of Gravity (ETG) which have become a sort of paradigm in the study of gravitational interactions [3]. These theories are essentially based on the corrections and enlargements of Einstein's theory of Gravity. The paradigm consists of adding higher order curvature invariants and non-minimally coupled scalar fields into dynamics resulting from effective action of quantum gravity. An excellent review on extended theories of gravity can be found in [4].

Though the inflationary model [1, 5, 6], described by a scalar field, known as inflaton, solves the problem of flatness, isotropy of microwave background radiation and unwanted relics, the question of where the scalar field comes from and why it undergoes such a peculiar phase transition from false to right vacuum still remains unanswered. Moreover, recent observations showed an accelerated mode of expansion of the present day Universe [7, 8]. This leads cosmologists to reconsider alternative possibilities.

As one of the way out many specialists considered spinor field as an alternative source. Being related to almost all stable elementary particles such as proton, electron and neutrino, spinor field, especially Dirac spin-$1/2$ play a principal role at the microlevel. However, in cosmology, the role of spinor field was generally considered to be restricted. Only recently, after some remarkable works by different authors [9–23], showing the important role that spinor fields play on the evolution of the Universe, the situation began to change. This change of attitude is directly related to some fundamental questions of modern cosmology:

(i) Problem of initial singularity: One of the problems of modern cosmology is the presence of initial singularity, which means the finiteness of time. The main purpose of introducing a nonlinear term in the spinor field Lagrangian is to study the possibility of the elimination of initial singularity. In a number of papers [11–15] it was shown that the introduction of spinor field with a suitable nonlinearity into the system indeed gives rise to singularity-free models of the Universe. It should be noted that the singularity-free solutions in the papers mentioned were obtained at the expense of dominant energy condition. Problem of singularity and its possible elimination exploiting spinor field were discussed in [24–27].

(ii) problem of isotropization: Although the Universe seems homogenous and isotropic at present, it does not necessarily mean that it is also suitable for a description of the early stages of the development of the Universe and there are no observational data guaranteeing the isotropy in the era prior to the recombination. In fact, there are theoretical arguments that support the existence of an anisotropic phase that approaches an isotropic one [28]. The observations from Cosmic Background Explorer's differential radiometer have detected and measured cosmic microwave background anisotropies in different angular scales. Recently Planck has compiled the most detailed map of the cosmic microwave background ever created. The new map renews our understanding of the Universes composition and evolution. The image of the cosmic microwave background (CMB) composed from the lights, imprinted on the sky when the Universe was just 380 000 years old shows tiny temperature fluctuations that correspond to regions of slightly different densities at very early times. These anisotropies are supposed to hide in their fold the entire history of cosmic evolution dating back to the recombination era and are being considered as indicative of the geometry and the content of the universe. There is widespread consensus among the cosmologists that cosmic microwave background anisotropies in small angular scales have the seeds of all future structure: the stars and galaxies of today. It was found that the introduction of nonlinear spinor field accelerates the isotropization process of the initially anisotropic Universe

[13, 14, 16, 25].

(iii) late time acceleration of the Universe: Some recent experiments detected an accelerated mode of expansion of the Universe [7, 8]. Detection and further experimental reconfirmation of current cosmic acceleration pose to cosmology a fundamental task of identifying and revealing the cause of such phenomenon. This fact can be reconciled with the theory if one assumes that the Universe id mostly filled with so-called dark energy. This form of matter (energy) is not observable in laboratory and it does not interact with electromagnetic radiation. These facts played decisive role in naming this object. In contrast to dark matter, dark energy is uniformly distributed over the space, does not intertwine under the influence of gravity in all scales and it has a strong negative pressure of the order of energy density. Based on these properties, cosmologists have suggested a number of dark energy models those are able to explain the current accelerated phase of expansion of the Universe. In this connection a series of papers appeared recently in the literature, where a spinor field was considered as an alternative model for dark energy [17–19, 21, 26].

It should be noted that most of the works mentioned above were carried out within the scope of Bianchi type-I cosmological model. Results obtained using a spinor field as a source of Bianchi type-I cosmological field can be summed up as follows: A suitable choice of spinor field nonlinearity

(i) *accelerates the isotropization process* [13, 14, 16];

(ii) *gives rise to a singularity-free Universe* [13–16];

(iii) *generates late time acceleration* [17–19, 21, 22].

Given the role that spinor field can play in the evolution of the Universe, question that naturally pops up is, if the spinor field can redraw the picture of evolution caused by perfect fluid and dark energy, is it possible to simulate perfect fluid and dark energy by means of a spinor field? Affirmative answer to this question was given in the a number of papers [29–33]. In those papers spinor description of matter such as perfect fluid and dark energy was given and the evolution of the Universe given by different Bianchi models was thoroughly studied. In almost all the papers the spinor field was considered to be time-dependent functions and its energy-momentum tensor was given by the diagonal elements only. Some latest study shows that due to the specific connection with gravitational field the energy-momentum tensor of the spinor field possesses non-trivial non-diagonal components as well. In this paper we study the role of non-diagonal components of the energy-momentum tensor of the spinor field in the evolution of the Universe. To our knowledge such study was never done previously. In section II we give the spinor field Lagrangian in details. In section III the system of Einstein-Dirac equations is solved for BI metric without engaging the non-diagonal components of energy-momentum tensor as it was done in previous works of many authors. In section IV we analyze the role of non-diagonal components of energy-momentum tensor on the evolution of the Universe.

It should also be noted that recently inflation has been studied within the scope of spinor theory as well. Basic theory of dark spinor inflation is presented in [34, 35]. The ELKO field in interaction through contortion with its own spin density was studied in [36], that was further developed in [37]. Conformal coupling of dark spinor field to gravity within the scope of FRW model was studied in [38].

Recently many authors studied the Dirac spinors within the scope of different cosmological models with torsion[24–27, 37, 39]. Dirac spinors in Bianchi type-I $f(R)$ cosmology with torsion was studied in [39], where it was shown that the dynamic behavior of the universe depends on the particular choice of the function $f(R)$. In this paper tt was highlighted that, despite the anisotropic background the Einstein tensor is diagonal, whereas, because of intrinsic feature of spinor field, the energy tensor is non-diagonal. Dirac field equations coupled to electrodynamics and torsion fields were investigated in [27]. It was shown that minimal coupling between the torsion tensor and Dirac spinors generates a spin-spin interaction [25]. In [26] it was shown that the spacetime torsion, generated by Dirac spinors, induces gravitational repulsion. Nonsingular Dirac particles in spacetime with torsion were studied in [24].

II. BASIC EQUATION

Let us consider the case when the anisotropic space-time is filled with nonlinear spinor field. The corresponding action can be given by

$$\mathscr{S}(g;\psi,\bar{\psi}) = \int L\sqrt{-g}d\Omega \tag{2.1}$$

with

$$L = L_{\rm g} + L_{\rm sp}. \tag{2.2}$$

Here $L_{\rm g}$ corresponds to the gravitational field

$$L_{\rm g} = \frac{R}{2\kappa}, \tag{2.3}$$

where R is the scalar curvature, $\kappa = 8\pi G$, with G being Einstein's gravitational constant and $L_{\rm sp}$ is the spinor field Lagrangian.

A. Gravitational field

The gravitational field in our case is given by a Bianchi type-I anisotropic space time:

$$ds^2 = dt^2 - a_1^2 dx^2 - a_2^2 dy^2 - a_3^2 dz^2, \tag{2.4}$$

with a_1, a_2 and a_3 being the functions of time only. It is the simplest anisotropic model of space-time. The reason for considering anisotropic model lays on the fact that though an isotropic FRW model describes the present day Universe with great accuracy, there are both theoretical arguments and observational data suggesting the existence of an anisotropic phase in the remote past.

The nonzero components of the Einstein tensor corresponding to the metric (2.4) are

$$G_1^1 = -\frac{\ddot{a}_2}{a_2} - \frac{\ddot{a}_3}{a_3} - \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3}, \tag{2.5a}$$

$$G_2^2 = -\frac{\ddot{a}_3}{a_3} - \frac{\ddot{a}_1}{a_1} - \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}, \tag{2.5b}$$

$$G_3^3 = -\frac{\ddot{a}_1}{a_1} - \frac{\ddot{a}_2}{a_2} - \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2}, \tag{2.5c}$$

$$G_1^1 = -\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} - \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} - \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}. \tag{2.5d}$$

B. Spinor field

For a spinor field ψ, the symmetry between ψ and $\bar{\psi}$ appears to demand that one should choose the symmetrized Lagrangian [40]. Keeping this in mind we choose the spinor field Lagrangian as [13]:

$$L_{\rm sp} = \frac{\imath}{2}\left[\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma^\mu\psi\right] - m_{\rm sp}\bar{\psi}\psi - F, \tag{2.6}$$

where the nonlinear term F describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants generated from the real bilinear forms of a spinor field.

Here for simplicity we consider the case when $F = F(S)$ with $S = \bar{\psi}\psi$. Here ∇_μ is the covariant derivative of spinor field:

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi, \quad \nabla_\mu \bar{\psi} = \frac{\partial \bar{\psi}}{\partial x^\mu} + \bar{\psi}\Gamma_\mu, \tag{2.7}$$

with Γ_μ being the spinor affine connection. In (2.6) γ's are the Dirac matrices in curve space-time and obey the following algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \tag{2.8}$$

and are connected with the flat space-time Dirac matrices $\bar{\gamma}$ in the following way

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab}, \quad \gamma_\mu(x) = e^a_\mu(x) \bar{\gamma}_a, \tag{2.9}$$

where $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$ and e^a_μ is a set of tetrad 4-vectors. The spinor affine connection matrices $\Gamma_\mu(x)$ are uniquely determined up to an additive multiple of the unit matrix by the equation

$$\frac{\partial \gamma_\nu}{\partial x^\mu} - \Gamma^\rho_{\nu\mu}\gamma_\rho - \Gamma_\mu \gamma_\nu + \gamma_\nu \Gamma_\mu = 0, \tag{2.10}$$

with the solution

$$\Gamma_\mu = \frac{1}{4}\bar{\gamma}_a \gamma^\nu \partial_\mu e^{(a)}_\nu - \frac{1}{4}\gamma_\rho \gamma^\nu \Gamma^\rho_{\mu\nu}, \tag{2.11}$$

C. Field equations

Variation of (2.1) with respect to the metric function $g_{\mu\nu}$ gives the Einstein field equation

$$G^\nu_\mu = R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R = -\kappa T^\nu_\mu, \tag{2.12}$$

where R^ν_μ and R are the Ricci tensor and Ricci scalar, respectively. Here T^ν_μ is the energy momentum tensor of the spinor field.

Varying (2.6) with respect to $\bar{\psi}(\psi)$ one finds the spinor field equations:

$$\imath\gamma^\mu \nabla_\mu \psi - m_{\mathrm{sp}}\psi - F_S \psi = 0, \tag{2.13a}$$

$$\imath\nabla_\mu \bar{\psi}\gamma^\mu + m_{\mathrm{sp}}\bar{\psi} + 2F_S\bar{\psi} = 0. \tag{2.13b}$$

Here we denote $F_S = dF/dS$.

D. Energy momentum tensor of the spinor field

The energy-momentum tensor of the spinor field is given by

$$T^\rho_\mu = \frac{\imath}{4} g^{\rho\nu}\left(\bar{\psi}\gamma_\mu \nabla_\nu \psi + \bar{\psi}\gamma_\nu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma_\nu \psi - \nabla_\nu \bar{\psi}\gamma_\mu \psi\right) - \delta^\rho_\mu L_{\mathrm{sp}} \tag{2.14}$$

where L_{sp} in view of (2.13) can be rewritten as

$$\begin{aligned} L_{\mathrm{sp}} &= \frac{\imath}{2}\left[\bar{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma^\mu \psi\right] - m_{\mathrm{sp}}\bar{\psi}\psi - F(K) \\ &= \frac{\imath}{2}\bar{\psi}[\gamma^\mu \nabla_\mu \psi - m_{\mathrm{sp}}\psi] - \frac{\imath}{2}[\nabla_\mu \bar{\psi}\gamma^\mu + m_{\mathrm{sp}}\bar{\psi}]\psi - F(S), \\ &= SF_S - F(S). \end{aligned} \tag{2.15}$$

Then in view of (2.7) the energy-momentum tensor of the spinor field can be written as

$$T_\mu{}^\rho = \frac{\imath}{4} g^{\rho\nu}\left(\bar\psi\gamma_\mu\partial_\nu\psi + \bar\psi\gamma_\nu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\nu\psi - \partial_\nu\bar\psi\gamma_\mu\psi\right)$$
$$- \frac{\imath}{4} g^{\rho\nu}\bar\psi\left(\gamma_\mu\Gamma_\nu + \Gamma_\nu\gamma_\mu + \gamma_\nu\Gamma_\mu + \Gamma_\mu\gamma_\nu\right)\psi - \delta^\rho_\mu\left(SF_S - F(S)\right). \tag{2.16}$$

As is seen from (2.16), is case if for a given metric Γ_μ's are different, there arise nontrivial non-diagonal components of the energy momentum tensor.

From the (2.11) one finds the following expressions for spinor affine connections:

$$\Gamma_0 = 0, \quad \Gamma_1 = \frac{\dot a_1}{2}\vec\gamma^1\vec\gamma^0, \quad \Gamma_2 = \frac{\dot a_2}{2}\vec\gamma^2\vec\gamma^0, \quad \Gamma_3 = \frac{\dot a_3}{2}\vec\gamma^3\vec\gamma^0. \tag{2.17}$$

We consider the case when the spinor field depends on t only. Then from (2.16) one finds

$$T_0^0 = \frac{\imath}{2} g^{00}\left(\bar\psi\gamma_0\dot\psi - \dot{\bar\psi}\gamma_0\psi\right) - L_{\rm sp}, \tag{2.18a}$$
$$T_1^1 = -\frac{\imath}{2} g^{11}\bar\psi\left(\gamma_1\Gamma_1 + \Gamma_1\gamma_1\right)\psi - L_{\rm sp}, \tag{2.18b}$$
$$T_2^2 = -\frac{\imath}{2} g^{22}\bar\psi\left(\gamma_2\Gamma_2 + \Gamma_2\gamma_2\right)\psi - L_{\rm sp}, \tag{2.18c}$$
$$T_3^3 = -\frac{\imath}{2} g^{33}\bar\psi\left(\gamma_3\Gamma_3 + \Gamma_3\gamma_3\right)\psi - L_{\rm sp}, \tag{2.18d}$$
$$T_1^0 = \frac{\imath}{4} g^{00}\left(\bar\psi\gamma_1\dot\psi - \dot{\bar\psi}\gamma_1\psi\right) - \frac{\imath}{4} g^{00}\bar\psi\left(\gamma_0\Gamma_1 + \Gamma_1\gamma_0\right)\psi, \tag{2.18e}$$
$$T_2^0 = \frac{\imath}{4} g^{00}\left(\bar\psi\gamma_2\dot\psi - \dot{\bar\psi}\gamma_2\psi\right) - \frac{\imath}{4} g^{00}\bar\psi\left(\gamma_0\Gamma_2 + \Gamma_2\gamma_0\right)\psi, \tag{2.18f}$$
$$T_3^0 = \frac{\imath}{4} g^{00}\left(\bar\psi\gamma_3\dot\psi - \dot{\bar\psi}\gamma_3\psi\right) - \frac{\imath}{4} g^{00}\bar\psi\left(\gamma_0\Gamma_3 + \Gamma_3\gamma_0\right)\psi, \tag{2.18g}$$
$$T_2^1 = -\frac{\imath}{4} g^{11}\bar\psi\left(\gamma_2\Gamma_1 + \Gamma_1\gamma_2 + \gamma_1\Gamma_2 + \Gamma_2\gamma_1\right)\psi, \tag{2.18h}$$
$$T_3^2 = -\frac{\imath}{4} g^{22}\bar\psi\left(\gamma_3\Gamma_2 + \Gamma_2\gamma_3 + \gamma_2\Gamma_3 + \Gamma_3\gamma_2\right)\psi, \tag{2.18i}$$
$$T_3^1 = -\frac{\imath}{4} g^{11}\bar\psi\left(\gamma_3\Gamma_1 + \Gamma_1\gamma_3 + \gamma_1\Gamma_3 + \Gamma_3\gamma_1\right)\psi. \tag{2.18j}$$

In this case after a little manipulations from (2.16) for the nontrivial components of the energy momentum tensor one finds [41]:

$$T_0^0 = m_{\rm sp}S + F(S), \tag{2.19a}$$
$$T_1^1 = T_2^2 = T_3^3 = F(S) - SF_S, \tag{2.19b}$$
$$T_2^1 = -\frac{\imath}{4}\frac{a_2}{a_1}\left(\frac{\dot a_1}{a_1} - \frac{\dot a_2}{a_2}\right)\bar\psi\vec\gamma^1\vec\gamma^2\vec\gamma^0\psi, \tag{2.19c}$$
$$T_3^2 = -\frac{\imath}{4}\frac{a_3}{a_2}\left(\frac{\dot a_2}{a_2} - \frac{\dot a_3}{a_3}\right)\bar\psi\vec\gamma^2\vec\gamma^3\vec\gamma^0\psi, \tag{2.19d}$$
$$T_3^1 = -\frac{\imath}{4}\frac{a_3}{a_1}\left(\frac{\dot a_3}{a_3} - \frac{\dot a_1}{a_1}\right)\bar\psi\vec\gamma^3\vec\gamma^1\vec\gamma^0\psi. \tag{2.19e}$$

As one sees from (2.18) and (2.19) the non-triviality of non-diagonal components of the energy momentum tensors, namely T_2^1, T_3^2 and T_3^1 is directly connected with the affine spinor connections Γ_i's.

III. SOLUTION TO THE FIELD EQUATIONS

In this section we solve the field equations. Let us begin with the spinor field equations. In view of (2.7) and (2.17) the spinor field equation (2.13a) takes the form

$$\imath\gamma^0\big(\dot{\psi}+\frac{1}{2}\frac{\dot{V}}{V}\psi\big)-m_{\rm sp}\psi-SF_S\psi=0. \tag{3.1a}$$

$$\imath\gamma^0\big(\dot{\psi}+\frac{1}{2}\frac{\dot{V}}{V}\psi\big)+m_{\rm sp}\psi+SF_S\bar{\psi}=0, \tag{3.1b}$$

where we define the volume scale as

$$V=a_1a_2a_3. \tag{3.2}$$

From (3.1) one easily finds

$$\dot{S}+\frac{\dot{V}}{V}S=0, \tag{3.3}$$

with the solution

$$S=\frac{V_0}{V},\quad V_0=\text{const}. \tag{3.4}$$

As we have already mentioned, ψ is a function of t only. We consider the 4-component spinor field given by

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}. \tag{3.5}$$

Taking into account that $\phi_i=\sqrt{V}\psi_i$ and defining $\mathscr{D}=SF_S$ and inserting (3.5) into (3.1a) in this case we find

$$\dot{\phi}_1+\imath\mathscr{D}\phi_1=0, \tag{3.6a}$$

$$\dot{\phi}_2+\imath\mathscr{D}\phi_2=0, \tag{3.6b}$$

$$\dot{\phi}_3-\imath\mathscr{D}\phi_3=0, \tag{3.6c}$$

$$\dot{\phi}_4-\imath\mathscr{D}\phi_4=0,. \tag{3.6d}$$

Here we also consider the massless spinor field setting $m_{\rm sp}=0$. The foregoing system of equations can be easily solved. Finally for the spinor field we obtain

$$\psi_1(t)=(C_1/\sqrt{V})\exp\left(-i\int\mathscr{D}dt\right), \tag{3.7a}$$

$$\psi_2(t)=(C_2/\sqrt{V})\exp\left(-i\int\mathscr{D}dt\right), \tag{3.7b}$$

$$\psi_3(t)=(C_3/\sqrt{V})\exp\left(i\int\mathscr{D}dt\right), \tag{3.7c}$$

$$\psi_4(t)=(C_4/\sqrt{V})\exp\left(i\int\mathscr{D}dt\right), \tag{3.7d}$$

with C_1,C_2,C_3,C_4 being the integration constants and related to V_0 as

$$C_1^*C_1+C_2^*C_2-C_3^*C_3-C_4^*C_4=V_0.$$

Thus we see that the components of the spinor field are in some functional dependence of V.

Let us now solve the gravitational field equations. On account of (2.5) and (2.19) the system of Einstein field equations (2.12) takes the form

$$\frac{\ddot{a}_2}{a_2}+\frac{\ddot{a}_3}{a_3}+\frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3}=\kappa\big(F(S)-SF_S\big), \tag{3.8a}$$
$$\frac{\ddot{a}_3}{a_3}+\frac{\ddot{a}_1}{a_1}+\frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}=\kappa\big(F(S)-SF_S\big), \tag{3.8b}$$
$$\frac{\ddot{a}_1}{a_1}+\frac{\ddot{a}_2}{a_2}+\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2}=\kappa\big(F(S)-SF_S\big), \tag{3.8c}$$
$$\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2}+\frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3}+\frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1}=\kappa\big(m_{\rm sp}S+F(S)\big), \tag{3.8d}$$

together with the additional constrains

$$\left(\frac{\dot{a}_1}{a_1}-\frac{\dot{a}_2}{a_2}\right)\bar{\psi}\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^0\psi=0, \tag{3.9a}$$
$$\left(\frac{\dot{a}_2}{a_2}-\frac{\dot{a}_3}{a_3}\right)\bar{\psi}\bar{\gamma}^2\bar{\gamma}^3\bar{\gamma}^0\psi=0, \tag{3.9b}$$
$$\left(\frac{\dot{a}_3}{a_3}-\frac{\dot{a}_1}{a_1}\right)\bar{\psi}\bar{\gamma}^3\bar{\gamma}^1\bar{\gamma}^0\psi=0. \tag{3.9c}$$

From (3.9) we see, it is possible to consider the case when we impose restrictions on the spinor field or they will be imposed on the metric functions. In what follows we will study the both situations in details. Note that in this case thanks to the Einstein equations the non-diagonal components of the energy momentum tensor become zero. But we can also set them zero in order to simulate different kind of fluid and dark energy those have only non-zero diagonal components. It should be noted that additional constrains analogous to (3.9) was also found it [39].

A. restrictions on spinor field

Let us first consider the case when the non-diagonal components of the energy momentum tensor impose restrictions on the spinor field. In this case from (3.9) we obtain

$$\bar{\psi}\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^0\psi=\bar{\psi}\bar{\gamma}^3\bar{\gamma}^1\bar{\gamma}^0\psi=\bar{\psi}\bar{\gamma}^2\bar{\gamma}^3\bar{\gamma}^0\psi=0. \tag{3.10}$$

The components of the spinor field in this case undergo some changes. It is found that in this case the integration constants in (3.7) should obey

$$C_1^*C_1-C_2^*C_2-C_3^*C_3+C_4^*C_4=0, \tag{3.11a}$$
$$C_1^*C_2-C_2^*C_1-C_3^*C_4+C_4^*C_3=0, \tag{3.11b}$$
$$C_1^*C_2+C_2^*C_1-C_3^*C_4-C_4^*C_3=0, \tag{3.11c}$$
$$C_1^*C_1+C_2^*C_2-C_3^*C_3-C_4^*C_4=V_0, \tag{3.11d}$$

which gives

$$C_1^*C_2-C_3^*C_4=C_2^*C_1-C_4^*C_3=0, \tag{3.12a}$$
$$C_1^*C_1-C_3^*C_3=C_2^*C_2-C_4^*C_4=\frac{V_0}{2}. \tag{3.12b}$$

In this case solving the Einstein equation on account of the fact that $T_1^1 = T_2^2 = T_3^3$ for the metric functions one finds [13]

$$a_i = D_i V^{1/3} \exp\left(X_i \int \frac{dt}{V}\right), \quad \prod_{i=1}^{3} D_i = 1, \quad \sum_{i=1}^{3} X_i = 0, \tag{3.13}$$

with D_i and X_i being the integration constants. Thus we see that the metric functions can be expressed in terms of V.

Summation of (3.8a), (3.8b), (3.8c) and 3 times (3.8d) leads to the equation for V [13]

$$\ddot{V} = \frac{3\kappa}{2}(T_0^0 + T_1^1)V = \frac{3\kappa}{2}(m_{\rm sp}S + 2F(S) - SF_S)V. \tag{3.14}$$

Since S is a function of V the right hand side of (3.14) is a function of V as well, hence can be solved in quadrature. Given the concrete form of view one finds the solution for V. Here we consider the cases, when the spinor field describes different kinds of well-known fluid and dark energy.

Let us now recall that in the unified nonlinear spinor theory of Heisenberg, the massive term remains absent, and according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter [42, 43]. In the nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines total energy (or mass) of the nonlinear field system. Moreover, it was established that only a massless spinor field with the Lagrangian (2.6) describes perfect fluid from phantom to ekpyrotic matter [29–33]. Thus without losing the generality we can consider the massless spinor field putting $m_{\rm sp} = 0$.

Let us consider the case when the spinor field Lagrangian (2.6) describes a berotropic fluid. Inserting (2.19a) and (2.19b) into the barotropic equation of state

$$p = W\varepsilon, \tag{3.15}$$

where W is a constant, one finds

$$SF_S = (1+W)F(S), \tag{3.16}$$

with the solution

$$F(S) = \lambda S^{(1+W)}, \quad \lambda = \text{const}. \tag{3.17}$$

Depending on the value of W (3.15) describes perfect fluid from phantom to ekpyrotic matter, namely

$$W = 0, \qquad (\text{dust}), \tag{3.18a}$$
$$W = 1/3, \qquad (\text{radiation}), \tag{3.18b}$$
$$W \in (1/3, 1), \qquad (\text{hard Universe}), \tag{3.18c}$$
$$W = 1, \qquad (\text{stiff matter}), \tag{3.18d}$$
$$W \in (-1/3, -1), \qquad (\text{quintessence}), \tag{3.18e}$$
$$W = -1, \qquad (\text{cosmological constant}), \tag{3.18f}$$
$$W < -1, \qquad (\text{phantom matter}), \tag{3.18g}$$
$$W > 1, \qquad (\text{ekpyrotic matter}). \tag{3.18h}$$

In account of it the spinor field Lagrangian now reads

$$L = \frac{i}{2}\left[\bar{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma^\mu \psi\right] - \lambda S^{(1+W)}. \tag{3.19}$$

Thus a massless spinor field with the Lagrangian (3.19) describes perfect fluid from phantom to ekpyrotic matter. Here the constant of integration λ can be viewed as constant of self-coupling. A detailed analysis of this study was given in [29–32].

In case of (3.19) we have

$$T_0^0 = \varepsilon = \lambda S^{(1+W)}, \tag{3.20a}$$

$$T_1^1 = -p = -W\varepsilon = -W\lambda S^{(1+W)}. \tag{3.20b}$$

Eq. (3.14) then takes the form

$$\ddot{V} = \frac{3\kappa}{2}\lambda V_0^{1+W}(1-W)V^{-W}, \tag{3.21}$$

with the solution in quadrature

$$\int \frac{dV}{\sqrt{3\kappa\lambda V_0^{1+W}V^{1-W}+C_1}} = t+t_0. \tag{3.22}$$

Here C_1 and t_0 are the integration constants.

Let us consider the case when the spinor field describes a Chaplygin gas described by a equation of state

$$p = -A/\varepsilon^{\alpha}, \tag{3.23}$$

where A is a positive constant and $0 \leq \alpha \leq 1$. Then in case of a massless spinor field for F one finds

$$\frac{F^{\alpha}dF}{F^{1+\alpha}-A} = \frac{dS}{S}, \tag{3.24}$$

with the solution [30–32]

$$F = \left(A+\lambda S^{(1+\alpha)}\right)^{1/(1+\alpha)}. \tag{3.25}$$

The Spinor field Lagrangian in this case takes the form

$$L = \frac{i}{2}\left[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi\right] - \left(A+\lambda S^{(1+\alpha)}\right)^{1/(1+\alpha)}. \tag{3.26}$$

In this case we have

$$T_0^0 = \varepsilon = \left(A+\lambda S^{(1+\alpha)}\right)^{1/(1+\alpha)}, \tag{3.27a}$$

$$T_1^1 = -p = A/\varepsilon^{\alpha} = A/\left(A+\lambda S^{(1+\alpha)}\right)^{\alpha/(1+\alpha)}. \tag{3.27b}$$

The equation for V now reads

$$\ddot{V} = \frac{3\kappa}{2}\left[\left(AV^{1+\alpha}+\lambda V_0^{1+\alpha}\right)^{1/(1+\alpha)} + AV^{1+\alpha}/\left(AV^{1+\alpha}+\lambda V_0^{1+\alpha}\right)^{\alpha/(1+\alpha)}\right], \tag{3.28}$$

with the solution

$$\int \frac{dV}{\sqrt{C_1+3\kappa V\left(AV^{1+\alpha}+\lambda V_0^{1+\alpha}\right)^{1/(1+\alpha)}}} = t+t_0, \quad C_1 = \text{const.} \quad t_0 = \text{const.} \tag{3.29}$$

Inserting $\alpha = 1$ we come to the result obtained in [44].

We also consider a modified Chaplygin gas given by the equation of state

$$p = W\varepsilon - A/\varepsilon^{\alpha}, \tag{3.30}$$

with W is a constant, $A>0$ and $0\leq\alpha\leq 1$. Inserting $p=SF_S-F$ and $\varepsilon=F$ (we consider here massless spinor field) we get

$$\frac{F^{\alpha}dF}{F^{1+\alpha}-A/(1+W)}=(1+W)\frac{dS}{S}, \tag{3.31}$$

with the solution

$$F=\left(\frac{A}{1+W}+\lambda S^{(1+\alpha)(1+W)}\right)^{1/(1+\alpha)}. \tag{3.32}$$

The Spinor field Lagrangian in this case takes the form

$$L=\frac{i}{2}\left[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi-\nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi\right]-\left(\frac{A}{1+W}+\lambda S^{(1+\alpha)(1+W)}\right)^{1/(1+\alpha)}. \tag{3.33}$$

In this case we have In this case we have

$$T_0^0=\varepsilon=\left(\frac{A}{1+W}+\lambda S^{(1+\alpha)(1+W)}\right)^{1/(1+\alpha)}, \tag{3.34a}$$

$$T_1^1=-p=-W\varepsilon+A/\varepsilon^{\alpha}=-W\left(\frac{A}{1+W}+\lambda S^{(1+\alpha)(1+W)}\right)^{1/(1+\alpha)} + A\left(\frac{A}{1+W}+\lambda S^{(1+\alpha)(1+W)}\right)^{-\alpha/(1+\alpha)}. \tag{3.34b}$$

The equation for V now reads

$$\ddot{V}=\frac{3\kappa}{2}\left[(1-W)\left(\frac{A}{1+W}V^{(1+\alpha)(1+W)}+\lambda_0\right)^{1/(1+\alpha)}V^{-W} + A\left(\frac{A}{1+W}V^{(1+\alpha)(1+W)}+\lambda_0\right)^{-\alpha/(1+\alpha)}V^{1+\alpha(1+W)}\right], \quad \lambda_0=\lambda V_0^{(1+\alpha)(1+W)} \tag{3.35}$$

with the solution

$$\int\frac{dV}{\sqrt{C_1+3\kappa V\left(\frac{A}{1+W}V^{(1+\alpha)(1+W)}+\lambda_0\right)^{1/(1+\alpha)}V^{1-W}}}=t+t_0, \quad C_1,t_0=\text{consts}. \tag{3.36}$$

Finally, it should be noted that a quintessence with a modified equation of state

$$p=W(\varepsilon-\varepsilon_{\text{cr}}), \quad W\in(-1,0), \tag{3.37}$$

where ε_{cr} some critical energy density, the spinor field nonlinearity takes the form

$$F=\lambda S^{1+W}+\frac{W}{1+W}\varepsilon_{\text{cr}}. \tag{3.38}$$

The spinor field Lagrangian in this case reads

$$L=\frac{i}{2}\left[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi-\nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi\right]-\lambda S^{(1+W)/2}-\frac{W}{1+W}\varepsilon_{\text{cr}}. \tag{3.39}$$

Setting $\varepsilon_{\rm cr}=0$ one gets (3.19). The purpose of introducing the modified EoS was to avoid the problem of eternal acceleration. Taking into account that

$$T_0^0=\lambda S^{(1+W)}+\frac{W}{1+W}\varepsilon_{\rm cr}, \tag{3.40a}$$

$$T_1^1=T_2^2=T_3^3=-\lambda W S^{(1+W)}+\frac{W}{1+W}\varepsilon_{\rm cr}, \tag{3.40b}$$

for V in this case we find

$$\ddot{V}=\frac{3\kappa}{2}\Big[\lambda V_0^{1+W}(1-W)V^{-W}+2W\varepsilon_{\rm cr}V/(1+W)\Big], \tag{3.41}$$

with the solution in quadrature

$$\int\frac{dV}{\sqrt{3\kappa\big[\lambda V_0^{1-W}V^{1-W}+W\varepsilon_{\rm cr}V^2/(1+W)\big]+C_1}}=t+t_0. \tag{3.42}$$

Here C_1 and t_0 are the integration constants. Comparing (3.42) with those with a negative Λ-term we see that $\varepsilon_{\rm cr}$ plays the role of a negative cosmological constant.

1. *Problem of isotropization*

Since the present-day Universe is surprisingly isotropic, it is important to see whether our anisotropic BI model evolves into an isotropic FRW model. Isotropization means that at large physical times t, when the volume factor V tends to infinity, the three scale factors $a_i(t)$ grow at the same rate. Two wide-spread definition of isotropization read

$$\mathscr{A}=\frac{1}{3}\sum_{i=1}^{3}\frac{H_i^2}{H^2}-1\to 0, \tag{3.43a}$$

$$\Sigma^2=\frac{1}{2}\mathscr{A}H^2\to 0. \tag{3.43b}$$

Here $\mathscr{A}$ and Σ^2 are the average anisotropy and shear, respectively. $H_i=\dot{a}_i/a_i$ is the directional Hubble parameter and $H=\dot{a}/a$ average Hubble parameter, where $a(t)=V^{1/3}$ is the average scale factor. Here we exploit the isotropization condition proposed in [45]

$$\frac{a_i}{a}\Big|_{t\to\infty}\to\text{const}. \tag{3.44}$$

Then by rescaling some of the coordinates, we can make $a_i/a\to 1$, and the metric will become manifestly isotropic at large t.

From (3.13) we find

$$\frac{a_i}{a}=\frac{a_i}{V^{1/3}}=D_i\exp\Big(X_i\int\frac{dt}{V}\Big). \tag{3.45}$$

As is seen from (3.13) in our case $a_i/a\to D_i=$ const as $V\to\infty$. Recall that the isotropic FRW model has same scale factor in all three directions, i.e., $a_1(t)=a_2(t)=a_3(t)=a(t)$. So for the BI universe to evolve into a FRW one the constants D_i's are likely to be identical, i.e., $D_1=D_2=D_3=1$. Moreover, the isotropic nature of the present Universe leads to the fact that the three other constants X_i should be close to zero as well, i.e., $|X_i|<<1$, $(i=1,2,3)$, so that $X_i\int[V(t)]^{-1}dt\to 0$

for $t < \infty$ (for $V(t) = t^n$ with $n > 1$ the integral tends to zero as $t \to \infty$ for any X_i). It can be concluded that the spinor field Lagrangian with $W < 1$ leads to the isotropization of the Universe as $t \to \infty$, moreover, in case of $W < 0$ the system undergoes an earlier isotropization.

Unfortunately, it is not the end of the story. Due to the specific behavior of the spinor field in curve space-time there are still some unresolved questions regarding this case. So before dealing with other cases let us review this case once again.

In doing so we recall that the expression (3.10) can be rewritten in the form

$$\bar{\psi}\bar{\gamma}^5\bar{\gamma}^1\psi = \bar{\psi}\bar{\gamma}^5\bar{\gamma}^2\psi = \bar{\psi}\bar{\gamma}^5\bar{\gamma}^3\psi = 0. \tag{3.46}$$

Recalling that there are 16 independent bilinear combinations:

$$S = \bar{\psi}\psi \qquad \text{(scalar)}, \tag{3.47a}$$
$$P = \imath\bar{\psi}\gamma^5\psi \qquad \text{(pseudoscalar)}, \tag{3.47b}$$
$$v^\mu = (\bar{\psi}\gamma^\mu\psi) \qquad \text{(vector)}, \tag{3.47c}$$
$$A^\mu = (\bar{\psi}\gamma^5\gamma^\mu\psi) \qquad \text{(pseudovector)}, \tag{3.47d}$$
$$T^{\mu\nu} = (\bar{\psi}\sigma^{\mu\nu}\psi) \qquad \text{(antisymmetric tensor)}, \tag{3.47e}$$

where $\sigma^{\mu\nu} = (\imath/2)[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]$ and 5 invariants, corresponding to these bilinear forms:

$$I = S^2, \tag{3.48a}$$
$$J = P^2, \tag{3.48b}$$
$$I_v = v_\mu v^\mu = (\bar{\psi}\gamma^\mu\psi)g_{\mu\nu}(\bar{\psi}\gamma^\nu\psi), \tag{3.48c}$$
$$I_A = A_\mu A^\mu = (\bar{\psi}\gamma^5\gamma^\mu\psi)g_{\mu\nu}(\bar{\psi}\gamma^5\gamma^\nu\psi), \tag{3.48d}$$
$$I_T = T_{\mu\nu}T^{\mu\nu} = (\bar{\psi}\sigma^{\mu\nu}\psi)g_{\mu\alpha}g_{\nu\beta}(\bar{\psi}\sigma^{\alpha\beta}\psi). \tag{3.48e}$$

on account of of (3.46) we find $A^1 = A^2 = A^3 = 0$. Then from the equality

$$A_\mu v^\mu = 0, \tag{3.49}$$

we find

$$A_0 v^0 = \bar{\psi}\gamma^5\gamma_0\psi\bar{\psi}\gamma^0\psi = \bar{\psi}\gamma^5\gamma^0\psi\bar{\psi}^*\psi = 0. \tag{3.50}$$

Since $\bar{\psi}^*\psi \neq 0$, from (3.50) follows that $A^0 = 0$, hence $I_A = 0$. But according to the Fierz identity $I_v = -I_A = I + J$ and $I_T = I - J$. Hence we obtain

$$I_A = -(S^2 + P^2) = 0, \tag{3.51}$$

which leads to the fact that

$$S = \bar{\psi}\psi = 0, \quad P = i\bar{\psi}\gamma^5\psi = 0. \tag{3.52}$$

This very fact, even without reference to Heisenberg, suggests that the spinor in this case should be massless.

But the question is whether with $S = 0$ the nonlinearity altogether vanishes? In case the nonlinearity becomes trivial, we get vacuum solution, with

$$V = V_1 t + V_2, \quad V_1, V_2 - \text{consts}. \tag{3.53}$$

and

$$a_i = D_i\big(V_1 t + V_2\big)^{\frac{1}{3}+\frac{X_i}{V_1}}, \tag{3.54}$$

In this case $\frac{a_i}{a}\Big|_{t\to\infty} = \Big(V_1 t + V_2\Big)^{X_i/V_1}\Big|_{t\to\infty} \nrightarrow$ const. It means in absence of nonlinearity no isotropization takes place.

Nevertheless, the case discussed above is worth studying. It shows how sensitive the spinor field may be to the gravitational one. Now the question is how to resolve this puzzle?

The problem we are facing now occurs as a result of non-triviality of the non-diagonal components of the energy momentum tensor of the spinor field, which imposes some severe restrictions either on spinor or on gravitational fields. Moreover, this non-triviality is wholly depends on the affine spinor connections, which is defined by the gravitational field. One of the possible solutions is to consider other type of metrics. As it will be shown later, even imposing total (which corresponds to FRW metric) or partial (together with spinor field that gives rise to LRS BI metric) restrictions on the metric functions one can obtain satisfactory solutions to the problem in question.

But the question is "Is there any way to solve the system within the scope of BI metric given by (2.4)?" In our view there may be the following possibilities:

- Study other types of nonlinearities;
- Consider spinor fields with larger number of components, such as dark spinor or ELKO with 8 components and spinors with 16 components;
- Introduce torsion into the system;
- Investigate models with spinor field equations of higher order.

It should be noted that in a recent paper [41] we have considered the case with with the nonlinear term being some arbitrary functions of invariants (3.48) generated from bilinear spinor forms (3.47). Even in that kind of generalization leads to the conclusion obtained in (3.52). So the nonlinearity should be more general.

Though less likely, there may still be some other way to interpret the results obtained here. Here is some very close situation, but as I have already mentioned its probability is very small. Nevertheless, let us write a few lines about that. It is well known that the linear spinor field the Lagrangian

$$L_{\rm spl} = \frac{\imath}{2}\left[\bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\bar{\psi}\gamma^{\mu}\psi\right] - m_{\rm sp}\bar{\psi}\psi \tag{3.55}$$

vanishes thank to the spinor field equations

$$\imath\gamma^{\mu}\nabla_{\mu}\psi - m_{\rm sp}\psi = 0, \quad \imath\nabla_{\mu}\bar{\psi}\gamma^{\mu} + m_{\rm sp}\bar{\psi} = 0. \tag{3.56}$$

But it does not mean that the Lagrangian is trivial. Or in Hamiltonian formalism we set, Hamiltonian $H = 0$ that gives additional constrains, though the Hamiltonian as a whole is non-trivial. So there might be some extraordinary interpretation of the nonlinear term being non-trivial, though its arguments becomes trivial under some specific conditions.

But all these proposals need further detailed investigations. We plan to study them in some of our forthcoming papers.

B. restrictions on metric functions

Here we study the other possibility is to keep the components of the spinor field unaltered. In this case from (3.9) one finds

$$\frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} = \frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} = \frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} = 0, \tag{3.57}$$

which can be rewritten as

$$\frac{\dot{a}_1}{a_1} = \frac{\dot{a}_2}{a_2} = \frac{\dot{a}_3}{a_3} \equiv \frac{\dot{a}}{a}. \tag{3.58}$$

Taking into account that

$$\frac{\ddot{a}_i}{a_i} = \frac{d}{dt}\left(\frac{\dot{a}_i}{a_i}\right) + \left(\frac{\dot{a}_i}{a_i}\right)^2 = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 = \frac{\ddot{a}}{a},$$

the system (3.8) can be written as a system of two equations:

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \kappa T_1^1, \tag{3.59a}$$

$$3\frac{\dot{a}^2}{a^2} = \kappa T_0^0. \tag{3.59b}$$

In order to find the solution that satisfies both (3.59a) and (3.59b) we rewrite (3.59a) in view of (3.59b) in the following form:

$$\ddot{a} = \frac{\kappa}{6}\left(3T_1^1 - T_0^0\right)a. \tag{3.60}$$

Thus in account of non-diagonal components of the spinor field, we though begin with Bianchi type-I space time, in reality solving the Einstein field equations for FRW model. Before solving the equation (3.60), let us go back to (3.13). Taking into account that

$$\frac{\dot{a}_i}{a_i} = \frac{\dot{V}}{3V} + \frac{X_i}{V}, \tag{3.61}$$

in view of (3.58) we find that

$$X_1 = X_2 = X_3 = 0. \tag{3.62}$$

The triviality of the integration constant X_i follows from the fact that $X_1 + X_2 + X_3 = 0$. Thus the solution (3.13) should be written as

$$a_i = D_i V^{1/3} = D_i a, \quad \prod_{i=1}^{3} D_i = 1, \tag{3.63}$$

which means it represents a tiny sector of the general solutions (3.13) which one obtains for the BI model in case of isotropic distribution of matter with trivial non-diagonal components of energy-momentum tensor, e.g., when the Universe is filled with perfect fluid, dark energy etc.

Let us now define $a = V^{1/3}$ for different cases. In doing so we recall that K in this case takes the form

$$S = \frac{a_0^3}{a^3}, \quad a_0 = \text{const.}. \tag{3.64}$$

Then then equation for a in case of the spinor field given by (3.19) takes the form

$$\ddot{a} = -\frac{\kappa\lambda}{6}(1+3W)a_0^{3(1+W)}a^{-(2+3W)}, \tag{3.65}$$

with the solution is quadrature

$$\int \frac{da}{\sqrt{(\kappa\lambda/3)a_0^{3(1+W)}a^{-(1+3W)} + E_1}} = t, \tag{3.66}$$

with E_1 being integration constant.

As far as Chaplygin scenario is concerned in this case we have

$$\ddot{a} = -\frac{\kappa}{6}\frac{2Aa^{3(1+\alpha)} - \lambda_0}{a^2\left(Aa^{3(1+\alpha)} + \lambda_0\right)^{\alpha/(1+\alpha)}}, \quad \lambda_0 = \lambda a_0^{3(1+\alpha)} \tag{3.67}$$

This equation is solved numerically and the result is presented in Fig. 2

For the modified Chaplygin gas we have the following equation for a

$$\ddot{a} = \frac{\kappa}{6}\left[(1-3W)\left(\frac{A}{1+W}a^{3(1+\alpha)(1+W)}+\lambda_0\right)^{1/(1+\alpha)}a^{-(2+3W)}\right.$$
$$\left.+A\left(\frac{A}{1+W}a^{3(1+\alpha)(1+W)}+\lambda_0\right)^{-\alpha/(1+\alpha)}a^{1+3\alpha(1+W)}\right], \quad \lambda_0 = \lambda a_0^{3(1+\alpha)(1+W)} \tag{3.68}$$

This equation is solved numerically and the result is presented in Fig. 3

Finally we consider the case with modified quintessence. Inserting (3.40a) and (3.40b) into (3.60) in this case we find

$$\ddot{a} = -\frac{\kappa}{6}\left[(3W+1)\lambda a_0^{3(1+W)}a^{-(3W+2)} - \frac{2W}{1+W}\varepsilon_{\rm cr}a\right], \tag{3.69}$$

with he solution

$$\int \frac{da}{\sqrt{(\kappa/3)\left[\lambda a_0^{3(1+W)}a^{-(3W+1)}+[W/(1+W)]\varepsilon_{\rm cr}a^2+E_2\right]}} = t, \quad E_2 = \text{const.} \tag{3.70}$$

It can be shown that in case of modified quintessence the pressure is sign alternating. As a result we have a cyclic mode of evolution.

It should be noted that the metric functions a_i in this case not necessarily be identic, rather one can write

$$a_1 = c_1 a, \quad a_2 = c_2 a, \quad a_3 = c_3 a, \quad c_1c_2c_3 = 1, \tag{3.71}$$

with c_i being some integration constants.

As far as isotropization is concerned, in this case from (3.71) we find

$$\frac{a_i}{a} = c_i = \text{const.}, \tag{3.72}$$

for any given time. Though the solutions for metric functions can be obtained solving Einstein equations for FRW space-time, depending on the constants it might not be isotropic from the very beginning, but in the course of time becomes isotropic. For the metric to be completely isotropic the constants c_i should be identical, i.e., $c_1 = c_2 = c_3$.

In what follows we illustrate the evolution of the Universe filled with quintessence, Chaplygin gas and quintessence with modified equation of state for two different cases: when the restrictions are imposed on the metric functions and when both spinor field and the metric functions were restricted. In Figures 1, 2, 3 and 4 we illustrated the evolution of the Universe filled with quintessence, Chaplygin gas, modified Chaplygin gas and quintessence with modified equation of state, respectively. The solid (red) line stands for the volume scale, when the restrictions were imposed on both the components of the spinor field and the metric functions. In this case the spacetime is given by a LRS BI model and the isotropization takes place asymptotically. The blue line shows the evolution of the Universe when the restrictions were impose on the metric functions. In this case the spacetime becomes isotropic from the very beginning and is described by a FRW cosmological model. Here we plot the volume scale as a^3, which a being the average scale factor.

As one sees, in case of early isotropization the Universe grows rapidly.

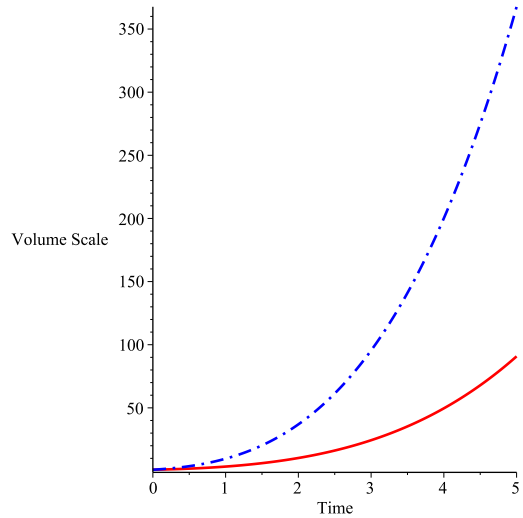

FIG. 1. Evolution of the Universe filled with quintessence. The solid (red) line stands for volume scale V, while the dash-dot (blue) line stands for a^3.

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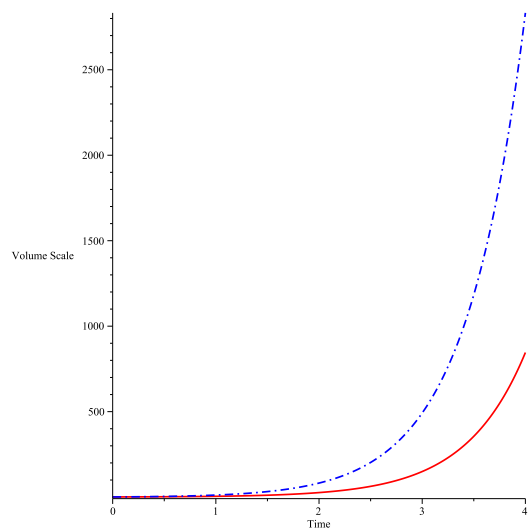

FIG. 2. Evolution of the Universe filled with Chaplygin gas. The solid (red) line stands for volume scale V, while the dash-dot (blue) line stands for a^3.

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C. restrictions on metric functions and Spinor field

As we have already seen, the restrictions imposed only one the spinor field may lead to vacuum solution, while the restrictions of metric functions give rise to isotropic FRW model from the very beginning.

As a third path one may offer a model by imposing restrictions on both metric functions and spinor field. In doing so, let us assume, say

$$\frac{\dot{a}_2}{a_2} - \frac{\dot{a}_3}{a_3} = 0, \tag{3.73}$$

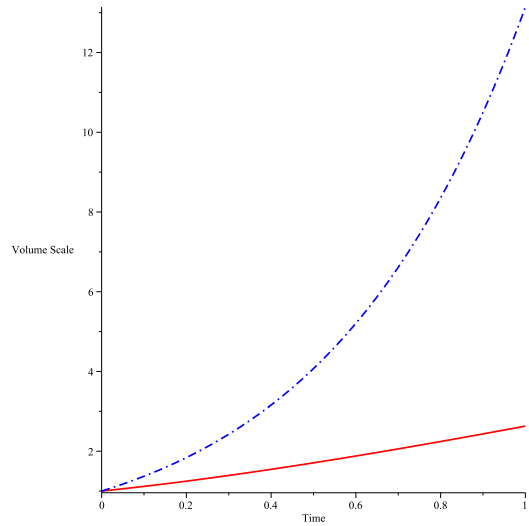

FIG. 3. Evolution of the Universe filled with modified Chaplygin gas. The solid (red) line stands for volume scale V, while the dash-dot (blue) line stands for a^3.

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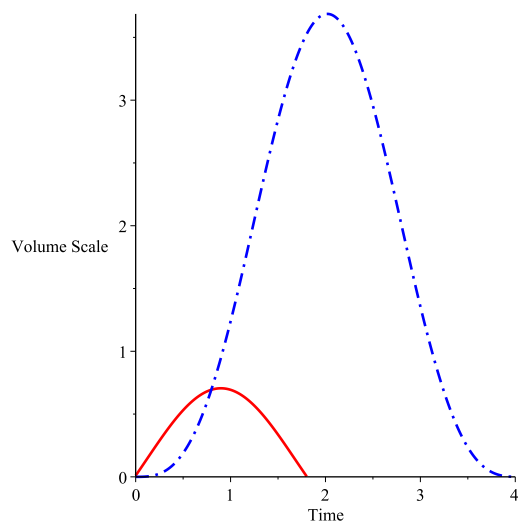

FIG. 4. Evolution of the Universe filled with quintessence with modified equation of state. The solid (red) line stands for volume scale V, while the dash-dot (blue) line stands for a^3.

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In this case we get a LRS Bianchi type-I space-time with the metric

$$ds^2 = dt^2 - a_1^2 dx^2 - a_2^2 \left[dy^2 + dz^2\right]. \tag{3.74}$$

Restrictions on the spinor field in this case look

$$\bar{\psi}\bar{\gamma}^5\bar{\gamma}^2\psi = \bar{\psi}\bar{\gamma}^2 5\bar{\gamma}^3\psi = 0. \tag{3.75}$$

Such type of restriction was earlier used in [39]. Under this condition we now have the following

relations between the coefficients of spinor field

$$C_1^*C_2 - C_3^*C_4 = C_2^*C_1 - C_4^*C_3, \tag{3.76a}$$

$$C_1^*C_1 - C_3^*C_3 = C_2^*C_2 - C_4^*C_4 = \frac{V_0}{2}. \tag{3.76b}$$

The Einstein Equations in this case read

$$2\frac{\ddot{a}_2}{a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 = \kappa\big(F(S) - SF_S\big), \tag{3.77a}$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} = \kappa\big(F(S) - SF_S\big), \tag{3.77b}$$

$$2\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \left(\frac{\dot{a}_2}{a_2}\right)^2 = \kappa F(S), \tag{3.77c}$$

where we set the spinor mass $m_{\rm sp} = 0$. As in previous case, defining

$$V = a_1 a_2^2, \tag{3.78}$$

we find

$$a_1 = D^2 V^{1/3} \exp[2X \int \frac{dt}{V}], \quad a_2 = (1/D) V^{1/3} \exp[-X \int \frac{dt}{V}], \tag{3.79}$$

with D and X being some arbitrary constants of integration. As far as V is concerned, summation of (3.77a), 2 times (3.77b) and 3 times (3.77c) gives

$$\ddot{V} = \frac{3\kappa}{2}\big(2F(S) - SF_S\big)\, V. \tag{3.80}$$

Further choosing the nonlinear term in the forms (3.17), (3.25), and (3.38), respectively, we obtain the analogical solutions as in first case. And as in that case here too we found that the isotropization process takes place asymptotically.

IV. PHYSICAL ASPECTS OF THE MODELS

In this section we discuss the physical aspects of the models considered above. Since the Bianchi type - I model within the spinor source needs further considerations, we begin with the FRW like case, when restrictions were imposed on the metric functions only. Let us note that though perfect fluid, quintessence etc. given by (3.18), (3.23), (3.30) and (3.37) can be simulated by the spinor field, it does not necessarily mean that we should confine to those models only. In this section we study the spinor field with a non-zero massive term and compare the results obtained with some recent observations. To begin with we write the EoS parameter ω as a ration between the pressure and energy density. Taking into account that $T_0^0 = \varepsilon$ and $T_1^1 = -p$ from (3.59) one obtains

$$\omega = \frac{p}{\varepsilon} = -\frac{2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}}{3\frac{\dot{a}^2}{a^2}}, \tag{4.1}$$

which on account of $q = -a\ddot{a}/\dot{a}^2$ gives the well known relation between the deceleration parameter q and EoS parameter ω:

$$q = \frac{3}{2}\big(\omega + \frac{1}{3}\big). \tag{4.2}$$

From (4.2) we see, for $\omega > -1/3$ the Universe expands with deceleration, while an accelerative mode of expansion takes place only when $\omega < -1/3$. For $\omega = -1/3$ the deceleration parameter

becomes trivial. In this case the metric function is either constant $a = const.$ or a linear function of time $a = C_1 t + C_2$.

On the other hand, from (2.19a) and (2.19b) one finds

$$\omega = \frac{p}{\varepsilon} = \frac{SF_S - F}{mS + F}. \tag{4.3}$$

In our previous papers [13, 14] we considered different types of spinor field nonlinearities. But given the fact that most of the established source fields are simulated by the spinor field nonlinearities given as power law, here we consider only that case setting $F = \lambda S^n$. In this case we find

$$\omega = \frac{\lambda(n-1)S^n}{mS + \lambda S^n}. \tag{4.4}$$

The relations (4.4) shows that only for $n < 2/3$ the spinor field in the given model can give rise to an accelerated mode of expansion.

Further recalling that $S = a_0^3/a^3$ from (4.4) we find

$$\omega = \frac{\lambda(n-1)}{\lambda + \tilde{m}a^{3(n-1)}}, \quad \tilde{m} = m/a_0^{3(n-1)}. \tag{4.5}$$

From (4.5) we see that for a massless spinor field the EoS parameter is a constant, namely $\omega = n - 1$, while if the spinor field has a nontrivial mass, the EoS parameter is time-dependent. Moreover, in absence of nonlinear term ($\lambda = 0$ and/or $n = 1$) the EoS parameter becomes trivial ($\omega = 0$). Note that this conclusion in no way contradicts our previous results. In order to simulate different matters the EoS parameter in (3.15) was taken to be constant. This very assumption leads to the spinor field Lagrangian with $m_{\rm sp} = 0$.

In what follows we study the case when the EoS parameter is time dependent. It can be a function of red-shift z or scale factor a (which it indeed is) as well. The red-shift dependence of ω can be linear like

$$\omega(z) = \omega_0 + \omega^{'} z, \tag{4.6}$$

with $\omega^{'} = \frac{d\omega}{dz}|_{z=0}$ (see Refs. [46, 47] or nonlinear as [48, 49]

$$\omega(z) = \omega_0 + \frac{\omega_1 z}{1+z}. \tag{4.7}$$

So, as for as the scale factor dependence of ω is concern, the parametrization

$$\omega(a) = \omega_0 + \omega_a(1-a), \tag{4.8}$$

where ω_0 is the present value ($a = 1$) and ω_a is the measure of the time variation $\omega^{'}$ is widely used in the literature [50].

So, if the present work is compared with experimental results obtained in [51–54], then one can conclude that the limit of ω provided by equation (4.5) may accommodated with the acceptable range of EoS parameter. As it was already noticed, the EoS parameter vanishes in absence of spinor field nonlinearity.

For the value of ω to be in consistent with observation [51], we have the following general condition

$$a_{[1]} < a < a_{[2]}, \tag{4.9}$$

where

$$a_{[1]} = \left[-\frac{(n+0.67)\lambda}{1.67\tilde{m}}\right]^{1/3(n-1)}, \quad a_{[2]} = \left[-\frac{(n-0.38)\lambda}{0.62\tilde{m}}\right]^{1/3(n-1)} \tag{4.10}$$

For this constrain, we obtain $-1.67 < \omega < -0.62$, which is in good agreement with the limit obtained from observational results coming from SNe Ia data [51].

For the value of ω to be consistent with observation [52], we have the following general condition

$$a_{[3]} < a < a_{[4]}, \tag{4.11}$$

where

$$a_{[3]} = \left[-\frac{(n+0.33)\lambda}{1.33\tilde{m}}\right]^{1/3(n-1)}, \quad a_{[4]} = \left[-\frac{(n-0.21)\lambda}{0.79\tilde{m}}\right]^{1/3(n-1)}. \tag{4.12}$$

For this constrain, we obtain $-1.33 < \omega < -0.79$, which is in good agreement with the limit obtained from observational results coming from SNe Ia data [52].

For the value of ω to be consistent with observation [53, 54], we have the following general condition

$$a_{[5]} < a < a_{[6]}, \tag{4.13}$$

where

$$a_{[5]} = \left[-\frac{(n+0.44)\lambda}{1.44\tilde{m}}\right]^{1/3(n-1)}, \quad a_{[6]} = \left[-\frac{(n-0.08)\lambda}{0.92\tilde{m}}\right]^{1/3(n-1)}. \tag{4.14}$$

For this constrain, we obtain $-1.44 < \omega < -0.92$, which is in good agreement with the limit obtained from observational results coming from SNe Ia data [53, 54].

We also observed that if

$$a_{[0]} = \left[-\frac{n\lambda}{\tilde{m}}\right]^{1/3(n-1)}, \tag{4.15}$$

then for $a = a_{[0]}$ we have $\omega = -1$, i.e., we have universe with cosmological constant. If $a < a_{[0]}$ the we have $\omega > -1$ that corresponds to quintessence, while for $a > a_{[0]}$ we have $\omega < -1$, i.e., Universe with phantom matter [55].

Since for the Bianchi type model given by (2.4) both the spinor mass and spinor field nonlinearity vanish, there is no need to carry out the foregoing analysis for this case. As far as LRS Bianchi type-I metric is concerned, one can compare the result with observational data in the same way, as it is done for FRW case. In this case $S = V_0/V$ from (4.4) we find

$$\omega = \frac{\lambda(n-1)}{\lambda + \tilde{m}V^{(n-1)}}, \quad \tilde{m} = m/V_0^{(n-1)}. \tag{4.16}$$

For the value of ω to be in consistent with observation [51], we have the following general condition

$$V_{[1]} < V < V_{[2]}, \tag{4.17}$$

where

$$V_{[1]} = \left[-\frac{(n+0.67)\lambda}{1.67\tilde{m}}\right]^{1/(n-1)}, \quad V_{[2]} = \left[-\frac{(n-0.38)\lambda}{0.62\tilde{m}}\right]^{1/(n-1)} \tag{4.18}$$

For this constrain, we obtain $-1.67 < \omega < -0.62$, which is in good agreement with the limit obtained from observational results coming from SNe Ia data [51].

For the value of ω to be consistent with observation [52], we have the following general condition

$$V_{[3]} < V < V_{[4]}, \tag{4.19}$$

where

$$V_{[3]} = \left[-\frac{(n+0.33)\lambda}{1.33\tilde{m}}\right]^{1/(n-1)}, \quad V_{[4]} = \left[-\frac{(n-0.21)\lambda}{0.79\tilde{m}}\right]^{1/(n-1)}. \tag{4.20}$$

For this constrain, we obtain $-1.33 < \omega < -0.79$, which is in good agreement with the limit obtained from observational results coming from SNe Ia data [52].

For the value of ω to be consistent with observation [53, 54], we have the following general condition

$$V_{[5]} < V < V_{[6]}, \tag{4.21}$$

where

$$V_{[5]} = \left[-\frac{(n+0.44)\lambda}{1.44\tilde{m}}\right]^{1/(n-1)}, \quad V_{[6]} = \left[-\frac{(n-0.08)\lambda}{0.92\tilde{m}}\right]^{1/(n-1)}. \tag{4.22}$$

For this constrain, we obtain $-1.44 < \omega < -0.92$, which is in good agreement with the limit obtained from observational results coming from SNe Ia data [53, 54].

We also observed that if

$$V_{[0]} = \left[-\frac{n\lambda}{\tilde{m}}\right]^{1/(n-1)}, \tag{4.23}$$

then for $V = V_{[0]}$ we have $\omega = -1$, i.e., we have universe with cosmological constant. If $V < V_{[0]}$ the we have $\omega > -1$ that corresponds to quintessence, while for $V > V_{[0]}$ we have $\omega < -1$, i.e., Universe with phantom matter [55].

V. CONCLUSION

Within the scope of Bianchi type-I space time we study the role of spinor field on the evolution of the Universe. It is shown that even in case of space independence of the spinor field it still possesses non-zero non-diagonal components of energy-momentum tensor thanks to its specific relation with gravitational field. This fact plays vital role on the evolution of the Universe. There might be three different scenarios.

In the first case only the components of the spinor field are affected leaving the space-time initially anisotropic that evolves into an isotropic one asymptotically. Unfortunately, due to the specific behavior of the spinor field the bilinear forms constructed from it becomes trivial, thus giving rise to a massless and linear spinor field Lagrangian. So this case presents a very tiny sector of spinor field.

According to the second scenario, where restrictions were imposed wholly on metric functions, they comes out to be proportional to each other right from the beginning,i.e.,

$$a_1 \sim a_2 \sim a_3, \tag{5.1}$$

and can be completely described by the Einstein field equations for FRW metric. As numerical analysis shows, in the second case the Universe expands rather rapidly that leads to the early isotropization of spacetime.

A third possibility was considered when the non-diagonal components of energy-momentum tensor influence both the spinor field and metric functions simultaneously. This case is described by a locally rotationally symmetric Bianchi type-I (LRS-BI) spacetime. In this case isotropization takes place asymptotically and the nonlinearity remains non-trivial.

The results obtained were compared to the recent observational data and the acceptable ranges for the EoS parameter were established. It was found that if the relation between the pressure and energy density obeys a barotropic equation of state, only a non-trivial spinor mass can give rise to a dynamic EoS parameter.

It should be noted that in case when the restrictions are imposed only on the components of the spinor field, though the system is solved completely, the bilinear spinor forms become trivial. So we need some alternative approach to this problem. Since this problem occurs due to the non-diagonal components of the energy momentum tensor of the spinor field which is directly related to spinor affine connection, it needs a very careful treatment. We plan to address this problem in some of our coming papers.

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