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GROMOV-WITTEN INVARIANTS FOR VARIETIES WITH $\mathbb{C}^*$ ACTION

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Abstract. For any smooth projective variety with a $\mathbb{C}^*$ action, we reduce the problem of computing its Gromov-Witten invariants to the similar problem for its fixed locus. Starting from the stacky version of variation of GIT for our variety, we construct the building blocks for the fixed loci of the moduli space of stable maps. We use this construction to compute their contribution to the virtual fundamental class.

1. Introduction

For any compact manifold with a one parameter group action, the Atiyah-Bott Localization Theorem allows one to recover an equivariant cohomology class from its pullbacks to the fixed loci. In the algebraic context a similar result was established by D.Edidin and W.Graham in [EG]. Starting from the first computations of Gromov-Witten invariants, localization proved to be a powerful tool in Gromov-Witten theory. Not long after the introduction of a rigorous algebraic-geometric definition for virtual fundamental classes by K.Behrend and B.Fantechi (see [BF]), T.Graber and R.Panharipande ([GP]) proved the virtual localization formula for moduli spaces with a $(\mathbb{C}^*)^k$ equivariant perfect obstruction theory. The main example of such moduli spaces at the time was the moduli space of stable maps for smooth projective varieties with a torus action.

However until now, this method has been applied only for varieties with strong torus action, i.e. with only finitely many orbits in dimensions 0 and 1. In this paper we develop a general procedure by which localization can be applied to compute equivariant Gromov-Witten invariants for a more general class of smooth projective varieties X with $\mathbb{C}^*$ actions.

Virtual localization reduces the virtual fundamental class $[\overline{M}_{g,n}(X,\beta)]^{vir}$ to a sum in the expected equivariant Chow group

$$[\overline{M}_{g,n}(X,\beta)]^{vir} = i_* \sum \frac{[F_\Gamma]^{vir}}{e^{\mathbb{C}^*}(N^{vir}_{F_\Gamma})}$$

where the summands correspond to the components F_Γ of the fixed point locus of $\overline{M}_{g,n}(X,\beta)$. The classes $[F_\Gamma]^{vir}$ and $e^{\mathbb{C}^*}(N^{vir}_{F_\Gamma})$ are obtained from the *fixed* and *moving* parts of the obstruction theory restricted to F_Γ.

Following the procedure developed in the case of target varieties with strong torus action, the components of the fixed point locus in the moduli space of stable maps can be indexed by decorated graphs. Each vertex v in such a graph contributes a moduli space $\overline{M}_{g_v,n_v}(X_v,\beta_v)$, where X_v is a component of the fixed locus of X. Each edge (u,v) corresponds to invariant curves whose image stretches between the fixed loci X_v and X_u. Thus an edge contributes a certain fixed component of a moduli space of stable maps in genus zero, with two marked points. We will denote such components

by $M_{\omega,2}$. The component of $\overline{M}_{g,n}(X,\beta)^{\mathbb{C}^*}$ associated to a decorated graph Γ is then a finite quotient of a fibre product of spaces mentioned above.

The virtual fundamental class of such a product splits into a product of contributions from the constituent moduli spaces, with additional special contributions of the cotangent classes ψ at the nodes. Thus computing the Gromov-Witten invariants of the X via localization requires two types of inputs:

(1) The Gromov-Witten invariants of the fixed point loci in X, with descendants;
(2) The contributions from the spaces $M_{\omega,2}$.

Assuming the former known, we focus on the later. We will show how to calculate this contribution in terms of the following data:

- The components of the fixed locus in X, their normal bundles and their decomposition induced by the $\mathbb{C}^*$–action;
- The Bialynicki-Birula decompositions of X given by the torus action;
- A stacky version of the variation of GIT for X, as well as for its Bialynicki-Birula strata.

Of all the spaces $M_{\omega,2}$, the most prominent is a component (or union of components) of the fixed point locus in the moduli space of stable maps with the homology class of the action map $t \to t \cdot x$ for generic $x \in X$. To explain how we constructed this particular $M_{\omega,2}$, we start by considering the natural Deligne-Mumford stack structures of the GIT quotients of X, and their weighted blow-ups corresponding to variation of GIT. Taking fiber products we obtain an entire hierarchy of spaces, and $M_{\omega,2}$ is the inverse limit of the induced inverse system. Each of the spaces in the system represents a moduli problem, and the universal families of all these moduli spaces duly form a hierarchy of their own. Furthermore, each universal family comes with its own evaluation map into a suitable target space birational to X. The universal family over $M_{\omega,2}$ is the inverse limit of the inverse system generated by these target spaces and the birational morphisms between them. The three networks thus obtained are connected into a network of triples, each triple consisting of a moduli space, universal family and evaluation map. This intricate structure allows us to compare the moving parts of the obstruction theories for these moduli spaces.

Furthermore, each of the three networks is organized hierarchically, with the structure of the weighted blow-ups at the basis readily decipherable in terms of the Bialynicki-Birula strata in X, their normal bundles and the weights induced by the $\mathbb{C}^*$–action. This allows us to compute the moving part of the virtual fundamental class for the main space $M_{\omega,2}$. The fixed part also results from the inverse system structure based on the formula proven in [M] (Theorem 6.3).

The remaining spaces $M_{\omega,2}$ can be divided into two categories. Some are obtained from the stacky points of the stacky GIT quotients mentioned earlier. More precisely, certain components of the inertia stacks of the above stacky quotients also form their own inverse systems, whose inverse limits are some of the desired spaces $M_{\omega,2}$. A second category of spaces parametrizes maps whose class is not a multiple of the generic orbit. Such stable maps stretch between components of the fixed locus of X other than the source and sink. In this case we employ stacky quotients of Bialinycki-Birula strata, which are pulled back through the networks of maps constructed above and intersected to yield the second category of spaces $M_{\omega,2}$.

The paper is organized as follows: In section 2 we set up some basic tools and notations required in indexing the components of the fixed locus of the moduli space of stable maps, and outline the main factors in the equivariant computation of the

virtual fundamental class. In section 3 we describe the main component $M_{\omega,2}$. We set up the three networks of spaces, expressing each as canonical components of the fixed locus of certain moduli spaces of weighted stable maps. We also record the structure of the most important weighted blow-up morphisms in some detail. The fourth section is dedicated to computations of relevant classes in equivariant K-theory.

2. The fixed locus of the moduli space of stable maps

Let X be a smooth projective variety with an algebraic $\mathbb{C}^*$–action such that for a generic point $x \in X$, the stabilizer $\mathrm{Stab}(x)$ is trivial.

The $\mathbb{C}^*$- action on X induces a natural $\mathbb{C}^*$- action on the moduli space of stable maps $\overline{M}_{0,n}(X,\beta)$. Our first goal is to describe the fixed point locus $\overline{M}_{0,n}(X,\beta)^{\mathbb{C}^*}$. We will present a way of indexing (unions of) components of $\overline{M}_{0,n}(X,\beta)^{\mathbb{C}^*}$ by suitably chosen graphs. We will start by defining the main building blocks.

We consider a $\mathbb{C}^*$–equivariant embedding $\iota : X \hookrightarrow \prod_{i=1}^m \mathbb{P}^{n_i}$ and projections $\pi_i : \prod_{k=1}^m \mathbb{P}^{n_k} \to \mathbb{P}^{n_i}$ giving ample line bundles $\mathcal{L}_i := (\pi_i \circ \iota)^* \mathcal{O}_{\mathbb{P}^{n_i}}(1)$, such that the classes $c_1(\mathcal{L}_i)$ with $i \in \{1, ..., n\}$ form a basis for $H^{1,1}(X)$.

For the $\mathbb{C}^*$–action

$$\begin{array}{rcl} \mathbb{C}^* \times \mathbb{P}^{n_i} & \to & \mathbb{P}^{n_i} \text{ given by} \\ (t, [z_0 : ... : z_{n_i}]) & \to & [t^{a_0} z_0 : ... : t^{a_{n_i}} z_{n_i}] \end{array}$$

we define the moment map $\mu_i : \mathbb{P}^{n_i} \to \mathbb{R}$ by

$$\mu_i([z_0 : ... : z_{n_i}]) := \frac{\sum_{j=0}^{n_i} a_j |z_j|^2}{\sum_{j=0}^{n_i} |z_j|^2}.$$

We note that μ_i maps the $\mathbb{C}^*$–fixed locus of $\mathbb{P}^{n_i}$ onto the set of weights $\{a_0, ..., a_{n_i}\}$.

Remark 2.1. For each $p \in \mathbb{P}^{n_i}$, the function $\nu : \mathbb{C}^* \to \mathbb{R}$ given by $\nu(t) = \mu_i(t \cdot p)$ is constant on circles $S^1(r) = \{t \in \mathbb{C}^*;\ |t| = r\}$ while $\nu_{|\mathbb{R}^*}$ is non-decreasing and

$$\begin{array}{rcl} \lim_{t\to 0} \mu_i(t \cdot p) & = & \min \{a_j;\ j \in \{1, ..., n_i\} \text{ and } z_i \neq 0\} =: m(p), \\ \lim_{t\to \infty} \mu_i(t \cdot p) & = & \max \{a_j;\ j \in \{1, ..., n_i\} \text{ and } z_i \neq 0\} =: M(p), \end{array}$$

so that the closure $\overline{O_p}$ of the orbit $O_p = \mathbb{C}^* p$ satisfies $\overline{\mu_i(O_p)} = [m(p), M(p)]$.

Furthermore, the map $\mathbb{C}^* \to \mathbb{P}^{n_i}$ given by $t \to t \cdot p$ extends uniquely to a map $f_p : \mathbb{P}^1 \to \mathbb{P}^{n_i}$ satisfying

$$\deg f_{p*}[\mathbb{P}^1] = \deg f_{p*}(\mathcal{O}_{\mathbb{P}^{n_i}}(1)) = M(p) - m(p).$$

Putting together the properties of μ_i above we obtain the following

Lemma 2.2. *Consider the map $\mu^m : \prod_{i=1}^m \mathbb{P}^{n_i} \to \mathbb{R}^m$ given by $\mu^m = (\mu_1, ..., \mu_m)$, and the isomorphism $\mathbb{Z}^m \to (H^{1,1}(X))^V \bigcap H_2(X, \mathbb{Z})$ given by the dual of the basis $\{c_1(\mathcal{L}_i);\ i \in \{1, ..., n\}\}$ of $H^{1,1}(X)$ associated to the embedding $\iota : X \hookrightarrow \prod_{i=1}^m \mathbb{P}^{n_i}$.*

For each $x \in X$, let $x_0 := \lim_{t\to 0} t \cdot x$ and $x_\infty := \lim_{t\to\infty} t \cdot x$ and let Stab (x) denote the stabilizer of x.

a) *The components of the fixed locus of X under the $\mathbb{C}^*$– action are mapped by $\mu^m : \prod_{i=1}^m \mathbb{P}^{n_i} \to \mathbb{R}^m$ into points on the lattice $\mathbb{Z}^m$, with coordinates given by the weights of the $\mathbb{C}^*$–action on the $\mathbb{P}^{n_i}$-s.*

b) *The map* $\mathbb{C}^* \to X$ *given by* $t \to t \cdot x$ *extends uniquely to a* $\mid Stab(x)\mid : 1$ *map* $f_x : \mathbb{P}^1 \to X$ *of class* $f_{x*}[\mathbb{P}^1]$ *represented as a vector*

$$f_{x*}[\mathbb{P}^1] = \mu^m(x_\infty) - \mu^m(x_0) \in \mathbb{Z}^m,$$

while the closure of the orbit O_x *satisfies*

$$[\overline{O_x}] = \frac{\mu^m(x_\infty) - \mu^m(x_0)}{\mid Stab(x)\mid}.$$

Proof. Via the projections $\pi_i : \prod_{k=1}^m \mathbb{P}^{n_k} \to \mathbb{P}^{n_i}$, the proof is reduced to the observations on μ_i in Remark 2.1. □

We define a partial order on $\mathbb{R}^m$ as follows: $u < v \Longleftrightarrow u_i < v_i$ for all $i \in \{1, ..., n\}$. Based on the following observations, we can construct an oriented graph with vertices in $\mathbb{Z}^m$ as follows:

Definition 2.3. The oriented graph associated to a $\mathbb{C}^*$-action on X:
We define a set of vertices $\mathcal{V} \subset \mathbb{Z}^m$ as the image through μ^m of the fixed point locus of X. For each $u \in \mathcal{V}$, we denote by

$$X_u = \{x \in X;\ \mu^m(x) = u \text{ and } t \cdot x = x \text{ for all } t \in \mathbb{C}^*\}.$$

We define a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ as follows: a pair $(u, v) \in \mathcal{E}$ if and only if

- $u < v$, and
- there is a $\mathbb{C}^*$–orbit $O_x \subseteq X$ such that $u = \mu^m(x_0)$ and $v = \mu^m(x_\infty)$.

Definition 2.4. The invariant curve classes associated to the graph $(\mathcal{V}, \mathcal{E})$: Let Ω denote the set of pairs (c, β) with $c \in \mathcal{V} \cup \mathcal{E}$ and $\beta \in H_2(X, \mathbb{Z})$ defined by the following conditions:

(i) If $c \in \mathcal{V}$, then $\beta \in j_{c*}H_2(X_c, \mathbb{Z})$ where $j_c : X_c \hookrightarrow X$.
(ii) If $c = (u, v) \in \mathcal{E}$, then $\beta = k[\overline{O_x}]$ where the orbit $O_x = \{t \cdot x | t \in \mathbb{C}^*\}$ satisfies $u, v \in \overline{\mu^m(O_x)}$, and $k \in \mathbb{N}$.

By Lemma 2.2, we have $\beta = k[\overline{O_x}] = \frac{k}{\mid \mathrm{Stab}(x)\mid}(v - u) \in \mathbb{Q}^m$.

Notation 2.5. For each $\omega \in \Omega$, we will denote its second component by β_ω. We will also denote by X_ω the fixed locus X_u whenever $\omega = (u, \beta) \in \Omega$.

Notation 2.6. For each $\omega = (u, \beta) \in \Omega \bigcap (\mathcal{V} \times H_2(X, \mathbb{Z}))$ and non-negative integer n, we denote

$$M_{\omega,n}(X) := \begin{cases} \overline{M}_{0,n}(X_u, \beta), \text{ if } n \geq 3, \text{ or } \beta \neq 0, \\ X_u \text{ otherwise.} \end{cases}$$

Definition 2.7. For each $\omega = ((u, v), \beta) \in \Omega \bigcap (\mathcal{E} \times H_2(X, \mathbb{Z}))$, we define $M_{\omega,2}(X)$ to be the substack of $\overline{M}_{0,2}(X, \beta)^{\mathbb{C}^*}$ parametrizing isomorphism classes of stable maps with marked points $[(C, \varphi, (s_0, s_\infty))]$ fixed by $\mathbb{C}^*$ which in addition satisfy the following properties:

(1) $C = \bigcup_{i=1}^s C_j$ is a connected chain of $\mathbb{P}^1$-s (possibly of length one). The marked points s_0, s_∞ lie in the first and last components of the chain, respectively, and $\mu^m \circ \varphi(s_0) = u$, while $\mu^m \circ \varphi(s_\infty) = v$.
(2) The map φ is flow-preserving: Given the marked points $s_0, s_\infty = s_s$ and the nodes $\{s_j\} = C_j \bigcap C_{j+1}$ for $j \in \{1, ..., s-1\}$, their images $x_j := \varphi(s_j)$ satisfy

$$x_j = \lim_{t \to 0} t \cdot x \text{ and } x_{j+1} = \lim_{t \to \infty} t \cdot x \text{ for generic } x \in \varphi(C_j).$$

(3) $\varphi_*[C] = \beta$ and the restriction of φ on each component C_j has class

$$\varphi_*[C_j] = \frac{\beta}{\mu^m(x_\infty) - \mu^m(x_0)}(\mu^m(x_j) - \mu^m(x_{j-1})) \in \mathbb{Z}^m.$$

With the notations above, each of the irreducible components $\varphi(C_j)$ is invariant under the $\mathbb{C}^*$–action on X, the points $x_j = \varphi(s_j)$ are exactly all the fixed points of the $\mathbb{C}^*$-action on $\operatorname{Im}\varphi$.

Note that $\frac{\beta}{\mu^m(x_\infty)-\mu^m(x_0)} = \frac{\beta}{(v-u)} \in \mathbb{Q}$ due to Lemma 2.2 and Definition 2.4.

Condition (3) is chosen so as to insure that the spaces thus constructed are the smallest building blocks suitable for assembling subspaces of the fixed point loci, e.g. $M_{\omega,2}(\prod_i \mathbb{P}^{n_i})$ is a connected component of $\overline{M}_{0,2}(\prod_i \mathbb{P}^{n_i}, \beta)^{\mathbb{C}^*}$.

We will group components of the fixed locus $\overline{M}_{0,n}(X,\beta)^{\mathbb{C}^*}$ in a way that is naturally compatible with the decomposition into components of $\overline{M}_{0,n}(\prod_i \mathbb{P}^{n_i}, \beta)^{\mathbb{C}^*}$. We index such (unions of) components by triples (T, τ, m) defined as follows:

Definition 2.8. A *minimal triple* (T, τ, m) is defined by the following properties:

(1) T is a tree, with $V(T)$ and $E(T)$ the sets of vertices and edges, respectively. For each vertex $v \in V(T)$, we denote by $d(v)$ the degree of v, i.e. the number of edges incident to v.

(2) $m : V(T) \cup E(T) \to \mathbb{Z}_{\geq 0}$ is such that $n = \sum_{v\in V(T)} m(v) + \sum_{e\in E(T)} m(e)$.

(3) $\tau : V(T) \cup E(T) \to \Omega$ is a map whose composition with the projection $\Omega \to \mathcal{V} \cup \mathcal{E}$ is a morphism of graphs, and satisfying

$$\beta = \sum_{v\in V(T)} \beta_{\tau(v)} + \sum_{e\in E(T)} \beta_{\tau(e)}.$$

(4) For each chain $\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{s-1}, v_s\}$ in T such that $d(v_j) = 2$ for $1 < j < s$, if

$\tau(v_1) = (u_1, \beta_1)$, $\tau(v_k) = (u_k, \beta_k)$ and $\tau(v_j) = (u_j, 0)$ for $1 < j < s$,

then for all possible elements $((u_1, u_s), \beta) \in \Omega$, there exists an index $j \in \{1, ..., s-1\}$ such that

$$\beta_{\tau(\{v_j,v_{j+1}\})} \neq \frac{\beta}{u_s - u_1}(u_{j+1} - u_j).$$

As before, we note that $((u_1, u_s), \beta) \in \Omega \Longrightarrow \frac{\beta}{u_s-u_1} \in \mathbb{Q}$.

Definition 2.9. For each triple (T, τ, m) as above we associate a moduli space $F_{(T,\tau,m)}$ defined as the fibre product of stable map spaces given by the following fibre square diagram:

$$\begin{array}{ccc} F_{(T,\tau,m)} & \longrightarrow & \prod_{v\in V(T)} M_{\tau(v),d(v)+m(v)} \times \prod_{e\in E(T)} M_{\tau(e),m(e)+2} \\ \downarrow & & \downarrow \prod_{i\in S} \mathrm{ev}_i \\ \Delta : \prod_{v\in V(T)} \prod_{i=1}^{d(v)} X_{\tau(v)} & \longrightarrow & \prod_{v\in V(T)} \prod_{i=1}^{d(v)} X_{\tau(v)} \times \prod_{e=\{v_1,v_2\}\in E(T)} (X_{\tau(v_1)} \times X_{\tau(v_2)}), \end{array}$$

where S is constructed by selecting the first $d(v)$ marked points from each of the factors of the product

$$\prod_{v\in V(T)} \overline{M}_{0,d(v)+m(v)}(X_{\tau(v)}, \beta_{\tau(v)})$$

and the last 2 marked points in each factor of $\prod_{e\in E(T)} M_{\tau(e),2}$, and ev_i are the corresponding evaluation maps.

Note that $\sum_{v\in V(T)} d(v) = 2|E(T)|$ and the factors of the product

$$\prod_{e\{v_1,v_2\}\in E(T)} (X_{\tau(v_1)} \times X_{\tau(v_2)})$$

are, up to a permutation, the same as those in $\prod_{v\in V(T)} \prod_{i=1}^{d(v)} X_{\tau(v)}$ so that the lower arrow Δ is the diagonal map followed by the permutation above.

Proposition 2.10. *The fixed point locus $\overline{M}_{0,n}(X,\beta)^{\mathbb{C}^*}$ is a disjoint union of spaces $F_{(T,\tau,m)}/A_{(T,\tau,m)}$, where the moduli spaces $F_{(T,\tau,m)}$ are indexed by all triples (T,τ,m) as in Definition 2.9, and $A_{(T,\tau,m)}$ is a finite group (the automorphism group associated to the data (T,τ,m)).*

We will dedicate the next section to a constructive proof of the representability of the functor $M_{\omega,2}$, and implicitly $F_{(T,\tau,m)}$. We will focus on the main case when *omega* corresponds to the class of the action map $t \to t\cdot x$ for $x \in X$ generic. The remaining cases will derive from here.

2.1. **Maps from curves in higher genus.** For moduli spaces of stable maps in general genus g, Definitions 2.8 and 2.9 and Proposition 2.10 can be modified accordingly to work with subspaces $F_{(\Gamma,\gamma,m)}$ of $\overline{M}_{g,n}(X,\beta)^{\mathbb{C}^*}$. In this context Γ is a graph and the function $\gamma : E(\Gamma)\bigcup V(\Gamma) \to \Omega \times \mathbb{Z}_{\geq 0}$ has an additional coordinate g_γ indexing the genus of a curve component, such that g is obtained by summing the values of g_γ for all the vertices and edges of Γ, together with the number of loops of the graph Γ. While every $v \in V(\Gamma)$ has an associated moduli space of stable maps $\overline{M}_{g_{\gamma(v)},m(v)+d(v)}(X_{\gamma(v)},\beta_{\gamma(v)})$ with $g_{\gamma(v)} \geq 0$, in the case of $e \in E(\Gamma)$, the associated space $M_{\gamma(e),m(e)+2}$ parametrizes maps with domains of genus 0. The focus of this article is on the contribution of these last spaces to the Gromov-Witten invariant calculations.

2.2. **Gromov-Witten invariants.** The virtual localization formula proved by Graber and Pandharipande in [GP] presents the virtual fundamental class $[\overline{M}_{g,n}(X,\beta)]^{vir}$ as a sum in the Chow ring $A_*^{\mathbb{C}^*}(\overline{M}_{g,n}(X,\beta)) \otimes \mathbb{Q}[t,\frac{1}{t}]$:

$$[\overline{M}_{g,n}(X,\beta)]^{vir} = i_* \sum \frac{[F_{(\Gamma,\gamma,m)}]^{vir}}{e^{\mathbb{C}^*}(N^{vir}_{F_{(\Gamma,\gamma,m)}})}$$

where $e^{\mathbb{C}^*}$ denotes the top equivariant Chern class.

For each $F_{(\Gamma,\gamma,m)}$, the class $[F_{(\Gamma,\gamma,m)}]^{vir}$ and $N^{vir}_{F_{(\Gamma,\gamma,m)}}$ are defined as follows: Given a perfect obstruction theory $(E^\bullet,\phi)$ for $\overline{M}_{g,n}(X,\beta)$ with an equivariant lift of the $\mathbb{C}^*$-action,

- The *virtual class* $[F_{(\Gamma,\gamma,m)}]^{vir}$ is calculated from the fixed part of the restriction $(E^\bullet_{(\Gamma,\gamma,m)},\phi_{(\Gamma,\gamma,m)})$ of the obstruction theory $(E^\bullet,\phi)$ to $F_{(\Gamma,\gamma,m)}$. This is itself a perfect obstruction theory ([GP]).
- The *virtual normal bundle* $N^{vir}_{F_{(\Gamma,\gamma,m)}}$ is a two–term complex defined as the moving part of $E_{\bullet,(\Gamma,\gamma,m)}$.

Furthermore one can write an explicit formula for each the terms $\frac{[F_{(\Gamma,\gamma,m)}]^{vir}}{e^{\mathbb{C}^*}(N^{vir}_{F_{(\Gamma,\gamma,m)}})}$ by applying the formula for the virtual class of the boundary strata from [B2]. Let

$V_2(T) \subset V(T)$ denote the degree 2 vertices v of tree T, with adjacent edges e_v^1 and e_v^2, and such that $\beta_{\gamma(v)} = 0$. Then $\frac{[F_{(\Gamma,\gamma,m)}]^{vir}}{e^{\mathbb{C}^*}(N^{vir}_{F_{(\Gamma,\gamma,m)}})}$ can be written as

$$\frac{[\Delta]\prod_{v\in V(T)}[M_{\gamma(v),d(v)+m(v)}]^{vir}\prod_{e\in E(T)}[M_{\gamma(e),2}]^{vir}}{\prod_{e\in E(T)} e^{\mathbb{C}^*}(N^{vir}_{M_{\gamma(e),2}})\prod_{v\in V_2(T)}(\psi_{e_v^1,v}+\psi_{e_v^2,v})\prod_{v\in V(T)\setminus V_2(T), v\in e\in E(T)}(\psi_{e,v}+\psi_{v,i})}.$$

Here i is a marked point for $M_{\gamma(v),d(v)+m(v)}$, chosen so that v and i have the same images through their respective evaluation maps. Then $\psi_{e,v}$ and $\psi_{v,i}$ denote the top equivariant Chern classes of the line bundles on $M_{\gamma(e),2}$ and $M_{\gamma(v),d(v)+m(v)}$, respectively, given by cotangent directions along the universal curve, restricted to the marked point corresponding to v, and the i-th marked point, respectively. As before, $[M_{\gamma(e),2}]^{vir}$ and $[M_{\gamma(v),d(v)+m(v)}]^{vir}$ denote the fixed part of the virtual class, respectively the virtual class of the corresponding moduli spaces.

For each $\omega = ((u,v),\beta) \in \Omega$ we define $c_{\omega,n_1,n_2} : H^*(X_u)[t,t^{-1}] \to H^*(X_v)[t,t^{-1}]$ by

$$c_{\omega,n_1,n_2}(\alpha) = ev_{2*}(ev_1^*(\alpha)\psi_1^{n_1}\psi_2^{n_2} \cap \frac{[M_{\omega,2}]^{vir}}{e^{\mathbb{C}^*}(N^{vir}_{M_{\omega,2}})}).$$

Here $n_1, n_2 \in \mathbb{Z}_{\geq 0}$. Computation of equivariant Gromov-Witten invariants with descendants for X can now be reduced to the computations of Gromov-Witten invariants with descendents for the fixed locus of X, together with calculating c_{ω,n_1,n_2}. In this paper we will show how to compute c_{ω,n_1,n_2} by describing both $[M_{\omega,2}]^{vir}$ and $N^{vir}_{M_{\omega,2}}$.

3. A construction of the main subspace $M_{\omega,2}$ of $\overline{M}_{0,2}(X,\beta)^{\mathbb{C}^*}$.

Consider a $\mathbb{C}^*$– equivariant embedding of X into $\mathbb{P}^N$ constructed by choosing a linearization on a very ample line bundle on X. By a slight abuse of notation, we will denote by μ both the moment map $\mu : \mathbb{P}^N \to \mathbb{R}$ and its restriction to X. We can choose the linearization so that $\mu(\mathbb{P}^N) = [0,d]$. (Changing the linearization modifies the moment map by a translation on $\mathbb{R}$.) We may assume without loss of generality that $\mu(X) = [0,d]$. Indeed, if $\mu(X) = [i,i']$ with $0 < i$ or $i' < d$, then the decomposition into strata induced by the $\mathbb{C}^*$–action on $\mathbb{P}^n$ would insure that X is embedded in a projective subspace invariant under the $\mathbb{C}^*$–action, and hence we may replace $\mathbb{P}^n$ with this subspace in our discussion.

Notation 3.1. Let I denote the image through μ of the fixed point locus of $\mathbb{P}^N$. For each $i \in I$, let $P_i = \{x \in \mathbb{P}^N;\ \mu(x) = i \text{ and } t\cdot x = x \text{ for all } t \in \mathbb{C}^*\}$ and $X_i = P_i \cap X$.

3.1. **Fixed loci in moduli spaces of weighted stable maps.** Let us consider the case when β is the class of the action map $t \to t\cdot x$ for generic $x \in X$. The oriented graph $(\mathcal{V},\mathcal{E})$ associated to the $\mathbb{C}^*$–action (Definition 2.3) has a minimal and maximal vertices u_0 and u_∞, connected to all other vertices by oriented paths. With the notations from the previous section, the images of the moment maps for $\prod_i \mathbb{P}^{n_i}$ and $\mathbb{P}^N$ are connected by a projection $\mathbb{R}^m \to \mathbb{R}$, which sends u_0 to 0 and u_∞ to d.

Our goal is to provide a concrete construction for the moduli space $M_{\omega,2}(X)$ introduced in Definition 2.7, in the main case when $\omega = ((u_0,u_\infty),\beta)$. The ambient space for this construction will be a moduli space of weighted stable maps. The term of *weighted stable maps* was coined in [AG] and [BM], where it referred to stable maps with weights on the marked points. We will be using this term in the more general sense where it refers to stable rational maps with weights on the marked points and the map (see for example [MM1]).

We will start by defining special fixed loci in moduli spaces of weighted stable maps. We will show that the simplest of these correspond to the GIT quotients of X by $\mathbb{C}^*$, and we will use the birational maps between them to construct $M_{\omega,2}(X)$ as an inverse limit of this family of spaces.

We recall the definition of the moduli spaces of weighted stable maps as in [MM1]:

Definition 3.2. For a rational number $a > 0$ and an n-tuple $\mathcal{A} = (a_1, ..., a_n) \in \mathbb{Q}^n$ such that $0 \leq a_j \leq 1$ and $\sum_{j=1}^n a_j + da > 2$, we define the moduli space of weighted stable maps $\overline{M}_{0,\mathcal{A}}(\mathbb{P}^N, d, a)$ to parametrize isomorphism classes of tuples

$$[(\pi\colon C \to B, \{s_j\}_{1\leq j\leq n}, \mathcal{L}, e)]$$

where $\pi : C \to B$ is a family of smooth (or at most nodal) rational curves, s_j are sections of π mapping to smooth points of the fibres, $\mathcal{L}$ is a line bundle on C of degree d on each fiber C_b, and $e : \mathcal{O}_C^{N+1} \to \mathcal{L}$ is a morphism of fiber bundles (specified up to isomorphisms of the target) such that:

(1) $\omega_{C|B}(\sum_{j=1}^n a_j s_j) \otimes \mathcal{L}^a$ is relatively ample over B,
(2) $\mathcal{G} := \operatorname{Coker} e$, restricted over each fiber C_b, is a skyscraper sheaf supported only on smooth points, and
(3) for any $s \in C_b$ and for any $J \subseteq \{1, ..., n\}$ (possibly empty) such that $s = s_j(b)$ for all $j \in J$, the following condition holds

$$\sum_{\jmath\in J} a_j + a \dim \mathcal{G}_s \leq 1.$$

For a more intuitive description of a weighted stable map, we note that the rational map $\varphi : C \dashrightarrow \mathbb{P}^N$ determined by e, when restricted to each fibre C_b, extends naturally to a well-defined map $\varphi_b : C_b \to \mathbb{P}^N$ of degree $\deg \varphi_b \leq d$. Whenever $\deg \varphi_b < d$, the curve C_b contains some special base-locus points (the support of $\mathcal{G}_b = \operatorname{Coker} e_b$), such that

$$\deg \varphi_b + \sum_{p\in \mathrm{Supp}\ (\mathcal{G}_b)} \dim \mathcal{G}_p = d. \tag{3.1}$$

Each point $p \in \mathrm{Supp}\ (\mathcal{G}_b)$ comes with a positive multiplicity $m_p := \dim \mathcal{G}_p$. Condition (1) determines the minimum possible degrees of φ_b on ending components ("tails") of C_b in terms of a and the weights of the marked points on the given components. Condition (2) determines the exact cases when the special sections s_j can intersect and the maximum possible multiplicity m_p of the intersection point. Conditions (1) and (2) are complementary: for a given subset of marked sections, the maximum possible multiplicity at the intersection is 1 less than the minimum possible degree on an ending component.

There is a natural map $\overline{M}_{0,n}(\mathbb{P}^N, d) \to \overline{M}_{0,\mathcal{A}}(\mathbb{P}^N, d, a)$. We denote by $\overline{M}_{0,\mathcal{A}}(X, \beta, a)$ the image of $\overline{M}_{0,n}(X, \beta)$ under this map.

Notation 3.3. Consider a real number a satisfying $0 < a \leq \frac{1}{d}$. For each pair of elements $i, i' \in I$ with $i < i'$, we will be using the set of weights on two marked points $\mathcal{A}(i, i') = (1 - ia, 1 - (d - i')a)$.

We note that due to the chosen weights, a point in $\overline{M}_{0,\mathcal{A}(i,i')}(\mathbb{P}^N, d, a)$ parametrizes a chain C of $\mathbb{P}^1$-s whose starting and ending components C_0 and C_∞ contain the marked points s_0 and s_∞ respectively, together with a map $\varphi : C \to \mathbb{P}^n$ non-constant on each component, and multiplicities $m_0 = \dim \mathcal{G}_0$ and $m_\infty = \dim \mathcal{G}_\infty$ such that

$$\deg \varphi_{|C_0} + m_0 > i \geq m_0,\ \deg \varphi_{|C_\infty} + m_\infty > d - i' \geq m_\infty. \tag{3.2}$$

The $\mathbb{C}^*$- action on X and the equivariant embedding into $\mathbb{P}^N$ also induce a natural $\mathbb{C}^*$- action on $\overline{M}_{0,\mathcal{A}}(X,\beta,a)$. Indeed, a linearization on the very ample line bundle on X gives an action on the bundle $\mathcal{O}_C^{N+1}$, and hence on a weighted stable map. Note that any two linearizations determine the same action on a weighted stable map. Indeed, two different linearizations associated to the same action on $\mathbb{P}^N$ differ by multiplication by a scalar on $\mathcal{O}_C^{N+1}$ and hence on $\mathcal{L}$.

Definition 3.4. We denote the elements of I by $0 = i_0 < i_1 < ... < i_k = d$. For each pair of indices $j, j' \in \{0, 1, ..., k\}$ we denote by $M_{(j,j')}(\mathbb{P}^N)$ the substack of $\overline{M}_{0,\mathcal{A}(i_j,i_{j'})}(\mathbb{P}^N, d, a)^{\mathbb{C}^*}$ representing classes of weighted stable maps

$$[C \to B, (s_0, s_\infty), \mathcal{L}, e]$$

fixed by $\mathbb{C}^*$ such that each fibre $C_b = \bigcup_{l=0}^r C_l$ is a connected chain of $\mathbb{P}^1$-s in addition satisfying the following conditions:

(1) The weighted stable map φ_b determined on each fibre C_b by e is *flow-preserving*: the images $x_l := \varphi_b(s_l)$ of the marked points $s_0, s_\infty = s_r$ and the nodes $\{s_l\} = C_l \bigcap C_{l-1}$ satisfy $x_l = \lim_{t\to 0} t \cdot x$ and $x_{l+1} = \lim_{t\to\infty} t \cdot x$ for generic $x \in \varphi_b(C_l)$.

(2) The map φ_b is *action-class-preserving*: $\deg \varphi_*[C_l] = \mu(x_{l+1}) - \mu(x_l)$ for each component C_l.

We define $M_{(j,j')}(X) := M_{(j,j')}(\mathbb{P}^N) \times_{\overline{M}_{0,\mathcal{A}(i_j,i_{j'})}(\mathbb{P}^N,d,a)} \overline{M}_{0,\mathcal{A}(i_j,i_{j'})}(X,\beta,a)$.

Let $\mathcal{U}_{(j,j')}(X)$ denote the universal family over $M_{(j,j')}(X)$.

In the case of spaces $M_{(j,j+1)}$, condition (2) is superfluous as in this case a weighted stable map $\varphi : C \to X$ maps $C \cong \mathbb{P}^1$ to just one orbit in X.

Lemma 3.5. *Let $i_j < i_{j+1}$ be two consecutive elements of I.*

For each closed point b in $M_{(j,j+1)}(X)$, the fibre over b of $\pi : \mathcal{U}_{(j,j+1)}(X) \to M_{(j,j+1)}(X)$ is isomorphic to $\mathbb{P}^1$ and the corresponding weighted stable map φ_b satisfies

$$Im\ \mu \circ \varphi_b \supset [i_j, i_{j+1}].$$

Proof. Let $(C, (s_0, s_\infty), \mathcal{L}, e)$ be a weighted stable map such that C is the fibre of the universal family over a closed point b in $\overline{M}_{0,\mathcal{A}(i_j,i_{j+1})}(X,\beta,a)$, and let $\varphi : C \to X$ be the well-defined map on C induced by e. Assume that $[(C, (s_0, s_\infty), \mathcal{L}, e)]$ is fixed by the $\mathbb{C}^*$-action induced from the action on X. Thus for each $t \in \mathbb{C}^*$ there exists an automorphism g_t of C and an isomorphism $\widetilde{g}_t : \mathcal{L} \to g_t^*\mathcal{L}$ making the following diagram commutative:

$$\begin{tikzcd} \mathcal{O}_C^{N+1} \arrow[r, "e"] \arrow[d, "t\cdot"] & \mathcal{L} \cong \mathcal{O}_C(d) \arrow[d, "\widetilde{g}_t"] \\ g_t^*\mathcal{O}_C^{N+1} \arrow[r, "g_t^*e"] & g_t^*\mathcal{L} \cong \mathcal{O}_C(d). \end{tikzcd} \tag{3.3}$$

It follows that Im φ is $\mathbb{C}^*$-equivariant, and hence $\varphi(\mathrm{Supp}(\mathrm{Coker}\, e))$ is a finite subset of the fixed point locus of X. Thus $\mathrm{Supp}(\mathrm{Coker}\, e) \subseteq \{s_0, s_\infty\}$. Indeed:

- No component of C is mapped by φ to the fixed point locus of X, due to condition (2) in Definition 3.4. (Note that condition (1) in Definition 3.2, together with the choice of weights insure that no component of C is contracted to a point by φ.) Thus each component can have at most two points mapped to the fixed point locus of X. Nodes are such points.

- No nodes can be in $\mathrm{Supp}(\mathrm{Coker}\, e)$ by the definition of weighted stable maps.
- $\varphi(s_0)$ and $\varphi(s_\infty)$ are fixed points.

Consider now a component $C_j \cong \mathbb{P}^1$ of C. Locally around the point $0 \in \mathbb{P}^1$, the diagram above is of the form

$$\begin{tikzcd} \mathbb{C}[z]^{N+1}_{(z)} \arrow[r] \arrow[d] & \mathbb{C}[z]_{(z)} \arrow[d] \\ \mathbb{C}[z]^{N+1}_{(z)} \arrow[r] & \mathbb{C}[z]_{(z)} \end{tikzcd} \quad \text{defined on basis elements by} \quad \begin{tikzcd} e_i \arrow[r] \arrow[d] & z^{\alpha_i} c_i \arrow[d] \\ t^{a_i} e_i \arrow[r] & t^{a_i} z^{\alpha_i} c_i = (t^r z)^{\alpha_i} c_i \end{tikzcd}$$

where $c_i \in \mathbb{C}$ must be a constant to insure diagram commutativity and $r \in \mathbb{Z}$ is determined by the isomorphism g_t. It follows that $a_i = r\alpha_i$ for all indices i for which $c_i \neq 0$.

With these data we have $\deg \varphi_*[C_l] = M - m$ where $M = \max\{\alpha_i;\, c_i \neq 0\}$ and $m = \min\{\alpha_i;\, c_i \neq 0\}$. On the other hand the points $x_l := \varphi(0)$ and $x_{l+1} = \varphi(\infty)$ satisfy $\mu(x_l) = \min\{a_i;\, c_i \neq 0\}$, $\mu(x_{l+1}) = \max\{a_i;\, c_i \neq 0\}$. Thus condition (2) in Definition 3.4 implies $r = 1$, hence also $\mu(x_l) = \dim \mathcal{G}_{x_l}$ and $d - \mu(x_{l+1}) = \dim \mathcal{G}_{x_{l+1}}$.

We are now ready to prove that $C \cong \mathbb{P}^1$ and

$$\mu \circ \varphi(C) = [\mu \circ \varphi(s_0), \mu \circ \varphi(s_\infty)] \supseteq [i_j, i_{j+1}].$$

Indeed, with the notations from Definition 3.4, in the cases $l = 0$ and $l = r$ inequalities (3.2) imply $\mu(x_0) \leq i_j$ and $\mu(x_\infty) \geq i_{j+1}$ as well as

$$\begin{aligned} \mu(x_1) &= \mu(x_0) + \deg \varphi_{|C_0} > i_j \text{ and} \\ \mu(x_r) &= \mu(x_\infty) + \deg \varphi_{|C_r} < i_{j+1}. \end{aligned}$$

As i_j and i_{j+1} are consecutive elements in I, it follows that $\mu(x_1) \geq i_{j+1}$ and $\mu(x_r) \leq i_j$. But the map φ is flow-preserving and $i_j < i_{j+1}$, so it must be that $r = 1$ so that $x_1 = \varphi(s_\infty)$, $x_r = \varphi(s_0)$ and $C \cong \mathbb{P}^1$.

□

Recall the following

Theorem 3.6. *(Theorem 8.3 in* [MFK] *for the GIT quotient associated to a point in the chamber (i_j, i_{j+1}).) Let $i_j < i_{j+1}$ be two consecutive points in I. Then there exists a linearization on the chosen very ample line bundle on X such that the sets of semistable and stable points are given by*

$$X^s_{(j,j+1)} = X^{ss}_{(j,j+1)} = \{x \in X;\, (i_j, i_{j+1}) \subset \mu(O_x)\}.$$

Thus the previous Lemma we have proven that any weighted stable map φ parametrized by a point in $M_{(j,j+1)}(X)$ satisfies $\varphi(C) \subset X^s_{(j,j+1)}$.

Theorem 3.7. *If $i_j < i_{j+1}$ are consecutive elements of I, then $M_{(j,j+1)}(X)$ is isomorphic to the stacky quotient $[X^s_{(j,j+1)}/\mathbb{C}^*]$ associated to the GIT quotient $X^s_{(j,j+1)}//\mathbb{C}^*$.*

Proof. Step 1. We build a map $[X^s_{(j,j+1)}/\mathbb{C}^*] \to M_{(j,j+1)}(X)$. To construct a family of weighted stable maps over $[X^s_{(j,j+1)}/\mathbb{C}^*]$, we first compactify $X^s_{(j,j+1)}$ by adding zero and infinity sections as follows: We consider the actions of $\mathbb{C}^*$ on $\mathbb{A}^1$ of weights 1 and -1, respectively:

$$\begin{aligned} \mathbb{C}^* \times \mathbb{A}^1 \to \mathbb{A}^1 \quad &\text{and} \quad \mathbb{C}^* \times \mathbb{A}^1 \to \mathbb{A}^1, \\ (t, u) \to tu \quad &\text{and} \quad (t, u) \to t^{-1}u. \end{aligned}$$

We define $\overline{X^s}_{(j,j+1)}$ to be the stack obtained by gluing the two stacky quotients $[(X^s_{(j,j+1)} \times \mathbb{A}^1)/\mathbb{C}^*]$ along $X^s_{(j,j+1)} = [(X^s_{(j,j+1)} \times \mathbb{C}^*)/\mathbb{C}^*]$. The natural map $\pi : \overline{X^s}_{(j,j+1)} \to [X^s_{(j,j+1)}/\mathbb{C}^*]$ admits two sections s_0 and s_∞ induced by the zero-section $[X^s_{(j,j+1)}/\mathbb{C}^*] \to [(X^s_{(j,j+1)} \times \mathbb{A}^1)/\mathbb{C}^*]$ in the two patches above.

The inclusion $X^s_{(j,j+1)} \hookrightarrow X$ gives a rational map $\varphi : \overline{X^s}_{(j,j+1)} \dashrightarrow X$ associated to a morphism of vector bundles $e : \mathcal{O}^{N+1} \to \mathcal{L}$ over $\overline{X^s}_{(j,j+1)}$. The line bundle $\mathcal{L}$ is the pull-back of $\mathcal{O}_{\mathbb{P}^N}(1)$ and is of degree d, the degree of the action map $t \to t \cdot x$ for x generic. Condition (1) in Definition 3.2 is satisfied due to the inequality $i_j < i_{j+1}$. The support of $\operatorname{Coker} e$ is included in $\operatorname{Im} s_0 \bigcup \operatorname{Im} s_\infty$ (Condition (2)).

When restricted over each fibre $C_b = \mathbb{P}^1$ of π, the rational map φ is of the form $t \to t \cdot x$ and extends to a map $\varphi_b = f_x : \mathbb{P}^1 \to X$, so that Lemma 2.1 and equation 3.1 together imply

$$\dim \operatorname{Coker} e_{s_0(b)} + \dim \operatorname{Coker} e_{s_\infty(b)} + \mu(x_\infty) - \mu(x_0) = d, \tag{3.4}$$

where $x_0 = \varphi_b(s_0(b))$, $x_\infty = \varphi_b(s_\infty(b))$ and therefore $\operatorname{Im} \mu \circ \varphi_b = [\mu(x_0), \mu(x_\infty)]$.

We claim that

$$\dim \operatorname{Coker} e_{s_0(b)} = \mu(x_0) \text{ and } \dim \operatorname{Coker} e_{s_\infty(b)} = d - \mu(x_\infty). \tag{3.5}$$

Consider fibres $C_b = \mathbb{P}^1$ of π such that $\operatorname{Coker} e_{s_0(b)} \neq 0$. The generic fibres among these satisfy $\operatorname{Coker} e_{s_\infty(b)} = 0$ and $\mu(x_\infty) = d$, and hence $\dim \operatorname{Coker} e_{s_0(b)} = \mu(x_0)$ by equation 3.4. Similarly, the generic fibres among those where $\operatorname{Coker} e_{s_\infty(b)} \neq 0$ satisfy $\operatorname{Coker} e_{s_0(b)} = 0$ and $\dim \operatorname{Coker} e_{s_\infty(b)} = d - \mu(x_\infty)$. Since $\dim \operatorname{Coker} e_{s_\infty}$ and $\dim \operatorname{Coker} e_{s_\infty}$ are upper semi-continuous, it follows that for any fibre C_b we have

$$\dim \operatorname{Coker} e_{s_0(b)} \geq \mu(x_0) \text{ and } \dim \operatorname{Coker} e_{s_\infty(b)} \geq d - \mu(x_\infty).$$

In conjunction with equation 3.4, this yields equations 3.5.

Finally, we recall the condition $(i_j, i_{j+1}) \subset \mu(O_x)$ for $x \in X^s_{(j,j+1)}$. On the other hand, for $x \in \operatorname{Im} \varphi$, $\mu(O_x) = (\mu(x_0), \mu(x_\infty))$ and hence

$$\mu(x_0) \leq i_j < i_{j+1} \leq \mu(x_\infty).$$

Together with equation 3.5, this suffices to check Condition (3) from Definition 3.2.

We have proven the existence of a map $[X^s_{(j,j+1)}/\mathbb{C}^*] \to M_{(j,j+1)}(X)$, which is part of a fibre square

$$\begin{array}{ccc} \overline{X^s}_{(j,j+1)} & \xrightarrow{\phi} & \mathcal{U}_{(j,j+1)}(X) \\ \downarrow & & \downarrow \\ [X^s_{(j,j+1)}/\mathbb{C}^*] & \longrightarrow & M_{(j,j+1)}(X) \end{array}$$

Step 2. The rational map $\mathcal{U}_{(j,j+1)}(X) \dashrightarrow X$ corresponds to a well defined map $\mathcal{U}_{(j,j+1)}(X) \setminus (\operatorname{Im} s_0 \bigcup \operatorname{Im} s_\infty) \to X^s_{(j,j+1)}$ which extends to $\mathcal{U}_{(j,j+1)}(X) \to \overline{X^s}_{(j,j+1)}$, the inverse of the map ϕ above.

Indeed, consider a family of weighted stable maps $[C \to B, (s_0, s_\infty), \mathcal{L}, e]$ given by a map $B \to M_{(j,j+1)}(X)$, and let $\varphi : C \dashrightarrow X$ be the corresponding rational map. Let $q \in C_b$ be a point where φ is not defined. The restriction $\varphi_{|C_b}$ extends to a map $\varphi_b : C_b \to X$. Since $[C \to B, (s_0, s_\infty), \mathcal{L}, e]$ is fixed by $\mathbb{C}^*$, it follows that the $\mathbb{C}^*$–orbit $O_{\varphi_b(q)} \subset \operatorname{Im} \varphi_b$, and $\varphi_b^{-1}(O_{\varphi_b(q)}) \subseteq \operatorname{Supp} \operatorname{Coker} e_b$. But $\operatorname{Coker} e_b$ is a skyscraper

sheaf, so $\varphi_b(q)$ must be a fixed point. Hence $\mathcal{U}_{(j,j+1)}(X) \setminus (\text{ Im } s_0 \bigcup \text{ Im } s_\infty) \to X$ is well defined, and the image is in $X^s_{(j,j+1)}$ by Lemma 3.5.

The map $\mathcal{U}_{(j,j+1)}(X) \setminus (\text{ Im } s_0 \bigcup \text{ Im } s_\infty) \to X^s_{(j,j+1)}$ is an isomorphism of $\mathbb{C}^*$– bundles which extends canonically to an isomorphism of $\mathbb{P}^1$–bundles. By naturality, this is the inverse of the map ϕ constructed above. □

Remark 3.8. By applying the arguments of Lemma 3.5 more generally for $j < j'$, we can characterize a closed point $[\varphi : C \to X, s_0, s_\infty]$ in $M_{(j,j')}(X)$ as follows: C is a chain of $\mathbb{P}^1$-s, on each of which φ is the action map $t \to t \cdot x$ for some $x \in X$, and such that, via the moment map μ,

- $\mu(\varphi(s_0)) \leq i_j$ and $\mu(\varphi(s_\infty)) \geq i_{j'}$;
- all the nodes of C are mapped between i_j and $i_{j'}$.

Corollary 3.9. *$M_{(j,j+1)}(\mathbb{P}^N)$ is a closed and open substack of $\overline{M}_{0,\mathcal{A}(i_j,i_{j+1})}(\mathbb{P}^N, d, a)^{\mathbb{C}^*}$.*

Proof. The proof above implies that $M_{(j,j+1)}(X) \cong [X^s{}_{(j,j+1)}/\mathbb{C}^*]$ is indeed a closed substack of $\overline{M}_{0,\mathcal{A}(i_j,i_{j+1})}(X, \beta, a)$. Furthermore, we know $\overline{M}_{0,\mathcal{A}(i_j,i_{j+1})}(\mathbb{P}^N, d, a)$ is a smooth stack and hence $\overline{M}_{0,\mathcal{A}(i_j,i_{j+1})}(\mathbb{P}^N, d, a)^{\mathbb{C}^*}$ is smooth, its connected components are irreducible. It remains to note that for a generic point p in $M_{(j,j+1)}(\mathbb{P}^N)$, all points in an open neighbourhood of p in $\overline{M}_{0,\mathcal{A}(i_j,i_{j+1})}(\mathbb{P}^N, d, a)^{\mathbb{C}^*}$ represent weighted stable maps which also satisfy conditions (1)-(2) in Definition 3.4.

□

Notation 3.10. For each element $i_j \in I$, we denote by $X_j := \mu^{-1}(i_j)^{\mathbb{C}^*}$ the corresponding component of the fixed point locus of X, and

$$X_j^- := \{x \in X;\ \lim_{t\to\infty} t \cdot x \in X_j\} \text{ and } X_j^+ := \{x \in X;\ \lim_{t\to 0} t \cdot x \in X_j\}.$$

The natural maps $\pi_j^- : X_j^- \to X_j$ and $\pi_j^+ : X_j^+ \to X_j$, obtained by taking limits $x \to \lim_{t\to 0} t \cdot x$ and $x \to \lim_{t\to\infty} t \cdot x$, are affine bundles over X_j.

It is known that the normal bundles satisfy $\mathcal{N}_{X_j^-|X} = \pi_j^{-*}\mathcal{N}_{X_j|X_j^+}$ and $\mathcal{N}_{X_j^+|X} = \pi_j^{+*}\mathcal{N}_{X_j|X_j^-}$. The $\mathbb{C}^*$ action on X induces two actions on $\mathcal{N}_{X_j^-|X}$: one which makes the projection $\mathcal{N}_{X_j^-|X} \to X_j^-$ equivariant, and a second one induced from the action on $\mathcal{N}_{X_j|X_j^+}$. The first action allows $\mathcal{N}_{X_j^-|X}$ to descend to a bundle $[\mathcal{N}_{X_j^-)|X}/\mathbb{C}^*]$ over $[(X_j^- \setminus X_j)/\mathbb{C}^*]$. In the second case, $\mathbb{C}^*$ acts on each fibre, leading to a decomposition of the bundle by eigenvalues (weights). This decomposition descends to $[\mathcal{N}_{X_j^-|X}/\mathbb{C}^*]$. Similarly for $\mathcal{N}_{X_j^+|X}$.

Next we describe morphisms between spaces $M_{(j,j')}(X)$ as weighted blow-ups.

The local structure of a weighted blow-up $\widetilde{Z} \to Z$ of smooth Deligne-Mumford stacks has been described in detail in [MM2], section 2, and we will employ the notations introduced there. In particular we recall the associated affine fibration

$$A := \text{Spec}\,(\oplus_{n\geq 0}\mathcal{I}_n/\mathcal{I}_{n+1}) \to Y$$

for a weighted blow-up $\widetilde{Z} \to Z$ along the locally embedded smooth stack Y, and the filtration $\{\mathcal{I}_n\}_{n\geq 1}$ of the ideal of Y in Z. Then the exceptional divisor $\widetilde{Y} = P^w(A)$, a weighted projective fibration over Y.

Theorem 3.11. *Assume that $i_j < i_{j+1} < i_{j+2}$ are three consecutive elements of I. Then the maps*

$$\begin{tikzcd} & M_{(j,j+2)}(X) \arrow[dl] \arrow[dr] & \\ M_{(j,j+1)}(X) & & M_{(j+1,j+2)}(X) \end{tikzcd}$$

are weighted blow-ups. The blow-up loci are

$$\begin{aligned} Y^-_{(j,j+1)} &= [(X^-_{j+1} \setminus X_{j+1})/\mathbb{C}^*] \hookrightarrow M_{(j,j+1)}(X) \text{ and} \\ Y^+_{(j+1,j+2)} &= [(X^+_{j+1} \setminus X_{j+1})/\mathbb{C}^*] \hookrightarrow M_{(j+1,j+2)}(X), \end{aligned}$$

and the exceptional divisor $E_{(j,j+2)} = Y^-_{(j,j+1)} \times_X Y^+_{(j+1,j+2)}$ is a weighted projective fibration over $Y^-_{(j,j+1)}$ (as defined in [MM2]*):*

$$E_{(j,j+2)} = [(A_{(j,j+1)} \setminus Y^-_{(j,j+1)})/\mathbb{C}^*],$$

where $A_{(j,j+1)}$ is an affine bundle over $Y^-_{(j,j+1)}$ such that the normal bundle of its zero section $\mathcal{N}_{Y^-_{(j,j+1)}|A_{(j,j+1)}}$ is obtained by descent from $\pi^*_{j+1}\mathcal{N}_{X_{j+1}|X^+_{j+1}}$ and the weights of the action on $\mathcal{N}_{Y^-_{(j,j+1)}|A_{(j,j+1)}}$ are induced from the $\mathbb{C}^*$–action on X.

The coarse moduli spaces of $M_{(j,j+2)}(X)$ is the fiber product over the following variation of GIT diagram

$$\begin{tikzcd} X^s_{(j,j+1)}//\mathbb{C}^* \arrow[dr] & & X^s_{(j+1,j+2)}//\mathbb{C}^* \arrow[dl] \\ & (X^s_{(j,j+1)} \bigcup X_{j+1} \bigcup X^s_{(j+1,j+2)})//\mathbb{C}^*. & \end{tikzcd}$$

Proof. The map $M_{(j,j+2)}(X) \to M_{(j,j+1)}(X)$ is obtained by restricting the map between moduli spaces of weighted stable maps $p : \overline{M}_{0,\mathcal{A}(i_j,i_{j+2})}(X,\beta) \to \overline{M}_{0,\mathcal{A}(i_j,i_{j+1})}(X,\beta)$ to those components of the fixed point loci satisfying conditions (1)-(2) in Definition 3.4. Indeed, these conditions are compatible with p so $p(M_{(j,j+2)}(X)) = M_{(j,j+1)}(X)$.

Consider first the target $\mathbb{P}^n$. We let $a_0 = \frac{d-i_j}{d}$, $a_\infty = \frac{i_{j+2}}{d}$ and $a = \frac{1}{d}$. The boundary divisor of $\overline{M}_{0,\mathcal{A}(i_j,i_{j+2})}(\mathbb{P}^n,d,a)$ having nontrivial intersection with $M_{(j,j+2)}(\mathbb{P}^n)$ is the image of the embedding

$$\iota : \overline{M}_{0,(a_0,1)}(\mathbb{P}^n, i_{j+1}, a) \times_{\mathbb{P}^n} \overline{M}_{0,(1,a_\infty)}(\mathbb{P}^n, d-i_{j+1}, a) \to \overline{M}_{0,\mathcal{A}(i_j,i_{j+2})}(\mathbb{P}^n,d,a).$$

We denote the non-trivial intersection of $\operatorname{Im}\iota$ with $M_{(j,j+2)}(X)$ by $E_{(j,j+2)}$. Its image in $M_{(j,j+1)}(X)$ is thus $Y^-_{(j,j+1)} := M_{(j,j+1)}(X) \bigcap \overline{M}_{0,(a_0,1)}(\mathbb{P}^n, i_{j+1}, a)$. Here the embedding

$$\overline{M}_{0,(a_0,1)}(\mathbb{P}^n, i_{j+1}, a) \hookrightarrow \overline{M}_{0,\mathcal{A}(i_j,i_{j+1})}(\mathbb{P}^n,d,a)$$

is given as follows: a tuple $(C, \{s_0, s_\infty\}, \mathcal{L}, e)$ representing a point in $\overline{M}_{0,(a_0,1)}(\mathbb{P}^n, i_{j+1}, a)$ gives a tuple $(C, \{s_0, s_\infty\}, \mathcal{L} \otimes \mathcal{O}(d-i_{j+1}), e')$ where e' is the composition of e with $\mathcal{L} \to \mathcal{L} \otimes \mathcal{O}(d-i_{j+1})$, and the last morphism has a zero with multiplicity $d - i_{j+1}$ at s_∞. If $(C, \{s_0, s_\infty\}, \mathcal{L} \otimes \mathcal{O}(d-i_{j+1}), e')$ represents a point in $M_{(j,j+1)}(X)$, then by equation (3.5),

$$\mu \circ \varphi(s_\infty) = d - \dim \operatorname{Coker} e_{s_\infty} \leq d - (d - i_{j+1}) = i_{j+1}.$$

On the other hand, $[i_j, i_{j+1}] \subseteq \operatorname{Im} \mu \circ \varphi$ by Theorem 3.7, hence $\varphi(s_\infty) \in X_{j+1}$ and $\operatorname{Im} \varphi \subseteq X^-_{j+1}$ (as $C \cong \mathbb{P}^1$). Furthermore, combining this with the proof of Theorem 3.7, we obtain

$$Y^-_{(j,j+1)} \cong [(X^-_{j+1} \setminus X_{j+1})/\mathbb{C}^*].$$

Similarly, we obtain that the image of $E_{(j,j+2)}$ in $M_{(j+1,j+2)}(X)$ is

$$Y^+_{(j+1,j+2)} = M_{(j+1,j+2)}(X) \bigcap \overline{M}_{0,(1,a_\infty)}(\mathbb{P}^n, d - i_{j+1}, a) \cong [X^+_{j+1}/\mathbb{C}^*]$$

and $E_{(j,j+2)} = Y^-_{(j,j+1)} \times_{X_{j+1}} Y^+_{(j+1,j+2)}$. Hence

$$\begin{aligned} E_{(j,j+2)} &\cong [(X^-_{j+1} \setminus X_{j+1})/\mathbb{C}^*] \times_{X_{j+1}} [(X^+_{j+1} \setminus X_{j+1})/\mathbb{C}^*] \\ &\cong [(X^-_{j+1} \setminus X_{j+1}) \times_{X_{j+1}} [(X^+_{j+1} \setminus X_{j+1})/\mathbb{C}^*]/\mathbb{C}^*]. \end{aligned}$$

$X^-_{j+1} \times_{X_{j+1}} [(X^+_{j+1} \setminus X_{j+1})/\mathbb{C}^*] \to [(X^+_{j+1} \setminus X_{j+1})/\mathbb{C}^*]$ is an affine bundle obtained as pull-back of $X^-_{j+1} \to X_{j+1}$ thus the normal bundle of its zero section is $\pi^{+*}_{j+1}(\mathcal{N}_{X_{j+1}|X^-_{j+1}})$, and the weights of the $\mathbb{C}^*$ action on $X^-_{j+1} \times_{X_{j+1}} [X^+_{j+1}/\mathbb{C}^*]$ are those induced on $\mathcal{N}_{X_{j+1}|X^-_{j+1}}$ by the action on X. □

Theorem 3.12. *Assume that $i_j < i_{j+1} < i_{j+2} < i_{j+3}$ are four consecutive elements of I. Then the following is a Cartesian diagram:*

$$\begin{tikzcd} & M_{(j,j+3)}(X) \arrow[dl] \arrow[dr] & \\ M_{(j,j+2)}(X) \arrow[dr] & & M_{(j+1,j+3)}(X) \arrow[dl] \\ & M_{(j+1,j+2)}(X). & \end{tikzcd}$$

Proof. We noted earlier the existence of morphisms $M_{j,k}(X) \to M_{j',k'}(X)$ whenever $j \leq j'$ and $k \geq k'$. They are obtained by restriction from the birational morphisms between the moduli spaces of weighted stable maps. Thus there exists a canonical map $a : M_{(j,j+3)}(X) \to M_{(j,j+2)}(X) \times_{M_{(j+1,j+2)}(X)} M_{(j+1,j+3)}(X)$. To construct its inverse, consider an object in $M_{(j,j+2)}(X) \times_{M_{(j+1,j+2)}(X)} M_{(j+1,j+3)}(X)(B)$ over a scheme B. To it there correspond tuples $(C_{(j,j+2)}, \{s_0, s_\infty\}, e_{(j,j+2)} : \mathcal{O}^{n+1}_{C_{(j,j+2)}} \to \mathcal{L}_{(j,j+2)})$ and $(C_{(j+1,j+3)}, \{s_0, s_\infty\}, e_{(j+1,j+3)} : \mathcal{O}^{n+1}_{C_{(j+1,j+3)}} \to \mathcal{L}_{(j+1,j+3)})$ together with a pair of isomorphism $(\sigma, \overline{\sigma})$:

$$\begin{tikzcd} C_{(j,j+2)} \arrow[d, "q^-"] & C_{(j+1,j+3)} \arrow[d, "q^+"] \\ C^-_{(j+1,j+2)} \arrow[r, "\sigma"] & C^+_{(j+1,j+2)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{O}^{n+1}_{C^-_{(j+1,j+2)}} \arrow[r, "e^-"] \arrow[d] & \mathcal{L}^-_{(j+1,j+2)} \arrow[d, "\cong\overline{\sigma}"] \\ \sigma^*\mathcal{O}^{n+1}_{C^+_{(j+1,j+2)}} \arrow[r, "\sigma^* e^+"] & \sigma^*\mathcal{L}^+_{(j+1,j+2)}. \end{tikzcd}$$

such that

• The rational map q^- contracts the locus D_0 where the line bundle

$$\omega_{C_{(j,j+2)}|B}\left(\frac{d - i_{j+1}}{d} s_0 + \frac{i_{j+2}}{d} s_\infty\right) \otimes \mathcal{L}^a_{(j,j+2)}$$

fails to be ample, and similarly, q^+ the locus D_∞ where

$$\omega_{C_{(j+1,j+3)}|B}(\frac{d-i_{j+1}}{d}s_0+\frac{i_{j+2}}{d}s_\infty)\otimes\mathcal{L}^a_{(j+1,j+3)}$$

fails to be ample,

• $\mathcal{L}^-_{(j+1,j+2)}$ is obtained by descent from $\mathcal{L}_{(j,j+2)}\otimes\mathcal{O}_{C_{(j,j+2)}}(i_{j+1}D_0)$, and $\mathcal{L}^+_{(j+1,j+2)}$ from $\mathcal{L}_{(j+1,j+3)}\otimes\mathcal{O}_{C_{(j+1,j+3)}}((d-i_{j+2})D_\infty)$,

• The second diagram is commutative.

We construct a flat family C over B by gluing $C_{(j,j+2)}\setminus$ Im s_∞ and $C_{(j+1,j+3)}\setminus$ Im s_0 along the open subschemes

$$C^-_{(j+1,j+2)}\setminus(\text{ Im } s_\infty\bigcup\text{ Im } s_0)\xrightarrow{\cong}C^+_{(j+1,j+2)}\setminus(\text{ Im } s_\infty\bigcup\text{ Im } s_0),$$

via the restriction of σ. By construction C inherits the subschemes D_0 and D_∞ from $C_{(j,j+2)}$ and $C_{(j+1,j+3)}$ respectively, so their contractions give maps p^+ and p^- fitting into a commutative diagram

$$\begin{tikzcd}
 & C \arrow[dl, "p^+"'] \arrow[dr, "p^-"] & \\
C_{(j,j+2)} \arrow[dr, "q^-"'] & & C_{(j+1,j+3)} \arrow[dl, "q^+"] \\
 & C_{(j+1,j+2)}. &
\end{tikzcd}$$

(as can be checked on the two patches).

As well, the restrictions of $e_{(j,j+2)}$ and $e_{(j+1,j+3)}$ on $C_{(j,j+2)}\setminus$ Im s_∞ and $C_{(j+1,j+3)}\setminus$ Im s_0 can be glued via the pullbacks of $\overline{\sigma}$ on $C^-_{(j+1,j+2)}\setminus(\text{ Im } s_\infty\bigcup\text{ Im } s_0)$ to form a morphism $e:\mathcal{O}_C^{n+1}\to\mathcal{L}$ of bundles over C. We have thus obtained a tuple $(C,\{s_0,s_\infty\},e:\mathcal{O}_C^{n+1}\to\mathcal{L})$ in $M_{(j,j+3)}(X)(B)$. Indeed, the conditions in Definition 3.4 are all inherited from the patches. To check that the stability properties in Definition 3.2 are also inherited from those on $C_{(j,j+2)}$ and $C_{(j+1,j+3)}$, it is enough to note that:

• The sections s_0 and s_∞ in C are inherited from $C_{(j,j+2)}$ and $C_{(j+1,j+3)}$, respectively.

• On each fibre C_b over $b\in B$, the component $C_{0,b}$ containing $s_0(b)$ is also inherited from $C_{(j,j+2)}$ so $\mathcal{L}=\mathcal{L}_{(j,j+2)}$ over $C_{0,b}$. Similarly $\mathcal{L}=\mathcal{L}_{(j+1,j+3)}$ over the component $C_{\infty,b}$ containing $s_\infty(b)$.

• By construction $\mathcal{L}\otimes\mathcal{O}_C(i_{j+1}D_0)$ descends to $\mathcal{L}_{(j+1,j+3)}$ via p^- and the composition

$$\mathcal{O}_C^{n+1}\xrightarrow{e}\mathcal{L}\longrightarrow\mathcal{L}\otimes\mathcal{O}_C(i_{j+1}D_0)$$

descends to $e_{(j+1,j+3)}$. Thus for the points b such that $C_{0,b}\subset D_0$ we have

$$\deg\mathcal{L}_{C_{0,b}}=i_{j+1}=\dim\operatorname{Coker}e_{(j+1,j+3),s_0(b))}$$

and similarly, $\deg\mathcal{L}_{C_{\infty,b}}=d-i_{j+2}=\dim\operatorname{Coker}e_{(j,j+2),s_\infty(b))}$ whenever $C_{\infty,b}\subset D_\infty$.

The canonical contractions p^+ and p^- insure that the map $M_{(j,j+2)}(X)\times_{M_{(j+1,j+2)}(X)}M_{(j+1,j+3)}(X)\to M_{(j,j+3)}(X)$ thus constructed is the inverse of a.

□

Corollary 3.13. *For all the elements $0=i_0<i_1<i_2<i_3<...<i_k=d$ of I, we obtain a net of Cartesian diagrams:*

$$\begin{tikzcd}
 & & M_{(0,k)}(X) \arrow[dl] \arrow[dr] & & \\
 & M_{(0,k-1)}(X) \arrow[dl] \arrow[dr] & & M_{(1,k)}(X) \arrow[dl] \arrow[dr] & \\
M_{(0,k-2)}(X) \arrow[dl] \arrow[dr] & & M_{(1,k-1)}(X) \arrow[dl] \arrow[dr] & & M_{(2,k)}(X) \arrow[dl] \arrow[dr] \\
M_{(0,k-3)}(X) & M_{(1,k-2)}(X) & & M_{(2,k-1)}(X) & & M_{(3,k)}(X)
\end{tikzcd}$$

At the top, $M_{(0,k)}(X) = \varprojlim M_{(j,j')}(X)$ is a closed and open substack of $\overline{M}_{0,2}(X,\beta)^{\mathbb{C}^*}$.

Proof. We note that the weights $\mathcal{A}(0,d) = (1,1)$ so $M_{(0,k)}(X)$ is indeed a closed substack of $\overline{M}_{0,2}(X,\beta)^{\mathbb{C}^*}$. Conditions (1)-(2) in Definition 3.4 are open in $\overline{M}_{0,2}(X,\beta)^{\mathbb{C}^*}$. Indeed, consider any family $\pi : C \to B$ with sections s_0 and s_∞ and stable map $\varphi : C \to X$, invariant under the $\mathbb{C}^*$-action, and consider a fibre C_b satisfying properties (1)-(2) in Definition 3.4. The nodes $s_0, s_1, ..., s_r = s_\infty$ of C are mapped to points x_j in the fixed point loci X_j for some $j \in \{0, 1, ..., k\}$. Let Z denote the set of indices j with this property.

Since the components of the fixed locus of X are closed and finitely many, there exists a neighborhood B' of b, such that all the nodes in the fibres of $C_{|B'} \to B'$ are mapped into the same components X_j-s with $j \in Z$. (We used the continuity of φ and properness of π.) Since $X_j^+/\mathbb{C}^*$ and $X_j^-/\mathbb{C}^*$ are compact and $\operatorname{Im} \varphi_b \bigcap X_j^{\pm} \neq \emptyset$, then B' can be chosen such that whenever $C_{b'} \bigcap X_j \neq \emptyset$ for some $b' \in B'$, then $C_{b'} \bigcap X_j^+ \neq \emptyset$ and $C_{b'} \bigcap X_j^- \neq \emptyset$. Thus the preservation of flow is an open condition. As well, since there are finitely many ways to split a fibre according to the components of the fixed locus of X where the nodes are mapped and according to the degrees of the components, it follows that fixing the degree of φ on components of the fibres intersecting given components of the fixed point locus of X gives an open condition as well.

□

For each $j, j' \in \{0, 1, ..., k\}$, the universal family $\mathcal{U}_{(j,j')}(X)$ over $M_{(j,j')}(X)$ admits a rational evaluation map $\mathrm{ev}_{(j,j')}$ into X defined everywhere except on the images of the sections s_0 and s_∞. The next step will be to replace X by a more suitable target $X_{(j,j')}$.

Notation 3.14. We will work inside the weighted stable map spaces with three marked points $\overline{M}_{0,\mathcal{A}(i,i',\Delta i)}(\mathbb{P}^n, d, a)$, where $0 < a < \frac{1}{2d+1}$ and $\mathcal{A}(i, i', \Delta i)$ represents the triple of weights

$$a_\infty(i) := 1 - (d - i - 1)a, \quad a_0(i') := 1 - (i' - 1)a, \quad \text{and } a_1(\Delta i) := (\Delta i - 1)a,$$

for $-1 \leq i < i' \leq d + 1$ and $\Delta i \geq i' - i$.

In particular, $\overline{M}_{0,\mathcal{A}(-1,d+1,d+2)}(\mathbb{P}^n, d, a) \cong \mathbb{P}^n_d := \mathbb{P}^{(n+1)(d+1)-1}$, is the "linear sigma model" representing trivial families of $\mathbb{P}^1$-s with degree d rational maps into $\mathbb{P}^n$. We

note that $\mathbb{P}^n_d$ admits a natural $\mathbb{C}^*$–action induced by the action with weights $(0,1)$ of $\mathbb{C}^*$ on the source $\mathbb{P}^1$.

Of special interest to us will be the cases when $\Delta i = i' - i$ and $0 \leq i < i' \leq d$.

Lemma 3.15. *Each of the moduli spaces $\overline{M}_{0,\mathcal{A}(i,i',i'-i)}(\mathbb{P}^n, d, a)$ with $0 \leq i < i' \leq d$ parametrizes families of $\mathbb{P}^1$-s with at least two disjoint sections and as such admits a natural $\mathbb{C}^*$–action induced from the source.*

Proof. Let $(\pi : C \to B, \{s_0, s_1, s_\infty\}, e : \mathcal{O}_C^{n+1} \to \mathcal{L})$ be a family of $\mathcal{A}(i, i', i'-i)$-weighted stable maps. We first note that all the fibres C_b of the family $\pi : C \to B$ are $C_b \cong \mathbb{P}^1$. Indeed, suppose C_b is reducible and let C' be a "tail" of C_b, namely a component such that $C_b \setminus C'$ is an open subset of C_b with exactly one point on the boundary. As $a_1(i'-i) + da \leq 1$, condition (1) in Definition 3.2 implies that C' must contain $s_0(b)$ or $s_\infty(b)$. If C_0 and C_∞ are the components of C_b containing $s_0(b)$ and $s_\infty(b)$ respectively, then we are in one of these three cases:

(i) $s_1(b)$ is in neither C_0 nor C_∞. Then the choice of weights and condition (1) in Definition 3.2 imply $\deg \mathcal{L}_{|C_0} \geq i'$ and $\deg \mathcal{L}_{|C_\infty} \geq d - i$, but $d - i + i' > d = \deg \mathcal{L}$, which is impossible.
(ii) $s_1(b) \in C_0$. Then $\deg \mathcal{L}_{|C_0} + \deg \mathcal{L}_{|C_\infty} > i + (d-i) = d = \deg \mathcal{L}$, which is impossible.
(iii) $s_1(b) \in C_\infty$. Then $\deg \mathcal{L}_{|C_0} + \deg \mathcal{L}_{|C_\infty} > i' + (d - i') = d = \deg \mathcal{L}$, which is impossible.

Hence $C_0 = C_\infty$. Condition (3) in Definition 3.2 insures that $s_0(b)$ and $s_\infty(b)$ are always distinct. Hence $C = \mathbb{P}(\pi_* \mathcal{O}_C(\text{ Im } s_0))$ which admits a natural $\mathbb{C}^*$–action with weights $(0,1)$ on the fibres of π. □

Remark 3.16. We note that for $a_1 := (i' - i - 1)a$, all values a_0 and a_∞ with

$$1 - (d-i)a < a_\infty \leq 1 - (d - i - 1)a \text{ and } \quad 1 - i'a < a_0 \leq 1 - (i' - 1)a,$$

define the same moduli space $\overline{M}_{0,\mathcal{A}'(i,i',i'-i)}(\mathbb{P}^n, d, a)$.

Lemma 3.17. *For positive integers $0 \leq i < i' < i'' < i''' \leq d$, there is a commutative diagram of rational maps and morphisms as illustrated below, such that the lowest square is Cartesian. All the spaces in the diagram below admit two $\mathbb{C}^*$–actions induced from the target X and source respectively, and the natural maps between them are $\mathbb{C}^*$–equivariant:*

$$\begin{tikzcd}
 & & \overline{M}_{0,\mathcal{A}(i,i''',i'''-i)}(\mathbb{P}^n,d,a) & & \\
 & \overline{M}_{0,\mathcal{A}(i,i''+\frac{1}{2},i''-i+\frac{1}{4})}(\mathbb{P}^n,d,a) \arrow[ur, dashed] \arrow[dl] & & \overline{M}_{0,\mathcal{A}(i'-\frac{1}{2},i''',i'''-i'+\frac{1}{4})}(\mathbb{P}^n,d,a) \arrow[ul, dashed] \arrow[dr] & \\
\overline{M}_{0,\mathcal{A}(i,i'',i''-i)}(\mathbb{P}^n,d,a) \arrow[dr, dashed] & & \overline{M}_{0,\mathcal{A}(i'-\frac{1}{2},i''+\frac{1}{2},i''-i'+\frac{1}{4})}(\mathbb{P}^n,d,a) \arrow[ul, dashed] \arrow[ur, dashed] \arrow[dl] \arrow[dr] & & \overline{M}_{0,\mathcal{A}(i',i''',i'''-i')}(\mathbb{P}^n,d,a) \arrow[ul, dashed] \\
 & \overline{M}_{0,\mathcal{A}(i'-\frac{1}{2},i'',i''-i'+\frac{1}{4})}(\mathbb{P}^n,d,a) \arrow[dr] & & \overline{M}_{0,\mathcal{A}(i',i''+\frac{1}{2},i''-i'+\frac{1}{4})}(\mathbb{P}^n,d,a) \arrow[dl] & \\
 & & \overline{M}_{0,\mathcal{A}(i',i'',i''-i')}(\mathbb{P}^n,d,a) & &
\end{tikzcd}$$

Proof. We first note the existence of a birational morphism

$$\overline{M}_{0,\mathcal{A}(i,i''+\frac{1}{2},i''-i+\frac{1}{4})}(\mathbb{P}^n,d,a) \to \overline{M}_{0,\mathcal{A}(i,i'',i''-i)}(\mathbb{P}^n,d,a).$$

(Indeed, even if not all the weights defining the first space are larger than the corresponding weights of $\mathcal{A}(i,i'',i''-i)$, they are all larger than other triples in the same chamber as specified in Remark 3.16.) The restriction of the map above to the exceptional locus is

$$\overline{M}_{0,(a_0(i''),1)}(\mathbb{P}^n,i'',a) \times_{\mathbb{P}^n} \overline{M}_{0,(1,a_1(i''-i),a_\infty(i))}(\mathbb{P}^n,d-i'',a) \to \overline{M}_{0,(a_0(i''),1)}(\mathbb{P}^n,i'',a),$$

representing weighted stable maps of splitting type:

(We note that the fractionary parts of the weights play no significant role in the moduli problem for the exceptional divisor and can be omitted here.)

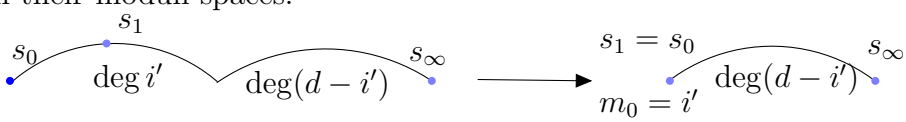
Indeed, redoing the casework from the proof of Lemma 3.15 in the case of the moduli space $\overline{M}_{0,\mathcal{A}(i,i''+\frac{1}{2},i''-i+\frac{1}{4})}(\mathbb{P}^n,d,a)$:

- If $s_1(b) \in C_\infty$, then
$$\deg \mathcal{L}_{|C_0} > i'' - \frac{1}{2} \text{ and } \deg \mathcal{L}_{|C_\infty} > d - i'' - \frac{1}{4},$$
yielding $\mathcal{L}_{|C_0} = i''$ and $\deg \mathcal{L}_{|C_\infty} = d - i''$ as in the exceptional divisor above.
- Whereas if $s_1(b) \in C_0$, then
$$\deg \mathcal{L}_{|C_0} > \frac{1}{4} + i \text{ and } \deg \mathcal{L}_{|C_\infty} > d - i - \frac{1}{4}$$
which is impossible.

The blow-up locus $\overline{M}_{0,(a_0(i''),1)}(\mathbb{P}^n,i'',a)$ represents objects in $\overline{M}_{0,\mathcal{A}(i,i'',i''-i)}(\mathbb{P}^n,d,a)$ for which $s_1 = s_\infty$ and $\dim \operatorname{Coker} e_{s_1} = d - i''$. This locus is fixed by the $\mathbb{C}^*$–action

defined in Lemma 3.15 and so there is an induced $\mathbb{C}^*$–action on the weighted blow-up $\overline{M}_{0,\mathcal{A}(i,i''+\frac{1}{2},i''-i+\frac{1}{4})}(\mathbb{P}^n,d,a)$.

The two other SW–oriented arrows are blow-downs whose exceptional loci represent the same splitting type as above. Similarly, the SE–oriented arrows are blow-downs which over the exceptional locus represent contractions of the following splitting type within their moduli spaces:



As the images of the sections s_0 and s_∞ are always disjoint, then so are the two different blow-up loci in $\overline{M}_{0,\mathcal{A}(i',i'',i''-i')}(\mathbb{P}^n,d,a)$, and so the result of these two successive blow-ups $\overline{M}_{0,\mathcal{A}(i'-\frac{1}{2},i''+\frac{1}{2},i''-i'+\frac{1}{4})}(\mathbb{P}^n,d,a)$ is the Cartesian product of the two individual blow-ups along those two loci. As noted earlier, the blow-up loci are fixed by the $\mathbb{C}^*$–action induced from the source and so $\overline{M}_{0,\mathcal{A}(i',i'',i''-i')}(\mathbb{P}^n,d,a)$ also inherits such a $\mathbb{C}^*$–action making all the morphisms in the diagram equivariant.

The NE–oriented rational map with target $\overline{M}_{0,\mathcal{A}(i,i''',i'''-i)}(\mathbb{P}^n,d,a)$ can be understood as obtained by successively blow-ing up the loci for which $i''\leq m_0<i'''$ (and their strict transforms), in decreasing order of m_0, following by blow-downs of all exceptional divisors which had resolved the cases $m_0>i''$. Similarly for the NW–oriented maps and $d-i'\leq m_\infty<d-i$. These loci are invariant under the $\mathbb{C}^*$–actions so the rational maps are equivariant where defined. The same arguments are valid for the remaining dotted arrows in the diagram. □

Definition 3.18. The two $\mathbb{C}^*$–actions induced from source and target respectively on the moduli spaces in Lemma 3.17 together form an action of $\mathbb{C}^*\times\mathbb{C}^*$ on these spaces. Consider the inverse diagonal action of $\mathbb{C}^*$ given by the embedding $\mathbb{C}^*\to\mathbb{C}^*\times\mathbb{C}^*$ where $t\to(t,t^{-1})$.

For each pair of positive integers (j,j') with $0<j<j'<k$ and their corresponding wall elements $i_j,i_{j'}\in I$ in the image of the moment map of X, we define $X_{(j,j')}$ to be the subspace of $(\overline{M}_{0,\mathcal{A}(i_j,i_{j'},i_{j'}-i_j)}(\mathbb{P}^n,d,a))^{\mathbb{C}^*}$ whose points parametrize $\mathbb{C}^*$-fixed weighted stable maps to X which are both flow-preserving and action-class-preserving, in the sense of Definition 3.4. Similarly to the embedding

$$X_{(j,j')}\hookrightarrow\overline{M}_{0,\mathcal{A}(i_j,i_{j'},i_{j'}-i_j)}(\mathbb{P}^n,d,a)^{\mathbb{C}^*},$$

we also define the following:

$$\begin{aligned}
&X_{(j,j'+\frac{1}{2})}\hookrightarrow\overline{M}_{0,\mathcal{A}(i_j,i_{j'}+\frac{1}{2},i_{j'}-i_j+\frac{1}{4})}(\mathbb{P}^n,d,a)^{\mathbb{C}^*},\\
&X_{(j-\frac{1}{2},j')}\hookrightarrow\overline{M}_{0,\mathcal{A}(i_j-\frac{1}{2},i_{j'},i_{j'}-i_j+\frac{1}{4})}(\mathbb{P}^n,d,a)^{\mathbb{C}^*},\ \text{and}\\
&X_{(j-\frac{1}{2},j'+\frac{1}{2})}\hookrightarrow\overline{M}_{0,\mathcal{A}(i_j-\frac{1}{2},i_{j'}+\frac{1}{2},i_{j'}-i_j+\frac{1}{4})}(\mathbb{P}^n,d,a)^{\mathbb{C}^*}.
\end{aligned}$$

Corollary 3.19. *Let j,j' be two integers with $0<j<j+1<j'-1<j'<k$ and consider the following walls in I:*

$$i=i_j,\ i'=i_{j+1},\ i''=i_{j'-1}\ \textit{and}\ i'''=i_{j'}.$$

The diagram in Lemma 3.17 restricted to the flow-preserving, action-class-preserving part of the fixed locus yields the following four Cartesian squares:

$$\begin{tikzcd}
 & & X_{(j,j')} & & \\
 & X_{(j,j'-\frac{1}{2})} \arrow[ur] & & X_{(j+\frac{1}{2},j')} \arrow[ul] & \\
X_{(j,j'-1)} \arrow[dr] & & X_{(j+\frac{1}{2},j'-\frac{1}{2})} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & & X_{(j+1,j')} \arrow[dl] \\
 & X_{(j+\frac{1}{2},j'-1)} \arrow[ul] \arrow[dr] & & X_{(j+1,j'-\frac{1}{2})} \arrow[ur] & \\
 & & X_{(j+1,j'-1)} \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

Proof. The conditions of flow and action-class preservations are compatible with the maps in Lemma 3.17 leading to the diagram above. We check that all the arrows in this diagram represent well defined morphism.

Indeed, the points where $\overline{M}_{0,\mathcal{A}(i,i''+\frac{1}{2},i''-i+\frac{1}{4})}(\mathbb{P}^n,d,a) \dashrightarrow \overline{M}_{0,\mathcal{A}(i,i''',i'''-i)}(\mathbb{P}^n,d,a)$ is not well defined correspond to the condition $i'' < m_0 < i'''$ on the moduli problem. This locus intersects $X_{(j,j'-\frac{1}{2})}$ trivially since $i'' = i_{j'-1}$ and $i''' = i_{j'}$ are consecutive walls. Hence $X_{(j,j'-\frac{1}{2})} \to X_{(j,j')}$ is everywhere well-defined, and similarly for all the other maps in the diagram.

We denote by $Z_{(j,j'-\frac{1}{2})}$ and $Z_{(j+\frac{1}{2},j')}$ the images of the exceptional loci of the two maps to $X_{(j,j')}$. Note that $Z_{(j,j'-\frac{1}{2})}$ and $Z_{(j+\frac{1}{2},j')}$ correspond to conditions $m_0 = i_{j'-1}$ and $m_\infty = d - i_{j+1}$ respectively, which cannot be satisfied simultaneously since $i_{j'-1} > i_{j+1}$. Hence the upper square in the diagram is Cartesian, as $X_{(j+\frac{1}{2},j'-\frac{1}{2})}$ can be thought of as constructed by gluing $X_{(j,j'-\frac{1}{2})}$ and $X_{(j+\frac{1}{2},j')}$ along $X_{(j,j')} \setminus (Z_{(j,j'-\frac{1}{2})} \bigcup Z_{(j+\frac{1}{2},j')})$. Similar arguments show that all the other squares in the diagram are Cartesian.

□

Lemma 3.20. *There is a $\mathbb{C}^*$-equivariant map $X_{(0,k)} \to X$ such that $X_{(0,k)}$ is the weighted blow-up of X along the source and the sink components of the fixed locus.*

Proof. For each point $x \in X$, the action-map $\varphi_x : \mathbb{C}^* \to X$, given by $t \to t \cdot x$, can be represented by a point in $\mathbb{P}^n_d$. We can define this explicitly as follows: Let $d_0, ..., d_n$ be the weights of the $\mathbb{C}^*$ action on $\mathbb{P}^n$ for which the embedding $X \hookrightarrow \mathbb{P}^n$ is $\mathbb{C}^*$–equivariant. Then we have an equivariant map

$$\begin{array}{ccc} X \hookrightarrow \mathbb{P}^n & \hookrightarrow & \mathbb{P}^n_d, \\ \left[(x_i)_{i\in\{0,...,n\}}\right] & \to & \left[(y_i^l)_{i\in\{0,...,n\},l\in\{0,...,n\}}\right] \end{array}$$

where $y_i^l = \begin{cases} x_i \text{ if } l = d_i, \\ 0 \text{ otherwise.} \end{cases}$

We think of $\mathbb{P}^n_d$ as a space of weighted stable maps with 3 marked points $\overline{M}_{\mathcal{A}}(\mathbb{P}^n,d,a)$ with $0 < a < \frac{1}{2d}$ and $0 < a_i < 1 - da$ as well as $a_i + a_l > 1$ for all $i, l \in \{0, \infty, 1\}$. Then with the embedding above, $\mathbb{P}^n$ is the flow and action-class–preserving part of the fixed locus of $\mathbb{P}^n_d$, for the inverse-diagonal action introduced earlier, and X plays the similar role for $\overline{M}_{\mathcal{A}}(X,\beta,a)$.

In the choice of weights for $\mathbb{P}^n_d$, we can choose a_1 small enough to justify the existence of a $\mathbb{C}^*$–equivariant birational morphism

$$\overline{M}_{\mathcal{A}(0,d,d)}(\mathbb{P}^n, d, a) \to \mathbb{P}^n_d,$$

Restricting to $\overline{M}_{\mathcal{A}(0,d,d)}(X, \beta, a)$, and then to the flow and action-class preserving parts of the fixed locus, we obtain a commutative diagram

$$\begin{array}{ccc} X_{(0,k)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbb{P}^n_{(0,k)} & \longrightarrow & \mathbb{P}^n \end{array}$$

of $\mathbb{C}^*$-equivariant morphisms. The action of $\mathbb{C}^*$ on the moduli spaces is $X_{(0,k)}$ and $\mathbb{P}^n_{(0,k)}$ is induced from the target, which in this case is the same as the one induced from the source of the weighted stable maps.

The morphism $\mathbb{P}^n_{(0,k)} \to \mathbb{P}^n$ can be described very explicitly in coordinates as the weighted blow-up of $\mathbb{P}^n$ along the source and the sink of $\mathbb{P}^n$, and similarly for $X_{(0,k)} \to X$. Indeed, the two exceptional divisors in $X_{(0,k)}$ parametrize orbits in X with $s_1 = s_0 \in X_0$ and $s_1 = s_\infty \in X_\infty$ respectively, while the map $X_{(0,k)} \to X$ is exactly the evaluation map at s_1. Thus the exceptional divisors are exactly the weighted projective fibrations

$$[(X_0^+ \setminus X_0)/\mathbb{C}^*] \to X_0 \text{ and } [(X_\infty^- \setminus X_\infty)/\mathbb{C}^*] \to X_\infty.$$

□

From now on, unless explicitly stated otherwise, all spaces of weighted stable maps and their subspaces will be considered with the $\mathbb{C}^*$–action induced from the action on the source.

Proposition 3.21. *Let j, j' be positive integers such that $0 < j < j' < k$ and let $\mathcal{U}_{(j,j')}(X)$ denote the universal family over $M_{(j,j')}(X)$, with evaluation map*

$$ev_{(j,j')} : \mathcal{U}_{(j,j')}(X) \dashrightarrow X$$

well defined everywhere except on the images of s_0 and s_∞. The following properties hold for $X_{(j,j')}$:

a) *There is a well defined, $\mathbb{C}^*$–equivariant map $\varphi_{(j,j')} : \mathcal{U}_{(j,j')}(X) \to X_{(j,j')}$.*
b) *There is a $\mathbb{C}^*$–equivariant birational map $X_{(j,j')} \dashrightarrow X$ defined everywhere outside the source and sink of $X_{(j,j')}$. Where defined, this map is an isomorphism on its image.*
c) *Via the above isomorphism, the restriction of $\varphi_{(j,j')}$ to $\mathcal{U}_{(j,j')} \setminus (\ Im\ s_0 \bigcup\ Im\ s_\infty)$ is identified with $ev_{(j,j')}$.*

Proof. a) For any positive real number a such that $0 \le a \le \frac{1}{d}$ we have

$$\mathcal{U}_{(j,j')}(X) \hookrightarrow \overline{M}_{\mathcal{A}^a(i_j, i_{j'})}(\mathbb{P}^n, d, a)$$

where $\mathcal{A}^a(i_j, i_{j'})$ can be chosen to be any triple (a_0, a_∞, a_1) satisfying

$a_0 \in (1 - a(i_j + 1), 1 - ai_j]$, $a_\infty \in (1 - a(d - i_{j'} + 1), 1 - a(d - i_{j'})]$, $0 \le a_1 < 1 - da$,

while for $0 \le a < \frac{1}{2d}$ we have $X_{(j,j')} \hookrightarrow \overline{M}_{\mathcal{A}(i_j, i'_j, i'_j - i_j)}(\mathbb{P}^n, d, a)$ with

$a_0 \in (1 - ai_{j'}, 1 - a(i_{j'} - 1)], a_\infty \in (1 - a(d - i_j - 1), 1 - a(d - i_j)], a_1 = a(i_{j'} - i_j - 1)$.

As $i_j + 1 \leq i_{j'}$ and by choice $a < \frac{1}{2d}$, the weights in $\mathcal{A}^a(i_j, i_{j'})$ are larger than those in $\mathcal{A}(i_j, i'_j, i'_j - i_j)$, justifying the existence of a birational morphism

$$\overline{M}_{\mathcal{A}^a(i_j,i_{j'})}(\mathbb{P}^n, d, a) \to \overline{M}_{\mathcal{A}(i_j,i'_j,i'_j-i_j)}(\mathbb{P}^n, d, a).$$

Restriction to the flow and action-class–preserving part of the fixed point locus in the moduli spaces of target X leads to a morphism $\varphi_{(j,j')} : \mathcal{U}_{(j,j')}(X) \to X_{(j,j')}$.

At the level of moduli problems, the morphism $\varphi_{(j,j')} : \mathcal{U}_{(j,j')}(X) \to X_{(j,j')}$ consists in contracting each chain of $\mathbb{P}^1$-s with marked point s_1 down to the component containing s_1, while adding the degrees of the contracted components to the new multiplicities of s_0 and s_∞.

b) The components of the fixed point locus of $X_{(j,j')}$ are the following:

- The source and sink, defined by the conditions $s_0 = s_1$ and $s_1 = s_\infty$, respectively.
- The remaining loci, parametrizing $\mathbb{P}^1$-s together with contractions to a fixed point of X, with $m_0 = i$ and $m_\infty = d - i$ for some $i \in I$.

We have already illustrated the existence of a chain of birational maps between $X_{(j,j')}$ and $X_{(0,k)}$, so that the composition $X_{(j,j')} \dashrightarrow X_{(0,k)}$ is defined everywhere except some subspaces of the source and sink. Indeed, in the diagram at Corollary 3.19, all rational maps which invert the NE and NW–oriented arrows have this property.

Finally, Lemma 3.20 shows that $X_{(0,k)}$ is the weighted blow-up of X along the source and the sink components of the fixed locus.

c) Outside of the source and sink, we can identify each point b of $X_{(j,j')}$ with $s_1(b) \in X$. Indeed, this map can be inverted by identifying $s_1(b)$ with its action-map $t \to t \cdot s_1(b)$. With this identification, the map $\varphi_{(j,j')}$ described in part (a) of the proof is identified with $\mathrm{ev}_{(j,j')}$ on $\mathcal{U}_{(j,j')} \setminus (\operatorname{Im} s_0 \bigcup \operatorname{Im} s_\infty)$. □

Remark 3.22. Let $(X_{(j,j')})_0$ and $(X_{(j,j')})_\infty$ denote the source and sink of $X_{(j,j')}$. Via the birational map $X_{(j,j')} \dashrightarrow X$, we can identify the open subspace $X_{(j,j')} \setminus (X_{(j,j')})_0 \bigcup (X_{(j,j')})_\infty)$ with $X^o_{(j,j')} := (\bigcup_{l' \leq j'} X^+_{l'}) \bigcap (\bigcup_{l \geq j} X^-_l)$. Then $X_{(j,j')} = \overline{X^o}_{(j,j')}$ is a compactification of this compatible with the $\mathbb{C}^*$–action, constructed like in the first part of the proof of Theorem 3.7. Indeed, the source and sink of $X_{(j,j')}$ are birational to the source and sink in $X_{(0,k)}$, and hence divisors (see Lemma 3.20). Hence $(X_{(j,j')})^+_0 \to (X_{(j,j')})_0$ is an affine fibration with $\mathbb{A}^1$ fibres, and similarly for $(X_{(j,j')})^-_\infty \to (X_{(j,j')})_\infty$.

Next we describe the maps between the target spaces $X_{(j,j')}$ in terms of the $\mathbb{C}^*$–action on X.

Notation 3.23. We denote by $\overline{X^+_{j'}}$ the closure of $X^+_{j'}$ in $X_{(j,j'+1)}$. As an alternate definition,

$$\overline{X^+_{j'}} \cong [\big((X^+_{j'} \times \mathbb{A}^1) \setminus (X_{j'} \times \{0\})\big) / \mathbb{C}^*],$$

up to isomorphisms the unique compactification of $X^+_{j'}$ to a projective fibration over $X_{j'}$. Indeed, $\overline{X^+_{j'}} \setminus X_{j'} \subset (X_{(j,j'+1)})^-_\infty$, which is an affine fibration with one-dimensional fibres over $(X_{(j,j'+1)})_\infty$, and so $\overline{X^+_{j'}} \setminus X_{j'}$ is an $\mathbb{A}^1$-fibration over its image in $(X_{(j,j'+1)})_\infty$. This image thus forms the divisor at infinity of $\overline{X^+_{j'}} \to X_{j'}$.

Recall the notation $Y^-_{(j'-1,j')} = [(X^-_{j'} - X_{j'})/\mathbb{C}^*]$.

Theorem 3.24. *a) The exceptional locus of the two birational morphisms*

$$X_{(j,j')} \longleftarrow X_{(j,j'+\frac{1}{2})} \longrightarrow X_{(j,j'+1)}$$

is $E := Y^-_{(j'-1,j')} \times_{X_{j'}} \overline{X^+_{j'}}$. *The restrictions of the two morphisms to the exceptional locus give the top two maps in the Cartesian diagram*

$$\begin{tikzcd}
 & Y^-_{(j'-1,j')} \times_{X_{j'}} \overline{X^+_{j'}} \arrow[dl, "\overline{g^+_{j'}}"'] \arrow[dr, "f^-_{j'}"] & \\
Y^-_{(j'-1,j')} \arrow[dr, "p^-_{j'}"'] & & \overline{X^+_{j'}} \arrow[dl, "\overline{q^+_{j'}}"] \\
 & X_{j'}. &
\end{tikzcd}$$

Both $p^-_{j'}$ *and* $\overline{q^+_{j'}}$ *are weighted projective fibrations over* $X_{j'}$.

b) Both morphisms $X_{(j,j')} \longleftarrow X_{(j,j'+\frac{1}{2})} \longrightarrow X_{(j,j'+1)}$ *are weighted blow-ups along the smooth loci* $Y^-_{(j'-1,j')}$ *and* $\overline{X^+_{j'}}$, *respectively, with the weights induced from the* $\mathbb{C}^*$*–action. All spaces are smooth Deligne-Mumford stacks.*

c) Let $\mathcal{F}$ *denote the conormal bundle of* E *in* $X_{(j,j'+\frac{1}{2})}$, *and let* π *denote the morphism* $X_{(j,j'+\frac{1}{2})} \to X_{(j,j'+1)}$. *The associated affine fibration over the blow-up locus* $\overline{X^+_{j'}}$, *defined as* $A = \mathrm{Spec}(\oplus_{n\geq 0}\pi_*\mathcal{F}^n)$, *satisfies the property*

$$A \otimes \mathcal{O}_{\overline{X^+_{j'}}}(-1) := \mathrm{Spec}(\oplus_{n\geq 0}\pi_*\mathcal{F}^n \otimes \mathcal{O}_{\overline{X^+_{j'}}}(-n)) \cong X^-_{j'} \times_{X_{j'}} \overline{X^+_{j'}},$$

with the natural $\mathbb{C}^*$*–action induced from the action on* X. *The normal bundle of the zero section in* A *is* $\overline{q^+_{j'}}^*\mathcal{N}^-_{X_{j'}|X} \otimes \mathcal{O}_{\overline{X^+_{j'}}}(1)$.

Similarly, if $B \to Y^-_{(j'-1,j')}$ *is the affine fibration associated to the weighted blow-up* $X_{(j,j'+\frac{1}{2})} \to X_{(j,j')}$, *then*

$$B \otimes \mathcal{O}_{Y^-_{(j'-1,j')}}(-1) \cong Y^-_{(j'-1,j')} \times_{X_{j'}} (X^+_{j'} \oplus \mathcal{O}_{X'_j}).$$

The normal bundle of the zero section in B *is* $p^{-*}_{j'}(\mathcal{N}^+_{X_{j'}|X} \oplus \mathcal{O}_{X_{j'}}) \otimes \mathcal{O}_{Y^-_{(j'-1,j')}}(1)$.

Proof. a) Following the proof of Lemma 3.17, the image of the exceptional locus of $X_{(j,j'+\frac{1}{2})} \to X_{(j,j'+1)}$ represents objects in $X_{(j,j'+1)}$ for which $\dim \mathrm{Coker}\, e_{s_0} = i_{j'}$. Via the identifications in Remark 3.22, removing the sink from this locus yields $X^+_{j'}$, so the locus is exactly $\overline{X^+_{j'}}$, a weighted projective fibration over $X_{j'}$.

On the other hand, the image of the exceptional locus of $X_{(j,j'+\frac{1}{2})} \to X_{(j,j')}$ represents objects in $X_{(j,j')}$ for which $s_1 = s_\infty$ and $\dim \mathrm{Coker}\, e_{s_\infty} = d - i_{j'}$. This can be identified with the sublocus of $M_{(j',j'+1)}(X)$ parametrizing orbits in $X^-_{j'}$. We know $M_{(j',j'+1)}(X) \cong [X^s_{(j',j'+1)}/\mathbb{C}^*]$ by Theorem 3.7 and by the same arguments, the blow-up locus is identified with $[(X^-_{j'} \setminus X_{j'})/\mathbb{C}^*]$, which is a weighted projective fibration over $X_{j'}$.

The shared exceptional locus for $X_{(j,j')} \leftarrow X_{(j,j'+\frac{1}{2})} \to X_{(j,j'+1)}$ is thus

$$[(X^-_{j'} \setminus X_{j'})/\mathbb{C}^*] \times_{X_{j'}} \overline{X^+_{j'}}.$$

This is a weighted projective fibration over the loci $[(X^-_{j'} \setminus X_{j'})/\mathbb{C}^*]$ and $\overline{X^+_{j'}}$; all three are smooth Deligne-Mumford stacks.

b) Recall the embedding $X_{(j,j'+1)} \hookrightarrow (\overline{M}_{0,\mathcal{A}(i_j,i_{j'+1},i_{j'+1}-i_j)}(\mathbb{P}^n,d,a))^{\mathbb{C}^*}$. For simplicity, we will denote by M the complement in $\overline{M}_{0,\mathcal{A}(i_j,i_{j'+1},i_{j'+1}-i_j)}(\mathbb{P}^n,d,a))^{\mathbb{C}^*}$ of the locus made by stable map spaces with $m_0 > i_{j'}$. Similarly, let $\widetilde{M}$ denote the preimage of M in $\overline{M}_{0,\mathcal{A}(i_j,i_{j'}+\frac{1}{2},i_{j'}-i_j+\frac{1}{4})}(\mathbb{P}^n,d,a)^{\mathbb{C}^*}$. Then as remarked in Corollary 3.19, $X_{(j,j'+1)} \hookrightarrow M$ and $X_{(j,j'+\frac{1}{2})} \hookrightarrow \widetilde{M}$.

We know that $\widetilde{M} \to M$ is a $\mathbb{C}^*$–equivariant weighted blow-up and the blow-up locus parametrizes weighted stable maps for which $m_0 = i_{j'}$. Thus in terms of the $\mathbb{C}^*$–action, the blow-up locus can be described as $\overline{M^+_{j'}}$, where

- $M_{j'}$ denotes the $\mathbb{C}^*$–fixed point locus parametrizing weighted stable maps for which $m_0 = i_{j'}$ and $m_\infty = d - i_{j'}$, the source being contracted to a fixed point of $\mathbb{P}^n$.
- $M^+_{j'} := \{c \in M;\ \lim_{t\to 0} t\cdot c \in M_{j'}\}$.
- $\overline{M^+_{j'}} := M^+_{j'} \bigcup \{\lim_{t\to\infty} t\cdot c;\ c \in M_{j'}\}$.

Indeed, for a weighted stable map $c = [(C, \{s_0, s_\infty, s_1\}, e : \mathcal{O}_C^{n+1} \to \mathcal{L})]$, we have $t\cdot c = [(C, \{s_0, s_\infty, t\cdot s_1\}, e : \mathcal{O}_C^{n+1} \to \mathcal{L})]$. If c is in the blow-up locus, then at $\lim_{t\to 0} t\cdot c$, the source contains a component C_{01} of degree 0 with $s_0, s_1 \in C_{01}$ (note that $s_0 \neq s_1$ due to the stability conditions). As well, $m_0 = i_{j'}$ and the stability conditions imply $s_\infty \in C_{01}$ and $m_\infty = d - i_{j'}$. On the other hand, $\lim_{t\to\infty} t\cdot c$ satisfies $s_1 = s_\infty$ and so is a point in the sink of M.

Let $\mathcal{I}$ denote the ideal sheaf of $M_{j'}$. The $\mathbb{C}^*$–action on M induces a natural splitting $\mathcal{I} = \mathcal{I}^+ \oplus \mathcal{I}^-$. Here $\mathcal{I}^+$ is the ideal of $\overline{M^+_{j'}}$ in M, and it admits a natural increasing filtration $\{\mathcal{I}_l\}_{l>0}$. Each term $\mathcal{I}_l$ can be described explicitly, in an étale neighborhood of $M_{j'}$ by $\mathbb{C}^*$–invariant quasi-affine schemes, as generated by the sections on which $\mathbb{C}^*$ acts with weights $\geq l$.

Consider the equivariant embedding $j : X_{(j,j'+1)} \hookrightarrow M$, so that, with the identifications from part (a), $j^{-1}\mathcal{I}^+$ is the ideal of $\overline{X^+_{j'}}$ in $X_{(j,j'+1)}$. The ideal $j^{-1}\mathcal{I}^+$ comes with a natural increasing filtration $\{j^{-1}\mathcal{I}_l\}_{l>0}$ compatible with the $\mathbb{C}^*$–action. The construction of a smooth Deligne-Mumford stack structure $\widetilde{X_{(j,j'+1)}}$ for the weighted blow-up given by this filtration depends on the two technical conditions from [MM2], Lemma 2.5, which for $\{j^{-1}\mathcal{I}_l\}_{l>0}$ reduce to checking

$$j^{-1}(\mathcal{I}_l \bigcap \mathcal{I}^2) = j^{-1}\mathcal{I}_l \bigcap j^{-1}\mathcal{I}^2.$$

We may localize outside the sink and reduce the condition above to $j^{-1}(\mathcal{I}_l \setminus \mathcal{I}_{l+1}) = j^{-1}(\mathcal{I}_l) \setminus j^{-1}(\mathcal{I}_{l+1})$ for each index $l \geq 1$, which indeed holds as $\mathcal{I}^+_{|X^+_{j'}}$ is graded by the weights of the $\mathbb{C}^*$-action, and this grading is preserved when restricting to $X_{(j,j'+1)}$.

Finally, we can identify $\widetilde{X_{(j,j'+1)}}$ with $X_{(j,j'+\frac{1}{2})}$ as they are both embedded in $\widetilde{M}$, they are both mapped birationally onto $X_{(j,j'+1)}$ and the maps restrict over the exceptional locus to $E := Y^-_{(j'-1,j')} \times_{X_{j'}} \overline{X^+_{j'}} \to \overline{X^+_{j'}}$.

c) Recall the weighted blow-up diagram

$$\begin{array}{ccc} \mathrm{Spec}(\oplus_{n\geq 0}\mathcal{F}^n) & \longrightarrow & \mathrm{Spec}(\oplus_{n\geq 0}\pi_*\mathcal{F}^n) \\ \uparrow & & \uparrow \\ E = Y^-_{(j'-1,j')} \times_{X_{j'}} \overline{X^+_{j'}} & \longrightarrow & \overline{X^+_{j'}} \end{array} \tag{3.6}$$

where the second line represents the restriction over the exceptional locus. ([MM2], Lemma 2.8).

Let $F := Y^-_{(j'-1,j')} \times_{X_{j'}} X^+_{j'}$, and let U be the space obtained by removing the sink from $X_{(j,j')}$. From the description of the weighted blow-up π in the proof of part (b) as induced by the weights of the $\mathbb{C}^*$–action, we can identify the restriction to U of the above diagram with the weighted blow-up diagram

$$\begin{array}{ccc} \mathrm{Bl}^w_{X_{j'}} X^-_{j'} \times_{X_{j'}} X^+_{j'} & \longrightarrow & X^-_{j'} \times_{X_{j'}} X^+_{j'} \\ \uparrow & & \uparrow \\ F = Y^-_{(j'-1,j')} \times_{X_{j'}} X^+_{j'} & \longrightarrow & X^+_{j'} \end{array}$$

(conform [MM2], Lemma 2.8).

Here $\mathrm{Bl}^w_{X_{j'}} X^-_{j'} \times_{X_{j'}} X^+_{j'} = \mathrm{Spec}(\oplus_{n\geq 0} g^+_{j'*}\mathcal{O}(n))$, where $\mathcal{O}(1)$ is canonically defined on $[(X^-_{j'} - X_{j'})/\mathbb{C}^*]$, the weighted projective fibration over $X_{j'}$, and $g^+_{j'}$ is the restriction of $\overline{g^+_{j'}}$ to F.

As a consequence, the normal bundle $\mathcal{N}_{E|X_{(j,j'+\frac{1}{2})}}$ can be written as

$$\mathcal{N}_{E|X_{(j,j'+\frac{1}{2})}} \cong \overline{g^+_{j'}}^*\mathcal{O}(-1) \otimes \mathcal{O}(-mD), \tag{3.7}$$

where $D = E - F$ and as such $\mathcal{O}(-D) = f^{-*}_{j'}\mathcal{O}_{\overline{X^+_{j'}}}(-1)$. Moreover, restriction over the fibres of $\overline{g^+_{j'}}$ gives $m = 1$.

After dualizing and push-forward of the above equation by $f^-_{j'}$ we obtain:

$$\begin{aligned} f^-_{j'*}\mathcal{N}^\vee_{E|X_{(j,j'+\frac{1}{2})}} \cong f^-_{j'*}(\overline{g^+_{j'}}^*\mathcal{O}(1) \otimes f^{-*}_{j'}\mathcal{O}_{\overline{X^+_{j'}}}(1)) &\cong f^-_{j'*}\overline{g^+_{j'}}^*\mathcal{O}(1) \otimes \mathcal{O}_{\overline{X^+_{j'}}}(1) \\ &\cong \overline{q^+_{j'}}^* p^-_{j'*}\mathcal{O}(1) \otimes \mathcal{O}_{\overline{X^+_{j'}}}(1), \end{aligned}$$

while $\mathrm{Spec}(\oplus_{n\geq 0} p^-_{j'*}\mathcal{O}(n))) \cong X^-_{j'}$. This leads to the property of the affine fibration A stated in part (c) of the theorem. Furthermore, the normal bundle of the zero section in $X^-_{j'}$ is $\mathcal{N}^-_{X_{j'}|X}$ (consistent with [MM2], Lemma 2.10. for the given filtration), hence the formula for the normal bundle of the zero section in A.

Similarly,

$$\begin{aligned} \overline{g^+_{j'}}_*\mathcal{N}^\vee_{E|X_{(j,j'+\frac{1}{2})}} \cong \overline{g^+_{j'}}_*(\overline{g^+_{j'}}^*\mathcal{O}(1) \otimes f^{-*}_{j'}\mathcal{O}_{\overline{X^+_{j'}}}(1)) &\cong \mathcal{O}(1) \otimes \overline{g^+_{j'}}_* f^{-*}_{j'}\mathcal{O}_{\overline{X^+_{j'}}}(1) \\ &\cong \mathcal{O}(1) \otimes p^{-*}_{j'}\overline{q^+_{j'}}_*\mathcal{O}_{\overline{X^+_{j'}}}(1). \end{aligned}$$

The weighted projective fibration $\overline{X^+_{j'}} = P^w(X^+_{j'} \oplus \mathcal{O}_{X'_j})$ corresponds to the affine fibration $\mathrm{Spec}(\oplus_{n\geq 0}\overline{q^+_{j'}}_*\mathcal{O}_{X^-_{j'}}(n)) \cong X^+_{j'} \oplus \mathcal{O}_{X'_j}$. The normal bundle of the zero section in this affine bundle is $\mathcal{N}^+_{X_{j'}|X} \oplus \mathcal{O}_{X_{j'}}$. Hence the properties of the fibration B stated in part (c) of the theorem.

□

Corollary 3.25. *With the notations introduced earlier, we have*

$$\mathcal{U}_{(j'-1,j')}(X) \cong X_{(j'-1,j')} \text{ and}$$

$$\mathcal{U}_{(j'-1,j'+1)}(X) \cong X_{(j'-1,j'+\frac{1}{2})} \times_{X_{(j'-1,j'+1)}} X_{(j'-\frac{1}{2},j'+1)} \cong X_{(j'-\frac{1}{2},j'+\frac{1}{2})}.$$

Proof. At the level of moduli problems of weighted stable maps, the morphism

$$\varphi_{(j,j')} : \mathcal{U}_{(j,j')}(X) \to X_{(j,j')}$$

corresponds to contracting those components of the domain curves which do not contain the section s_1. However, in the case of both spaces $\mathcal{U}_{(j'-1,j')}(X)$ and $X_{(j'-1,j')}$ the curves parametrized are $\mathbb{P}^1$-s, hence $\mathcal{U}_{(j'-1,j')}(X) \cong X_{(j'-1,j')}$.

On the other hand the morphism

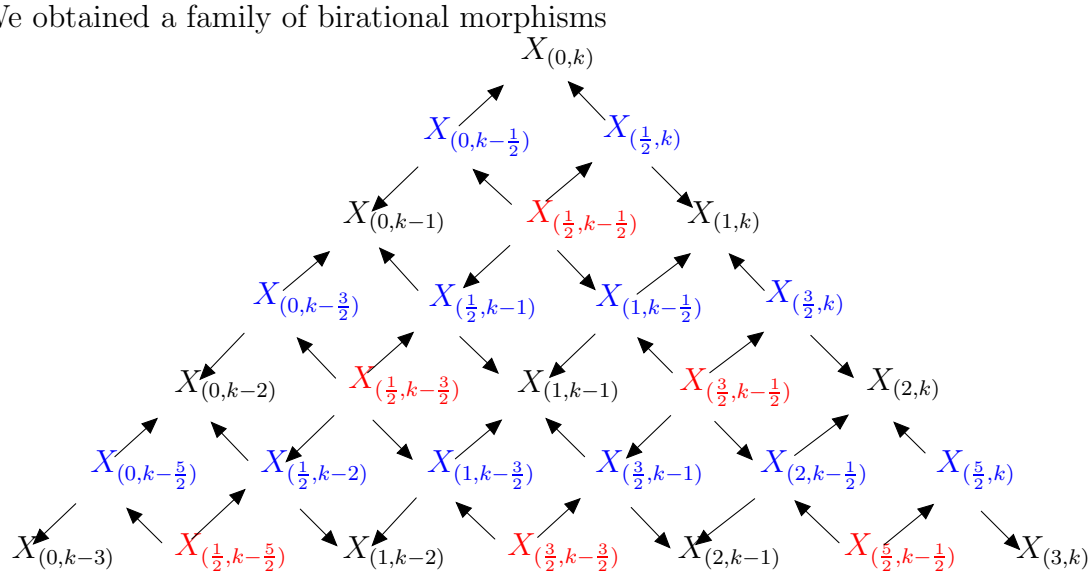
$$\varphi_{(j'-1,j'+1)} : \mathcal{U}_{(j'-1,j'+1)}(X) \to X_{(j'-1,j'+1)}$$

is a composition of two weighted blow-ups with blow-up loci $\overline{X^+_{j'}}$ and $\overline{X^-_{j'}}$. Blowing up $X_{(j'-1,j'+1)}$ along $\overline{X^+_{j'}}$ yields $X_{(j'-\frac{1}{2},j'+1)}$ while blowing up $X_{(j'-1,j'+1)}$ along $\overline{X^-_{j'}}$ yields $X_{(j'-1,j'+\frac{1}{2})}$. The two blow-up loci intersect transversely along $X_{j'}$. Hence

$$\mathcal{U}_{(j'-1,j'+1)}(X) \cong X_{(j'-1,j'+\frac{1}{2})} \times_{X_{(j'-1,j'+1)}} X_{(j'-\frac{1}{2},j'+1)} \cong X_{(j'-\frac{1}{2},j'+\frac{1}{2})}.$$

□

We obtained a family of birational morphisms



which we would like to compare to that of Corollary 3.13.

Notation 3.26. For every j, j', l, l' such that $0 \le j \le l < l' \le j' \le k$ we will employ the following notations for the natural morphisms between moduli spaces: $m^{j,j'}_{l,l'} : M_{(j,j')}(X) \to M_{(l,l')}(X)$, $u^{j,j'}_{l,l'} : \mathcal{U}_{(j,j')}(X) \to \mathcal{U}_{(l,l')}(X)$, $p_{(j,j')} : \mathcal{U}_{(j,j')}(X) \to M_{(j,j')}(X)$, $\varphi_{(j,j')} : \mathcal{U}_{(j,j')}(X) \to X_{(j,j')}$. For suitable choices of j, j', l, l', we will denote $v^{j,j'}_{l,l'} : X_{(j,j')}(X) \to X_{(l,l')}$. For simplicity of notations, in cases when there is no danger of confusion we may omit the indices of the morphisms, preserving only the those of the domain and target spaces.

Theorem 3.27. *For a pair of positive integers (j, j') such that $0 < j < j' < k$, consider the triangle formed by all the spaces $X_{(l,l')}$ above such that*

$$0 \leq j \leq l < l' \leq j' \leq k.$$

We denote by $I(j, j')$ the inverse family generated by all the morphisms between these spaces found in the triangle above, their compositions and fibred products.

There are natural isomorphisms as follows:

a) $\mathcal{U}_{(j,j'+1)}(X) \cong \mathcal{U}_{(j,j')}(X) \times_{X_{(j'-1,j')}} X_{(j'-\frac{1}{2},j'+\frac{1}{2})}$.

b) *The universal family $\mathcal{U}_{(j,j')}(X)$ over $M_{(j,j')}(X)$ is isomorphic with the inverse limit of the inverse family $I(j, j')$.*

Proof. We first claim that $\mathcal{U}_{(j,j'+1)}(X)$ is naturally isomorphic to

$$m_{j,j'}^{j,j'+1*}\mathcal{U}_{(j,j')}(X) \times_{m_{j'-1,j'}^{j,j'+1*}\mathcal{U}_{(j'-1,j')}(X)} m_{j'-1,j'+1}^{j,j'+1*}\mathcal{U}_{(j'-1,j'+1)}(X). \tag{3.8}$$

Indeed, within each fibre over a point b in $M_{(j,j'+1)}(X)$, both maps

$$\begin{aligned} \mathcal{U}_{(j,j'+1)}(X) &\to m_{j,j'}^{j,j'+1*}\mathcal{U}_{(j,j')}(X) \text{ and} \\ m_{j'-1,j'+1}^{j,j'+1*}\mathcal{U}_{(j'-1,j'+1)}(X) &\to m_{j'-1,j'}^{j,j'+1*}\mathcal{U}_{(j'-1,j')}(X) \end{aligned}$$

contract exactly those irreducible components $\mathcal{C}_\infty$ which contain $s_\infty(b)$ and such that $\mathcal{C}_\infty \setminus s_\infty(b)$ is mapped into $X_{j'}^+$. The maps

$$\begin{aligned} \mathcal{U}_{(j,j'+1)}(X) &\to m_{j'-1,j'+1}^{j,j'+1*}\mathcal{U}_{(j'-1,j'+1)}(X) \text{ and} \\ m_{j,j'}^{j,j'+1*}\mathcal{U}_{(j,j')}(X) &\to m_{j'-1,j'}^{j,j'+1*}\mathcal{U}_{(j'-1,j')}(X) \end{aligned}$$

contract exactly those (unions of) components $\mathcal{C}'$ which do not contain $s_\infty(b)$ and whose image in X is sent by the moment map to $[0, i_{j'-1}]$. Thus $\mathcal{U}_{(j,j'+1)}(X)$ can be obtained by gluing the complements of the contraction loci of the spaces $m_{j,j'}^{j,j'+1*}\mathcal{U}_{(j,j')}(X)$ and $m_{j'-1,j'+1}^{j,j'+1*}\mathcal{U}_{(j'-1,j'+1)}(X)$ along their image in $m_{j'-1,j'}^{j,j'+1*}\mathcal{U}_{(j'-1,j')}(X)$.

Furthermore, by definition, the pullback $m_{j'-1,j'+1}^{j,j'+1*}\mathcal{U}_{(j'-1,j'+1)}(X)$ is naturally isomorphic to

$$m_{j'-1,j'}^{j,j'+1*}\mathcal{U}_{(j'-1,j')}(X) \times_{m_{j'-1,j'}^{j'-1,j'+1*}\mathcal{U}_{(j'-1,j')}(X)} \mathcal{U}_{(j'-1,j'+1)}(X), \tag{3.9}$$

and on the other hand

$$m_{j,j'}^{j,j'+1*}\mathcal{U}_{(j,j')}(X) \cong \mathcal{U}_{(j,j')}(X) \times_{\mathcal{U}_{(j'-1,j')}(X)} m_{j'-1,j'}^{j'-1,j'+1*}\mathcal{U}_{(j'-1,j')}(X) \tag{3.10}$$

follows directly from the isomorphism

$$M_{(j,j'+1)}(X) \cong M_{(j,j')}(X) \times_{M_{(j'-1,j')}(X)} M_{(j'-1,j'+1)}(X)$$

(see Corollary 3.13). Putting together (3.8), (3.9), and (3.10) we obtain a natural isomorphism

$$\mathcal{U}_{(j,j'+1)}(X) \cong \mathcal{U}_{(j,j')}(X) \times_{\mathcal{U}_{(j'-1,j')}(X)} \mathcal{U}_{(j'-1,j'+1)}(X). \tag{3.11}$$

By Corollary 3.25, we have $\mathcal{U}_{(j'-1,j')}(X) \cong X_{(j'-1,j')}$ and

$$\mathcal{U}_{(j'-1,j'+1)}(X) \cong X_{(j'-1,j'+\frac{1}{2})} \times_{X_{(j'-1,j'+1)}} X_{(j'-\frac{1}{2},j'+1)} \cong X_{(j'-\frac{1}{2},j'+\frac{1}{2})}.$$

Hence equation (3.11) becomes:

$$\mathcal{U}_{(j,j'+1)}(X) \cong \mathcal{U}_{(j,j')}(X) \times_{X_{(j'-1,j')}} X_{(j'-\frac{1}{2},j'+\frac{1}{2})},$$

which proves part (a).

To prove that the universal family $\mathcal{U}_{(j,j')}(X)$ over $M_{(j,j')}(X)$ is isomorphic with the inverse limit of the inverse family $I(j,j')$, we first show that there exist natural maps $\mathcal{U}_{(j,j')}(X) \to X_{(l,l')}$ to all the spaces in the family $I(j,j')$. Indeed, the weight choices for $\mathcal{U}_{(j,j')}(X)$

$$a_0 = 1 - ai_j,\ a_\infty = 1 - a(d - i_{j'}),\ 0 \le a_1 < 1 - da$$

are larger than the respective weights for $X_{(l-\frac{1}{2},l'+\frac{1}{2})} \hookrightarrow \overline{M}_{0,\mathcal{A}(i_l-\frac{1}{2},i_{l'}+\frac{1}{2},i_{l'}-i_l+\frac{1}{4})}(\mathbb{P}^n,d,a)$ whenever $0 \le j < l \ge l' < j' \le k$:

$$a_0 := 1 - (i_{l'} - \tfrac{1}{2})a,\ a_\infty = 1 - (d - i_l - \tfrac{1}{2})a,\ a_1 = (i_{l'} - i_l - \tfrac{3}{4})a,$$

which confirms the existence of a morphism $\mathcal{U}_{(j,j'+1)}(X) \to X_{(l+\frac{1}{2},l'-\frac{1}{2})}$ for the given ranges of l, l'. Furthermore, each space in the family $I(j,j')$ is the targer of a map in the inverse system, with domain of the form $X_{(l+\frac{1}{2},l'-\frac{1}{2})}$ for suitable l, l' as above.

To finalize part (b), we apply induction on $j' > j$: In the case when $j' = j+1$, we know $\mathcal{U}_{(j,j+1)}(X) \cong X_{(j,j+1)}$. We now assume that $\mathcal{U}_{(j,j')}(X)$ does satisfy the universal property of the inverse limit of the inverse family $I(j,j')$, and prove the same for $\mathcal{U}_{(j,j'+1)}(X)$ and the inverse family $I(j,j'+1)$. Indeed, any scheme having compatible morphisms into all the spaces $X_{(l,l')}$ of $I(j,j'+1)$ will, in particular, have a natural morphism to $\mathcal{U}_{(j,j')}(X)$ (by the induction hypothesis), and to $X_{(j'-\frac{1}{2},j'+\frac{1}{2})}$, and hence, by part (a), to $\mathcal{U}_{(j,j'+1)}(X)$. The naturality insures that $\mathcal{U}_{(j,j'+1)}(X)$ is indeed the inverse limit of the family.

□

4. Computations in equivariant K–theory

Let β be the class of the map $\mathbb{P}^1 \to X$ given by $t \to t \cdot x$ for x generic in X. In the previous section we have described in detail the fixed locus $M_{(0,k)}(X) = M_{\omega,2}$, where $\omega = ((u_0, u_\infty), \beta)$, and u_0, u_∞ are the source and sink of the graph associated to the action. We will now discuss the contribution of the fixed locus $M_{\omega,2} = M_{(0,k)}(X)$ to the computation of the virtual fundamental class of $\overline{M}_{0,2}(X,\beta)$. This contribution consists in two components: a fixed part $[M_{\omega,2}]^{vir}$ and a moving part contribution $e^{\mathbb{C}^*}(N^{vir}_{M_{\omega,2}})$. The fixed part of the virtual fundamental class can be recovered as a special case of Theorem 6.3 in [M]:

Consider the Cartesian diagram

$$\begin{array}{ccc} M_{(0,k)}(X) & \longrightarrow & \prod_{j=0}^{k-2} M_{(j,j+2)}(X) \\ \downarrow & & \downarrow p \\ \Delta : \prod_{j=0}^{k-1} M_{(j,j+1)}(X) & \longrightarrow & \prod_{j=0}^{k-2} M_{(j,j+1)}(X) \times M_{(j+1,j+2)}(X) \end{array}$$

Here Δ is the diagonal map and $p = (m^{j,j+2}_{j,j+1}, m^{j,j+2}_{j+1,j+2})_j$. In this context,

$$[M_{(0,k)}(X)]^{vir} = p^*([\Delta]).$$

This is indeed a cycle in $M_{(0,k)}(X)$ as $M_{(0,k)}(X) = p^{-1}(\text{ Im } \Delta)$.

For the remainder of this section we will focus on the moving part.

The virtual normal bundle $N^{vir}_{M_{(0,k)}(X)}$ satisfies

$$[N^{vir}_{M_{(0,k)}(X)}] = \chi_{\neq 0}(R^\bullet p_{(0,k)*}(ev^*(\mathcal{T}_X))),$$

in the equivariant K-theory of $M_{(0,k)}(X)$. The equivariant K-groups come with a grading given by the $\mathbb{C}^*$-action, and for any complex $N^\bullet$, the term $\chi_{\neq 0}(N^\bullet)$ represents the positive degree part of $\chi(N^\bullet)$. The formula above follows directly from [B2], [GP] together with [M]. In general one should also consider the contribution from the relative cotangent complex of $m : M_{0,2}(X,\beta) \to \mathfrak{M}_{0,2}$, where $\mathfrak{M}_{0,2}$ is the Artin stack of rational curves. However the relative cotangent complex above does not contribute to the moving part for $M_{(0,k)}(X)$ because, given any map parametrized by a point in $M_{(0,k)}(X)$, the deformations of the domain curve that keep s_0 and s_∞ fixed are $\mathbb{C}^*$-equivariant.

Our strategy for computing $\chi_{\neq 0}(R^\bullet p_{(0,k)*}(ev^*(\mathcal{T}_X)))$ will be the following: we will write the class of $ev^*(\mathcal{T}_X)$ in the equivariant K-theory of $\mathcal{U}_{(0,k)}(X)$ as a sum of classes supported on the fixed point locus of $\mathcal{U}_{(0,k)}(X)$ together with classes which are mainly pulled-back from $M_{(0,k)}(X)$ or some substrata of $M_{(0,k)}(X)$. These last classes will not contribute to $\chi_{\neq 0}(R^\bullet p_{(0,k)*}(ev^*(\mathcal{T}_X)))$ as their push-forward via $p_{(0,k)}$ will have trivial $\mathbb{C}^*$-action.

Proposition 4.1. *Recall that $\varphi_{(l,l')} : \mathcal{U}_{(l,l')} \to X_{(l,l')}$ is the natural "evaluation map" and for $j \leq l < l' \leq j'$, we denoted by $u^{j,j'}_{l,l'} : \mathcal{U}_{(j,j')}(X) \to \mathcal{U}_{(l,l')}(X)$ the natural maps between the universal families over $m^{j,j'}_{l,l'} : M_{(j,j')}(X) \to M_{(l,l')}(X)$.*

Let $g^{jj'}_{ll'} := \varphi_{(l,l')} \circ u^{l,l'}_{j,j'}$. For any $(j+1) < l \leq l' \leq j'$, the following equation holds in the equivariant K-groups of $\mathcal{U}_{(j,j')}(X)$:

$$[g^{jj'*}_{(j+1)l}(\mathcal{T}_{X_{(j+1,l)}})] - [g^{jj'*}_{jl}(\mathcal{T}_{X_{(j,l)}})] = [g^{jj'*}_{(j+1)l'}(\mathcal{T}_{X_{(j+1,l')}})] - [g^{jj'*}_{jl'}(\mathcal{T}_{X_{(j,l')}})],$$

where $\mathcal{T}_{X_{(j,l)}}$ is the tangent bundle to $X_{(j,l)}$.

Proof. We will identify the component X_{j+1} of the fixed locus in X with its preimage in $X_{(j,l)}$. Using the stratification given by the $\mathbb{C}^*$-action on $X_{j,l}$ we denoted

$$\overline{X^-_{j+1}} = \overline{\{x \in X_{(j,l)} \text{ such that } \lim_{t\to\infty} tx \in X_{j+1}\}}.$$

There is an open set of $X_{j,l}$ containing $\overline{X^-_{j+1}}$ on which the natural birational map between $X_{(j,l)}$ and $X_{(j,l')}$ is an isomorphism. On the other hand $g^{jj'}_{(j+1)l} * (\mathcal{T}_{X_{(j+1,l)}})$ and $g^{jj'}_{jl}(\mathcal{T}_{X_{(j,l)}})$ are isomorphic outside $g^{j(j'-1)}_{jl}(\overline{X^-_{j+1}})$, and also in a neighborhood U around $g^{j(j'-1)}_{jl}(\overline{X^-_{j+1}})$,

$$g^{jj'*}_{(j+1)l}(\mathcal{T}_{X_{(j+1,l)}})_{|U} = g^{jj'*}_{(j+1)l'}(\mathcal{T}_{X_{(j+1,l')}})_{|U}$$

and

$$g^{jj'*}_{jl}(\mathcal{T}_{X_{(j,l)}})_{|U} = g^{jj'*}_{jl'}(\mathcal{T}_{X_{(j,l')}})_{|U}.$$

□

The following proposition will be our main tool in comparing classes of tangent bundles and their pull-backs:

Proposition 4.2. *Let Z denote a smooth Deligne-Mumford stack and let Y be a smooth substack. For a suitable increasing filtration of the ideal sheaf of Y in X, we denote by $f : \widetilde{Z} \to Z$ the weighted blow-up of Z along Y (as defined in* [MM2]*, section 2.1). Let $N_{Y|Z} = \oplus_w Q_w$ be the decomposition of the normal bundle after the weights associated to the weighted blow-up, where w denotes the weight of Q_w.*

We will denote by $g : E \to Y$ the restriction of f to the exceptional divisor and by $i : E \hookrightarrow \widetilde{Z}$ the embedding of the exceptional divisor into $\widetilde{Z}$.

The following equation holds in the K-theory of $\widetilde{Z}$ with rational coefficients:

$$[\mathcal{T}_{\widetilde{Z}}] - [f^*(\mathcal{T}_Z)] = i_*([\mathcal{O}_E(E)] - \sum_w \sum_{0\le j<w} [g^*(Q_w)\otimes \mathcal{O}_E(-jE)]).$$

Moreover, if $\mathbb{C}^*$ acts on Z and the embedding of Y in Z is equivariant, then weighted blow-up map is equivariant then the same formula holds in the equivariant K-theory with rational coefficients.

Proof. The proposition follows directly from Proposition 2.12 and Theorem 2.13 from [MM2] after translating from the Chow classes to classes in K-theory, and using the short exact sequences on $\widetilde{Z}$:

$$0 \to \mathcal{O}(-(j+1)E) \longrightarrow \mathcal{O}(-jE) \longrightarrow \mathcal{O}_E(-jE) \to 0$$

for $0 \le j < w$. □

We apply the the above proposition in the context given by the following commutative diagram of weighted blow-ups

$$\begin{tikzcd}
 & & X_{(j,j+2)} & & \\
 & X_{(j,j+\frac{3}{2})} \arrow[ur, "v^{j,j+\frac{3}{2}}_{j,j+2}"] & & X_{(j+\frac{1}{2},j+2)} \arrow[ul, "v^{j+\frac{1}{2},j+2}_{j,j+2}"'] & \\
X_{(j,j+1)} \arrow[d, "\cong"] & & X_{(j+\frac{1}{2},j+\frac{3}{2})} \arrow[ul, "v^{j,j+\frac{3}{2}}_{j,j+1}"'] \arrow[ur, "\varphi_{(j,j+2)}"] \arrow[ll, "u^{j,j+2}_{j,j+1}"'] \arrow[rr, "u^{j,j+2}_{j+1,j+2}"] \arrow[d, "\cong"] \arrow[dl] \arrow[dr] & & X_{(j+1,j+2)} \arrow[ur, "v^{j+\frac{1}{2},j+2}_{j+1,j+2}"] \arrow[d, "\cong"'] \\
\mathcal{U}_{(j,j+1)}(X) \arrow[d, "p_{(j,j+1)}"] & m^{j,j+2*}_{j,j+1}\mathcal{U}_{(j,j+1)}(X) \arrow[l, "H"'] \arrow[dr, "q"] & \mathcal{U}_{(j,j+2)}(X) \arrow[l, "b"] \arrow[r] \arrow[d, "p_{(j,j+2)}"] & m^{j,j+2*}_{j,j+1}\mathcal{U}_{(j+1,j+2)}(X) \arrow[r] \arrow[dl] & \mathcal{U}_{(j+1,j+2)}(X) \arrow[d, "p_{(j+1,j+2)}"] \\
M_{(j,j+1)}(X) & & M_{(j,j+2)}(X) \arrow[ll, "m^{j,j+2}_{j,j+1}"'] \arrow[rr, "m^{j,j+2}_{j+1,j+2}"] & & M_{(j+1,j+2)}(X)
\end{tikzcd}$$

With the identification $\mathcal{U}_{(j,j+2)} \simeq X_{(j+\frac{1}{2},j+\frac{3}{2})}$, we can write $\varphi_{(j,j+2)} : \mathcal{U}_{(j,j+2)} \to X_{(j,j+2)}$ as the composition of 2 weighted blow-ups. As well, using $\mathcal{U}_{(j,j+1)} \simeq X_{(j,j+1)}$, note that $u^{j,j+2}_{j,j+1}$ factorizes through $m^{j,(j+2)*}_{j,(j+1)}(\mathcal{U}_{(j,j+1)})$, where $\mathcal{U}_{(j,j+2)} \to m^{j,j+2*}_{j,j+1}(\mathcal{U}_{(j,j+1)})$ is the standard blow-up of a codimension 2 stratum and $m^{j,j+2*}_{j,j+1}(\mathcal{U}_{(j,j+1)}) \to \mathcal{U}_{(j,j+1)}$ is a weighted blow-up.

X_{j+1} is a component of the fixed locus in $X_{(j,j+2)}$, included in the two blow-up loci $\overline{X^+_{j+1}}$ and $\overline{X^-_{j+1}}$. Let $\mathcal{N}_{X_{j+1}|X^+_{j+1}} = \oplus_w Q^+_{w,j+1}$ and $\mathcal{N}_{X_{j+1}|X^-_{j+1}} = \oplus_w Q^-_{w,j+1}$ denote the positive and negative parts of the normal bundle of X_{j+1} in $X_{(j,j+2)}$ with their weight decompositions. The two exceptional divisors T^+_{j+1} and T^-_{j+1} in $\mathcal{U}_{(j,j+2)}$ intersect after $\mathbb{P}^w(\mathcal{N}_{X_{j+1|X^+_{j+1}}}) \times_{X_j} \mathbb{P}^w(\mathcal{N}_{X_{j+1|X^-_{j+1}}})$. We will denote by i^+_{j+1}, i^-_{j+1} and i^0_{j+1} the embeddings of T^+_{j+1}, T^-_{j+1} and $T^+_{j+1} \cap T^-_{j+1}$ respectively in $\mathcal{U}_{(j,j+2)}$, and by $\oplus_w \widetilde{Q}^-_{w,j+1}$ and $\oplus_w \widetilde{Q}^+_{w,j+1}$ the pullbacks of the normal bundles of $\overline{X^+_{j+1}}$ and $\overline{X^-_{j+1}}$ to T^+_{j+1} and T^-_{j+1} respectively.

For every $0 \le j \le k-2$ we will denote by $\widetilde{T}^+_j$, $\widetilde{T}^-_j$ and $\widetilde{T}_j$ the pseudo-divisors given by the pull-backs to $\mathcal{U}_{(0,k)}(X)$ of T^+_j, T^-_j and $T^+_j + T^-_j$ via $u^{0,k}_{j,j+2}$.

The complex

$$(\mathcal{O}_{\mathcal{U}_{(0,k)}(X)} \to u^{0,k*}_{j,j+2}\mathcal{O}_{\mathcal{U}_{(j,j+2)}(X)}(T^+_j)) =: \mathcal{O}_{\widetilde{T}^+_j}$$

is supported on $\widetilde{T}^+_j$. Similarly we define $\mathcal{O}_{\widetilde{T}^-_j}$ and $\mathcal{O}_{\widetilde{T}_j}$. Let $D_j := \widetilde{T}^+_j \cap \widetilde{T}^-_j$ and $\mathcal{O}_{D_j} = (\mathcal{O}_{\mathcal{U}_{(0,k)}(X)} \oplus \mathcal{O}_{\mathcal{U}_{(0,k)}(X)} \to u^{0,k*}_{j,j+2}\mathcal{O}_{\mathcal{U}_{(j,j+2)}(X)}(T^+_j) \oplus u^{0,k*}_{j,j+2}\mathcal{O}_{\mathcal{U}_{(j,j+2)}(X)}(T^-_j))$. We will denote by $Q^{\pm}_{w|D_j} := \oplus u^{0,k*}_{j,j+2}\varphi^*_{(j,j+2)}(Q^{\pm}_w) \otimes \mathcal{O}_{D_j}$.

Proposition 4.3. *With the notations from above, we have:*

a) The following relation holds on $\mathcal{U}_{(j,j+2)}(X)$:

$$\begin{aligned}
[T\mathcal{U}_{(j,j+2)}(X)] \;-\; & [\varphi^*_{(j,j+2)}\mathcal{T}_{X_{(j,j+2)}}] = \\
= \;& i^+_{j+1*}([\mathcal{O}(T^+_{j+1})] - \sum_w \sum_{0\le l<w} [\widetilde{Q}^-_{w,j+1} \otimes \mathcal{O}(-lT^+_{j+1})]) \\
+ \;& i^-_{j+1*}([\mathcal{O}(T^-_{j+1})] - \sum_w \sum_{0\le l<w} [\widetilde{Q}^+_{w,j+1} \otimes \mathcal{O}(-lT^-_{j+1})]).
\end{aligned}$$

$$\begin{aligned}
b)[\varphi^*_{(j,j+2)}\mathcal{T}_{X_{(j,j+2)}}] \;-\; & [u^{j,j+2*}_{j,j+1}\mathcal{T}_{X_{(j,j+1)}}] = [\mathcal{O}(s_\infty)] - [u^{j,j+2*}_{j,j+1}\mathcal{O}(s_\infty)] - \\
- \;& i^0_{j+1*}(\sum_w \sum_{0\le l<w} \sum_{1\le m\le w-l} [Q^+_{w,j+1} \otimes \mathcal{O}(mT^+_{j+1} - lT^-_{j+1})]) \\
- \;& i^0_{j+1*}(\sum_w \sum_{0\le l<w} \sum_{1\le m\le w-l} [Q^-_{w,j+1} \otimes \mathcal{O}(mT^-_{j+1} - lT^+_{j+1})]) + \\
+ \;& i^-_{j+1*}c_- + i^+_{j+1*}c_+
\end{aligned}$$

where c_-, c_+ are classes pulled-back from the boundary divisor $E_{(j,j+2)}$ in $M_{(j,j+2)}(X)$ to T^-_{j+1} and T^+_{j+1}, respectively. Both T^-_{j+1} and T^+_{j+1} are $\mathbb{P}^1$-bundles over $E_{(j,j+2)}$, and $E_{(j,j+2)} \cong T^-_{j+1} \bigcap T^+_{j+1}$.

We denoted by $\mathcal{O}(s_\infty)$ the line bundle of the divisor given by the image of s_∞, in the relevant universal family. By a slight abuse of notation, we denoted by $Q^{\pm}_{w,j+1}$ the pull-back of $Q^{\pm}_{w,j+1}$ to $T^+_j \cap T^-_j$.

c) Let $P^+_{w,l,j+1} := -p_{(j,j+2)*}i^0_{j+1*}(\sum_{1\le m\le w-l}[Q^+_{w,j+1} \otimes \mathcal{O}(mT^+_{j+1} - lT^-_{j+1})])$. The pull-back of $P^+_{w,l,j+1}$ to the exceptional divisor $E_{(j,j+2)}$ of $M_{(j,j+2)}(X)$ is

$$[Q^+_{w,j+1} \otimes \mathcal{O}(lT^+_{j+1})] - [Q^+_{w,j+1} \otimes \mathcal{O}(-lT^-_{j+1})].$$

Assuming that there are vector bundles $K^+_{w,j+1}$, $L^{\pm}_{j+1}$ on $M_{(j,j+2)}(X)$ whose pullbacks to $E_{(j,j+2)}$ are $Q^+_{w,j+1}$ and $\mathcal{O}(T^{\pm}_{j+1})$ respectively, and such that $\mathcal{O}(E_{(j,j+2)}) = \mathcal{O}(L^+_{j+1} + L^-_{j+1})$ then

$$P^+_{w,l,j+1} = [K^+_{w,j+1} \otimes \mathcal{O}(lL^+_{j+1})] - [K^+_{w,j+1} \otimes \mathcal{O}(-lL^-_{j+1})].$$

d) The following relation holds in the equivariant K-theory of $M_{(0,k)}(X)$:

$$\begin{aligned}
\chi_{\neq 0}(p_{(0,k)*}[ev^*(\mathcal{T}_X)]) \;=\; & \sum_w \sum_{0\le l<w} [Q^+_{w,0} \otimes \mathcal{O}(ls_0)] + \sum_w \sum_{0\le l<w} [Q_{w,k}]^- \otimes \mathcal{O}(ls_\infty)] \\
+ \;& \sum_{j=1}^{k-1} \sum_w \sum_{0\le l<w} P^+_{w,l,j} + \sum_{j=1}^{k-1} \sum_w \sum_{0\le l<w} P^-_{w,l,j}.
\end{aligned}$$

Proof. a)This is just a direct application of Proposition 4.2. We use the fact that the strata $\overline{X^+_{j+1}}$ and $\overline{X^-_{j+1}}$ are transverse to each other and so that the normal bundle of the strict transform of $\overline{X^+_{j+1}}$ in $X_{(j+\frac{1}{2},j+2)} = Bl^w_{\overline{X^-_{j+1}}} X_{(j,j+2)}$ is the pull-back of the normal bundle of $\overline{X}^+_{j+1}$ in $X_{(j,j+2)}$.

b) Denote by $\widetilde{\overline{X^-_{j+1}}}$ the strict transform of $\overline{X^-_{j+1}}$ in $X_{(j,j+\frac{3}{2})}$, and by E the exceptional divisor for the morphism $X_{(j,j+\frac{3}{2})} \to X_{(j,j+2)}$. The blow-down morphism $v^{j,j+\frac{3}{2}}_{j,j+1} : X_{(j,j+\frac{3}{2})} \to X_{(j,j+1)}$ has the same exceptional divisor E. The normal bundle N of $v^{j,j+\frac{3}{2}}_{j,j+1}(\widetilde{\overline{X^-_{j+1}}}) \cong \overline{X^-_{j+1}}$ in $X_{(j,j+1)}$ is the pull-back of the normal bundle of $Y^-_{(j,j+1)} = p_{j,j+1}(v^{j,j+\frac{3}{2}}_{j,j+1}(\widetilde{\overline{X^-_{j+1}}}))$ in $M_{(j,j+1)}(X)$. As the blow-up locus of $v^{j,j+\frac{3}{2}*}_{j,j+1}$ is embedded in $v^{j,j+\frac{3}{2}}_{j,j+1}(\widetilde{\overline{X^-_{j+1}}})$, by local computations

$$\begin{aligned} u^{j,j+2*}_{j,j+1} N &= v^{j+\frac{1}{2},j+\frac{3}{2}*}_{j,j+\frac{3}{2}} v^{j,j+\frac{3}{2}*}_{j,j+1}(N) = \oplus_w \widetilde{Q}^+_{w,j+1} \otimes \mathcal{O}(wT^+_{j+1}) \\ &= q^{-*}_{j,j+2} m^{j,j+2*}_{j,j+1} \mathcal{N}_{Y^-_{(j,j+1)}|M_{(j,j+2)}(X)}, \end{aligned}$$

where $q^{-*}_{j,j+2} : T^-_{j+1} \to E_{(j,j+2)}$ is the projection to the exceptional divisor in $M_{j,j+2}(X)$.

Furthermore, $T^+_{j+1} = v^{j+\frac{1}{2},j+\frac{3}{2}*}_{j,j+\frac{3}{2}}(E)$ and so $Q^+_{w,j+1} \otimes \mathcal{O}(-lT^-_{i+1} + (w-l)T^+_{j+1})$ is the pull-back of a class from the exceptional divisor in $M_{j,j+2}(X)$.

We use now that

$$\begin{aligned} i^-_{j+1*}([\widetilde{Q}^+_{w,j+1} \otimes \mathcal{O}(-lT^-_{j+1})] &- [\widetilde{Q}^+_{w,j+1} \otimes \mathcal{O}(-lT^-_{i+1} + (w-l)T^+_{j+1})]) = \\ &- \sum_{0<m\le w-l} i^0_{j+1*}([Q^+_{w,j+1} \otimes \mathcal{O}(-lT^-_{i+1} + mT^+_{j+1})]) \end{aligned}$$

which follows from successive application of short exact sequences

$$0 \to \mathcal{O}((m-1)T^+_{j+1}) \longrightarrow \mathcal{O}(mT^+_{j+1}) \longrightarrow \mathcal{O}_{T^+_{j+1}}(mT^+_{j+1}) \to 0$$

tensored with the bundles above.

The formula now follows from the application of Proposition 4.2 for the morphism $\mathcal{U}_{(j,j+2)} \to X_{(j,j+\frac{3}{2})}$, combined with formula (a) and the computations above.

c) The composition $p_{(j,j+2)} \circ i^0_{j+1} : T^+_{j+1} \cap T^-_{j+1} \to M_{(j,j+2)}(X)$ is an isomorphism on its image $E_{(j,j+2)}$.

Assume that there are vector bundles $K^+_{w,j+1}$, $L^\pm_{j+1}$ on $M_{(j,j+2)}(X)$ whose pullbacks to $E_{(j,j+2)}$ are $Q^+_{w,j+1}$ and $\mathcal{O}(T^\pm_{j+1})$ respectively, and such that $\mathcal{O}(E_{(j,j+2)}) = \mathcal{O}(L^+_{j+1} + L^-_{j+1})$, From the short exact sequence

$$0 \to \mathcal{O} \longrightarrow \mathcal{O}(E_{(j,j+2)}) \longrightarrow \mathcal{O}_{E_{(j,j+2)}}(E_{(j,j+2)}) \to 0$$

tensored with relevant vector bundles, we get

$$\begin{aligned} [K^+_{w,j+1} \otimes \mathcal{O}(kL^+_{j+1})] &- [K^+_{w,j+1} \otimes \mathcal{O}(-kL^-_{j+1})] = \\ &= i_*(\sum_{k=1}^{w} \sum_{l=1}^{k-1} [Q^+_{w,j+1} \otimes \mathcal{O}(kT^+_{j+1} - l\mathcal{O}(E_{(j,j+2)}))], \end{aligned}$$

for the embedding $i : E_{(j,j+2)} \hookrightarrow M_{(j,j+2)}(X)$.

After a change of variables $m = k - l$, the last term above becomes $P^+_{w,l,j+1}$.

After replacing $M_{(j,j+2)}(X)$ by the normal bundle of $E_{(j,j+2)}$ in $M_{(j,j+2)}(X)$, the assumptions are satisfied in this new ambient space. We can now get the formula for the pull-back of $P^+_{w,l,j+1}$ to $E_{(j,j+2)}$ by applying the arguments above in this new context, and then pulling-back to $E_{(j,j+2)}$.

$$0 \to \mathcal{O}(-(j+1)E) \longrightarrow \mathcal{O}(-jE) \longrightarrow \mathcal{O}_E(-jE) \to 0$$

For part d) we apply induction on k. At the initial step we use the relative Euler sequence for $\mathbb{P}^1$-bundles to get the relative tangent bundle of $\mathcal{U}_{(0,1)}(X) \to M_{(0,1)}(X)$ as $\mathcal{O}(s_0) \oplus \mathcal{O}(s_\infty)/\mathcal{O}_{\mathcal{U}_{(0,1)}(X)}$. Then we apply formula (b) to get the formula for $k = 2$. The induction step from k to $k+1$ uses the same formula (b) for $j = k-1$ and Proposition 4.1.

□

In order to obtain a nice closed formula for the moving part $e^{\mathbb{C}^*}(\mathcal{N}^{vir}_{M_{0,k}(X)})$, we will assume that the conditions of part c) of the previous Proposition are satisfied. That is, we assume there are classes of vector bundles $K^\pm_{w,j}$ and line bundles $L^\pm_{w,j}$ on $M_{0,k}(X)$ such that $K^\pm_{w,j} \otimes \mathcal{O}(D_j) = m^{0,k*}_{i,i+2}(Q^\pm_{w,j}) \otimes \mathcal{O}(D_j)$ and $L^\pm_j \otimes \mathcal{O}(D_j) = m^{0,k*}_{i,i+2}\mathcal{O}(T^\pm_j) \otimes \mathcal{O}(D_j)$. We denote by $c_K(t)$ the Chern polynomial of the vector bundle K. Let $p_N(t) := t^{\deg c_N} c_N(\frac{1}{t})$ and $\xi^\pm_j := c_1(L^\pm_j)$.

Theorem 4.4. *With the above notations*

$$e^{\mathbb{C}^*}(\chi_{\neq 0}(R^\bullet p_{(0,k)*}(ev^*(\mathcal{T}_X)))) = \prod_w \prod_{l=0}^{w-1} p_{Q^+_{w,0}}(w - lt - l\psi_1) \prod_w \prod_{l=0}^{w-1} p_{Q^-_{w,k}}((-w+l)t - l\psi_2)$$

$$\prod_{j=1}^{k-1} \frac{\prod_w \prod_{l=0}^{w-1} p_{K^+_{w,j}}((w-l)t + \xi^+_j) \prod_w \prod_{l=0}^{w-1} p_{K^-_{w,j}}((-w+l)t + \xi^-_j)}{\prod_w \prod_{l=0}^{w-1} p_{K^+_{w,j}}((w-l)t - \xi^-_j) \prod_w \prod_{l=0}^{w-1} p_{K^-_{w,j}}((-w+l)t - \xi^+_j)}.$$

4.1. **Moduli spaces $M_{\omega,2}$ and their virtual classes.** Let's consider first the case $\omega = ((u_0, u_\infty), n\beta)$, where β is the class of the generic curve. This case can be treated in the same way as $\omega = ((u_0, u_\infty), \beta)$ by composing the action of $\mathbb{C}^*$ on X with the map $\mathbb{C}^* \times X \to \mathbb{C}^* \times X$ which is the quotient by the cyclic subgroup $\mathbb{Z}_n$ on the first component, and identity on the second. We now have a new action of $\mathbb{C}^*$ on X. In this case there is a morphism $M_{(0,k)}(X) \to M_{\omega,2}$ whose fiber over points in $M_{\omega,2}$ parameterizing irreducible curves is $B\mathbb{Z}_n = [\text{point}/\mathbb{Z}_n]$, while over points parameterizing a chain of k curves, the fiber is $B(\mathbb{Z}_n)^k$. For this reason we will denote $M_{\omega,2} =: (M_{(0,k)}(X)/\mathbb{Z}_n)$.

In this case, the fixed part of the virtual class works is calculated be the same formula as in the previous section, with

$$\Delta : (M_{(j,j+1)}(X)/\mathbb{Z}_n) \to \prod_{j=0}^{k-2} (M_{(j,j+1)}(X)/\mathbb{Z}_n) \times (M_{(j+1,j+2)}/\mathbb{Z}_n)$$

and $p : \prod_{j=0}^{k-2}(M_{(j,j+2)}/\mathbb{Z}_n) \to \prod_{j=0}^{k-2}(M_{(j,j+1)}(X)/\mathbb{Z}_n) \times (M_{(j+1,j+2)}/\mathbb{Z}_n)$

Similarly, the virtual normal bundle of $M_{\omega,2}$ is

$$\begin{aligned}\chi_{\neq 0}(R^\bullet p^n_{(0,k)*}ev^*(\mathcal{T}_X)) &= \sum_w \sum_{0\le l<nw} [Q^+_{w,0}\otimes \mathcal{O}(ls_0)] + \sum_w \sum_{0\le l<nw} [Q^-_{w,l}]\otimes \mathcal{O}(ls_\infty)] \\ &+ \sum_{j=1}^{k-1}\sum_w \sum_{0\le l<nw} P^+_{w,l,j} + \sum_{j=1}^{k-1}\sum_w \sum_{0\le l<nw} P^-_{w,l,j}.\end{aligned}$$

We consider now $M_{\omega,2}$ in the case when $\omega = ((u,v), n\beta)$ when $\beta = v-u$ is not the class of the generic orbit and $n\in\mathbb{Z}_{>0}$. The choice of the moment map $\mu : X\to\mathbb{R}$ depends on the choice of the line bundle which induces a linear map $p : H^{1,1}(X)^\vee\to\mathbb{R}$ such that $\mu = p\circ\mu^m$. If $p(u) = i_j$ and $p(v) = i_q$ are the j-th and the q-th walls of μ, then there is a natural closed embedding $M_{\omega,2}\subset (M_{(j,q)}(X)/\mathbb{Z}_n)$. Moreover, the fixed part of the virtual class of $M_{\omega,2}$ is $h^*_{j+1,q}\mathbb{P}^w(N_{X_j|X_j^+})\cap h^*_{j,q-1}(\mathbb{P}^w(N_{X_q|X_q^-})$ for the natural maps $h_{j+1,q} : (M_{(j,q)}(X)/\mathbb{Z}_n)\to (M_{(j+1,q)}(X)/\mathbb{Z}_n)$ and $h_{j,q-1} : (M_{(j,q)}(X)/\mathbb{Z}_n)\to (M_{(j,q-1)}(X)/\mathbb{Z}_n)$.

The class of the virtual normal bundle is

$$\begin{aligned}&\chi_{\neq 0}(R^\bullet p^n_{(j,q)*}ev^*(\mathcal{T}_X)) = \\ &= \sum_w \sum_{0\le l<nw} [Q^+_{w,j}\otimes \mathcal{O}(l\widetilde{T_j}^+)] - \sum_w \sum_{0\le l<nw} [Q^-_{w,j}\otimes \mathcal{O}(-l\widetilde{T_j}^+)] \\ &+ \sum_w \sum_{0\le l<nw} [Q_{w,q}]^-\otimes \mathcal{O}(l\widetilde{T_q}^-)] - \sum_w \sum_{0\le l<nw} [Q_{w,q}]^+\otimes \mathcal{O}(-l\widetilde{T_q}^-)] \\ &+ \sum_{s=j+1}^{q-1}\sum_w \sum_{0\le l<nw} P^+_{w,l,s} + \sum_{s=j+1}^{q-1}\sum_w \sum_{0\le l<w} P^-_{w,l,s}\end{aligned}$$

These formulae follow from those in Proposition 4.3 together with the product axiom of [B2].

Finally, we consider $M_{\omega,2}$ in the case when $\omega = ((u,v), \frac{n}{m}\beta)$ for $\beta = v-u$ and $n,m\in\mathbb{Z}_{>0}$. There is a smooth variety $X_m\subset X$ such that $X_m = \{x\in X | \mathbb{Z}_m\subset \mathrm{Stab}(x)\}$. Moreover $\mathbb{Z}_m$ acts nontrivially on the fibers of the normal bundle $N_{X_m|X}$. As such the only contribution of $R^\bullet p_*(ev^*(TX))$ to the fixed part of the virtual class of $M_{\omega,2}$ comes from $R^\bullet p_*(ev^*(TX_m))$. The formula for the class of the virtual normal bundle of $M_{\omega,2}$ is

$$\begin{aligned}&\chi_{\neq 0}(R^\bullet p^{n/m}_{(j,q)*}ev^*(\mathcal{T}_X)) = \\ &= \sum_w \sum_{0\le l<\frac{n}{m}w} [Q^+_{w,j}\otimes \mathcal{O}(lm\widetilde{T_j}^+)] - \sum_w \sum_{0\le l<\frac{n}{m}w} [Q^-_{w,j}\otimes \mathcal{O}(-lm\widetilde{T_j}^+)] \\ &+ \sum_w \sum_{0\le l<\frac{n}{m}w} [Q_{w,q}]^-\otimes \mathcal{O}(lm\widetilde{T_q}^-)] - \sum_w \sum_{0\le l<\frac{n}{m}w} [Q_{w,q}]^+\otimes \mathcal{O}(-lm\widetilde{T_q}^-)] \\ &+ \sum_{s=j+1}^{q-1}\sum_w \sum_{0\le l<\frac{n}{m}w} P^+_{w,lm,s} + \sum_{s=j+1}^{q-1}\sum_w \sum_{0\le l<\frac{n}{m}w} P^-_{w,lm,s}.\end{aligned}$$

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