Probability density cloud as a geometrical tool to describe statistics of scattered light

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First-order statistics of scattered light is described using the representation of probability density cloud which visualizes a two-dimensional distribution of complex amplitude. The geometrical parameters of the cloud are studied in details and are set in relation to the statistical properties of phase. The moment generating function for intensity is obtained in closed form through these parameters. Example of exponentially modified normal distribution is given to illustrate the functioning of this geometrical approach.

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1. INTRODUCTION

In the title of the historical article by Lord Rayleigh: "On the resultant of a large number of vibrations of the same pitch and arbitrary phase" [1], the emphasis falls on the word "arbitrary", for it is always tempting for a scientist to find a model, an equation describing the widest range of phenomena. Unfortunately, the larger the range of phenomena the model covers, the heavier it looks mathematically, while a compact, ready-for-use expression is often derived for a particular case. The Rayleigh distribution we know is valid only when the phase spans equiprobably the whole range of the possible values from zero to 2π. The generalization, which Lord Rayleigh makes in the last part of his paper, comprises only certain class of phase distributions possessing the $\pi/2$ periodicity. In spite of this limitation, the neat result he has obtained (in his turn being inspired by the book of Émile Verdet [2]) has become the mold for the theory of light statistics for over one century since [3].

The model considered by Lord Rayleigh refers to so called strong diffuser. There is another group of random media and reflecting smooth surfaces combined in a model of weak diffuser [4]. In this model the phase is confined within a range smaller than 2π and the distribution is assumed to be normal (Gaussian). For volume scattering this assumption can be justified: multiple phase shifts along a propagation path permit implementation of the central limit theorem. More generally, ubiquitous use of normal distribution is rather due to mathematical convenience than to some physical fundamentals: any deviation from the normal distribution brings about extensive mathematical difficulties related to complicated expressions involving special functions [5]. Application requires a simplified mathematics; for example, determination of surface roughness by optical methods relies on the Gaussian model of phase randomness [6]. However, it does not mean that non-Gaussian statistics did not bring an attention. Beckmann [7] first drew attention to non-Gaussian distribution of surface height and many other authors followed his theoretical study.

Extensive experimental evidence shows that different materials and different polishing processes produce various types of surface height distribution. It can be either symmetrical or non-symmetrical, skewed positively or negatively [8, 9]. Some artificially generated biological surfaces have negative skewed height distribution with one or two peaks [10]. In spite of availability of various theoretical distributions, the good old Gaussian model remains the favorite when it comes to analysis of experimental data.

Probably one of the reasons why mathematical results of speckle theory are not used in practice is the tendency of many authors to obtain the result for intensity distribution in closed form omitting intermediate demonstrative explanations. Such a descriptive tool is probability cloud: graphical representation of a two dimensional probability density function (2D-PDF) for complex amplitude. This representation is implicitly utilized by Lord Rayleigh in his paper. The plots of the 2D-PDF are presented in many canonical books on light statistics [3], but little attention has been payed to the geometrical properties of the cloud: its position, extension, elongation and orientation. The exception is the study by Uozumi and Asakura [11], where this tool is employed to describe on- and off-axis statistics of light in the near-field. The tool of probability cloud provides a cognitive link between stochastic events in pupil and image planes. On the one side, the geometry of the cloud is related to the distribution of phase and, on the other side, it allows the quantitative prediction of the first order statistics of intensity.

The present paper follows this geometrical approach to evaluate statistical behavior of scattered light using the representation of the probability density cloud. From arbitrary phase distribution we derive expressions for the geometrical parameters of the cloud: its position, extension, elongation and orientation. The focus of our study is the relation of these parameters to the phase statistics. Section 2 describes this relation in the most general way by first setting the task into a physical context and then by introducing a mathematical formalism. Section 3 gives an illustration on example of exponentially modified normal phase distribution. This is only an example, not a restriction of our approach. This model analogically to the Gamma-distribution can be parametrized between the normal and the exponential distribution laws. In Section 4 we return to the general description and work with the statistics of intensity. We give an expression for the moment generating function in closed form through the geometrical parameters of the 2D-PDF. As our main subject is the cloud itself, the material of Section 4 is condensed.

Our study challenges to obtain the result for indeed arbitrary phase distribution. The only "Gaussian" limitation we preserve is an assumption on a large number of contributors, which means that we deal with a Gaussian complex random variable and its elliptical 2D-PDF. The non-Gaussian speckles we have discussed in our previous work [12]. The difference between Gaussian speckles and Gaussian phases is explained in Section 2B.

2. PROBABILITY CLOUD

A. From diffraction integral to a random phasor sum

We start by setting our task in a context of diffractive optics. Consider a monochromatic optical wave with a randomly distorted wavefront $\varphi(\boldsymbol{r})$. Its complex amplitude in the pupil plane of an imaging system is given by

$$\boldsymbol{F}(\boldsymbol{r}) = P(\boldsymbol{r}) e^{j\varphi(\boldsymbol{r})}, \tag{1}$$

where $\boldsymbol{r}$ is a position in the pupil plane and $P(\boldsymbol{r})$ is a pupil function. The amplitude of the wave is assumed undistorted. Phase randomness might be caused by a turbulent media through which the wave has propagated or by a surface from which the light has reflected. In the second case Eq. (1) describes the complex field directly after the surface and $P(\boldsymbol{r})$ is the shape of an illumination spot. It can also be the case when the pupil of the imaging system introduces the phase distortions of its own, like a multi-segmented primary mirror of a giant optical telescope [13].

The complex amplitude in the focal plane of the imaging system (coordinate $\boldsymbol{w}$) is given by the Fourier transform:

$$\boldsymbol{U}(\boldsymbol{w}) = \frac{1}{\lambda f} \iint \boldsymbol{F}(\boldsymbol{r}) e^{-j\frac{2\pi}{\lambda f}\boldsymbol{r}\boldsymbol{w}} d^2 r. \tag{2}$$

The same expression is valid for the complex amplitude in the far-field if the wave reflects from a surface and freely propagates in assumption that scattering angles are small, i.e. in the paraxial approximation. Then the focal distance f is replaced by a propagation distance z.

Intensity in the image plane $|\boldsymbol{U}(\boldsymbol{w})|^2$ is a random field. Statistical properties of this field are determined by the following factors:

1. First-order statistics of the wavefront distortion $\varphi(\boldsymbol{r})$, including its spatial stationarity or non-stationarity;
2. A ratio between the size of the pupil (or of the illumination spot) and the correlation radius of the complex field $e^{j\varphi(\boldsymbol{r})}$;
3. Observation point (coordinate $\boldsymbol{w}$).

Further conditions must be set. First, the observation point is chosen to be on-axis, so the diffraction integral becomes

$$\boldsymbol{U}(0) = \frac{1}{\lambda f} \iint \boldsymbol{F}(\boldsymbol{r}) d^2 r. \tag{3}$$

Second, it is assumed that the correlation radius of the complex field $e^{j\varphi(\boldsymbol{r})}$ is smaller than the size of the pupil and the pupil includes many coherent cells. Third, it is assumed that the random field $\varphi(\boldsymbol{r})$ is isotropic and spatially stationary, so that its statistical properties depend neither on direction nor on coordinate on the pupil. Forth, the illumination is regarded to be uniform. These assumptions allow splitting the pupil into n identical zones and the diffraction integral after some normalization turns into a random phasor sum with n phasors:

$$\boldsymbol{A} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e^{j\varphi_i}. \tag{4}$$

The random values φ_i are identically distributed which results from the assumption on spatial stationarity. They are also mutually uncorrelated which is ensured by the size of the zone. The size of the zone defines also the number of phasors – an important parameter in the study. Definition of the zone and therefore determination of the number of phasors involves a second-order statistics on phase: its autocorrelation function. Some authors prefer to work directly with the autocorrelation function, assuming a Gaussian-correlated phase [11, 14]. Nevertheless, this approach does not remove a mathematical heaviness of expressions involved. We prefer to work with the random phasor sum, keeping the number of phasors n as a free parameter. If the source of randomness is telescope segmentation, this issue does not arise.

B. Statistics of phase and statistics of complex amplitude

In theory of diffused light many different things are said to be Gaussian. We have: Gaussian speckles, Gaussian distribution of phase, Gaussian autocorrelation function of phase and even Gaussian beam. We can avoid questions about the autocorrelation function and the beam by introducing the notion of phasors. Although we have assume a uniform illumination, Eq.(4) can be modified taking into account a non-uniform beam shape. However, we must clarify the difference between Gaussian speckles and Gaussian phases. The former concerns the number of phasors n: if it is large enough that the central limit theorem can be applied to the complex random variable $\boldsymbol{A}$, speckles are called Gaussian. The later refers to the shape of the probability density function of phases φ_i (P-PDF). Although in many studies both assumptions are often made together, they are two different issues. The number of phasors is defined by the pupil size and the correlation radius of the complex field, i.e. by the second order statistics. The P-PDF is the first order statistics. Here we adopt the first assumption on a large number of phasors, but allow the P-PDF to be arbitrary.

The principal input is the characteristic function of phase – statistical average of the complex value $e^{jk\varphi}$ related to the P-PDF by the Fourier transform:

$$\boldsymbol{\Phi}(k) = \left\langle e^{jk\varphi} \right\rangle = \int p(\varphi) e^{jk\varphi} d\varphi. \tag{5}$$

We use only two values of the characteristics function:

$$\boldsymbol{V} = \boldsymbol{\Phi}(1), \quad \boldsymbol{W} = \boldsymbol{\Phi}(2) - \boldsymbol{\Phi}^2(1). \tag{6}$$

In general, $\boldsymbol{V}$ and $\boldsymbol{W}$ are complex. We represent them through the moduli and the angles:

$$\boldsymbol{V} = V e^{j\theta_V}, \; \boldsymbol{W} = W e^{j\theta_W}. \tag{7}$$

Suppose the phase is symmetrically distributed around $\varphi=\varphi_0$. In this case $\theta_V=\varphi_0$ and $\theta_W=2\varphi_0+\pi$. Symmetric phase distribution with zero mean gives $\theta_V=0$ and $\theta_W=\pi$.

Assumption on a large number of phasor allows employing the central limit theorem. The 2D-PDF is a mutual distribution law for the real $x=Re\boldsymbol{A}$ and the imaginary $y=Im\boldsymbol{A}$ parts of $\boldsymbol{A}$ [3]:

$$p_{\boldsymbol{A}}(x,y)=\frac{1}{2\pi\sqrt{\sigma_R^2\sigma_I^2-C_{RI}^2}}\times\exp\left\{-\left[\sigma_I^2(x-E_R)^2+\sigma_R^2(y-E_I)^2-2C_{RI}(x-E_R)(y-E_I)\right]/\left[2\left(\sigma_R^2\sigma_I^2-C_{RI}^2\right)\right]\right\}. \tag{8}$$

The last expression includes the following statistics moments (we use < . > to symbolize statistical averaging):

$$\begin{aligned}E_R&=\langle x\rangle, E_I=\langle y\rangle,\\ \sigma_R^2&=\langle x^2\rangle-\langle x\rangle^2,\\ \sigma_I^2&=\langle y^2\rangle-\langle y\rangle^2,\\ C_{RI}&=\langle\left(x-\langle x\rangle\right)\left(y-\langle y\rangle\right)\rangle.\end{aligned} \tag{9}$$

We relate these moments to the moduli and angles of $\boldsymbol{V}$ and $\boldsymbol{W}$:

$$\begin{aligned}E_R&=\sqrt{nV^2}\cos\theta_V,\ E_I=\sqrt{nV^2}\sin\theta_V,\\ \sigma_R^2&=\frac{1}{2}\left(1-V^2+W\cos\theta_W\right),\\ \sigma_I^2&=\frac{1}{2}\left(1-V^2-W\cos\theta_W\right),\\ C_{RI}&=\frac{W}{2}\sin\theta_W.\end{aligned} \tag{10}$$

Equations (10), although they do not seem to bring any simplification, are very helpful when later on we connect geometry of the 2D-PDF with the properties of the P-PDF. Note, that the structure of Eq.(10) is not particular to the model of random phasor sum. Expressions for E_R, E_I, σ_R, σ_I and C_{RI} derived directly from the diffraction integrals, including also the near-field diffraction, have the same structure, although $\boldsymbol{V}$ and $\boldsymbol{W}$ might have a physical meaning other than phase characteristic function [11, 15]. In this sense, result of the next sub-section is not limited by the framework of random phasor sum model.

C. Geometry of the cloud

Consider the 2D-PDF given by Eq. (8). The numerator of the exponent is a second-order polynomial

$$Q(x,y)=\sigma_I^2(x-E_R)^2+\sigma_R^2(y-E_I)^2-2C_{RI}(x-E_R)(y-E_I), \tag{11}$$

and the contours of constant probability density answering condition *Q(x,y)=const* are conic sections. The determinant of the quadratic part of the polynomial is

$$D=C_{RI}^2-\sigma_R^2\sigma_I^2. \tag{12}$$

The sign of the determinant defines the type of conic section, i.e. geometry of the contours of constant probability density. Due to the Schwarz's inequality $C_{RI}^2\le\sigma_R^2\sigma_I^2$ the determinant is non-positive. The contours of constant probability density are circles or ellipses when $D<0$ and line or point when $D=0$. Circle, line and point are forms of ellipse, so we speak about elliptical complex random variable.

One-sigma boundary analog in two dimensions is a contour on which $p_A(x,y)=p_{max}e^{-1/2}$. This contour is an ellipse satisfying equation

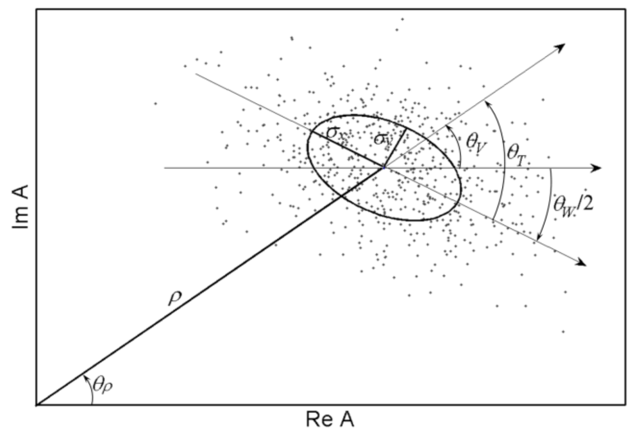

Fig. 1. Two-dimensional probability density cloud and its geometrical parameters: distance to the center, halve-axes and the angles. The major axis is the largest one and its half-length is always σ_x regardless the orientation.

$$\frac{Q(x,y)}{-D}=1 \tag{13}$$

and the 2D-PDF is therefore written as

$$p_A(x,y)=\frac{1}{2\pi\sqrt{-D}}\exp\left[-\frac{Q(x,y)}{-2D}\right]. \tag{14}$$

Geometry of the 2D-PDF can be formulated through the geometry of the ellipse from Eq.(13) and is characterized by: position of center (ρ, θ_ρ), halves of the major and minor axis (σ_x, σ_y) and either by the orientation or by the inclination (Figure 1). Orientation angle is the angle between the major axis and the *Ox* (*Re**A*** - axis). It equals $\theta_W/2$. Inclination angle is the angle between the major axes and the line passing through the center of the ellipse and the center of coordinates (θ_T).

Note that the halve-axis (σ_x, σ_y) differ from the variances (σ_R, σ_I), but the sum of squares is preserved:

$$\Sigma=\sigma_x^2+\sigma_y^2=\sigma_R^2+\sigma_I^2. \tag{15}$$

This parameter is the total size of the ellipse: extension of the 2D-PDF. Together with the total size, the eccentricity is important:

$$e=\left(\sigma_x^2-\sigma_y^2\right)/\left(\sigma_x^2+\sigma_y^2\right). \tag{16}$$

It gives the degree of elongation. The 2D-PDF is fully described if the following five parameters are known: ρ, θ_ρ, Σ, e, and θ_T.

Now we relay these five geometrical parameters to the two values of characteristic function using Eq. (10). The total size is related to the modulus of $\boldsymbol{V}$:

$$\Sigma=1-V^2, \tag{17}$$

while the eccentricity – to the total size and the modulus of $\boldsymbol{W}$:

$$e=W/\Sigma. \tag{18}$$

The only parameter containing the number of phasors is the modulus of distance to the center which also depends on the total size:

$$\rho=\sqrt{nV^2}=\sqrt{n(1-\Sigma)}. \tag{19}$$

The angular position of the center equals θ_V, while the inclination angle is a sum of the orientation angle and θ_V:

$$\theta_\rho = \theta_V\,,\ \theta_T = \theta_V + \theta_W/2. \tag{20}$$

The calculations leading to these results are not difficult and we do not present them here.

From Eq. (15) and (17) it follows that the total size varies between zero and one. Eq. (19) relates the size of the cloud to its location: if the number of phasors is preserved, the cloud shrinks while moving away from the center of coordinates. At the farthest distance equal to $\sqrt{n}$ the total size is zero and the cloud shrinks into a point. When the center of the cloud coincides with the center of coordinates, the total size equals one and the cloud is at its maximal spread.

Three other parameters, the halve-axes and the determinant, are algebraic expressions of the total size and the eccentricity:

$$\begin{aligned} D &= -\frac{1}{4}\Sigma^2\left(1-e^2\right),\\ \sigma_x^2 &= \frac{\Sigma}{2}(1+e),\\ \sigma_y^2 &= \frac{\Sigma}{2}(1-e). \end{aligned} \tag{21}$$

The eccentricity changes between zero and one. It equals zero when the halve-axes are identical: $\sigma_x^2=\sigma_y^2=\Sigma/2$. The cloud is then circular, but it does not necessary mean that it is centered at the center of coordinates. In general, position and extension of the cloud are not related to the degree of its elongation quantified by eccentricity. For the circular cloud shape, the determinant of the corresponding quadratic form is minimal: $D=-\Sigma^2/4$. In the opposite situation, when the eccentricity equals one, $\sigma_y=0$ and $\sigma_x^2=\Sigma$. Two shapes of the cloud are possible: line ($\Sigma\neq0$) and point ($\Sigma=0$). The determinant of the corresponding quadratic form is zero.

To conclude this section we present equation of the ellipse in parameters n, Σ, e, θ_ρ and θ_T. Instead of substituting Eq. (10) into Eq. (13) we find it from consideration of geometry. We know that the curve we are looking for is an ellipse with the halve axes from Eq.(21):

$$\frac{2x''^2}{\Sigma(1+e)} + \frac{2y''^2}{\Sigma(1-e)} = 1, \tag{22}$$

rotated by the angle $\theta_W/2$:

$$\begin{aligned} x'' &= x'\cos(\theta_W/2) - y'\sin(\theta_W/2),\\ y'' &= x'\sin(\theta_W/2) + y'\cos(\theta_W/2). \end{aligned} \tag{23}$$

and shifted by the vector $(\rho\cos\theta_\rho,\ \rho\sin\theta_\rho)$:

$$\begin{aligned} x' &= x - \rho\cos\theta_\rho,\\ y' &= y - \rho\sin\theta_\rho. \end{aligned} \tag{24}$$

Combining the last three equations and introducing the polar coordinates ($x=A\cos\theta_A$, $y=A\sin\theta_A$) after some trigonometric manipulations we obtain that the curve of one-sigma boundary for the 2D-PDF is given by:

$$\frac{2\left(A\cos\theta' - \rho\cos\theta_T\right)^2}{\Sigma(1+e)} + \frac{2\left(A\sin\theta' - \rho\sin\theta_T\right)^2}{\Sigma(1-e)} = 1. \tag{25}$$

where $\theta' = \theta_A + \theta_W/2 = \theta_A + \theta_T - \theta_\rho$ and $\rho = \sqrt{n(1-\Sigma)}$. It is the same curve as the one given by Eq. (13), but written in the polar coordinates through the geometrical parameters of the cloud. We will come back to this result in Section 4.

3. EXAMPLE

A. Exponentially modified normal P-PDF

Suppose that phase distortions are governed by two independent random processes: the first process is normally distributed with zero mean and standard deviation $\sigma \ge 0$ and the second process is exponentially distributed with $\tau \ge 0$:

$$\begin{aligned} p_1(\varphi) &= \left(2\pi\sigma^2\right)^{-1/2} e^{-\varphi^2/2\sigma^2},\\ p_2(\varphi) &= \begin{cases} \tau^{-1}e^{-\varepsilon\varphi/\tau}, \varepsilon\varphi \ge 0,\\ 0, \varepsilon\varphi < 0, \end{cases} \end{aligned} \tag{26}$$

$\varepsilon=\pm1$. The resultant phase is a sum of these two processes and follows exponentially modified normal distribution law. From the basic theory of random process, we know that the residual probability density function is a convolution of two initial probability density functions. Although for our purpose it is not necessary to know the P-PDF itself, for the completeness we write it down. The exponentially modified normal P-PDF has the following shape:

$$\begin{aligned} p(\varphi) = \frac{1}{2\tau}\exp&\left[\frac{1}{2\tau}\left(\frac{\sigma^2}{\tau} - 2\varepsilon\varphi\right)\right]\\ &\times \operatorname{erfc}\left[\frac{1}{\sqrt{2\sigma^2}}\left(\frac{\sigma^2}{\tau} - \varepsilon\varphi\right)\right]. \end{aligned} \tag{27}$$

where erfc(.) is a complementary error function and parameter ε determines the direction of the modification. When $\varepsilon=1$ the P-PDF is positively skewed, when $\varepsilon=-1$ the P-PDF is negatively skewed. The first three central moments of the phase are

$$\begin{aligned} \langle\varphi\rangle &= \varepsilon\tau,\\ \left\langle\left(\varphi - \langle\varphi\rangle\right)^2\right\rangle &= \sigma^2 + \tau^2,\\ \left\langle\left(\varphi - \langle\varphi\rangle\right)^3\right\rangle &= 2\varepsilon\tau^3. \end{aligned} \tag{28}$$

Zero mean of the initial normal distribution is shifted by the exponential process to the direction defined by ε. The variance is a sum of the two variances as we have assumed that the two processes are independent. The third central moment included in skewness depends only on τ and its sign is determined by ε.

To describe the geometrical properties of the cloud we need to know only two values of the characteristic function. Expression for a characteristic function is often much easier to obtain than expressions for a PDF because the characteristic function of a sum of independent random processes is a product of the characteristic functions. For the exponentially modified normal distribution the characteristic function is:

$$\boldsymbol{\Phi}(k) = e^{-\frac{k^2\sigma^2}{2}}\left(1 - ik\varepsilon\tau\right)^{-1}. \tag{29}$$

The complex values we need to calculate are

$$\begin{aligned} \boldsymbol{V} &= e^{-\frac{\sigma^2}{2}}\left(1 - i\varepsilon\tau\right)^{-1},\\ \boldsymbol{W} &= e^{-2\sigma^2}\left(1 - i2\varepsilon\tau\right)^{-1} - e^{-\sigma^2}\left(1 - i\varepsilon\tau\right)^{-2}. \end{aligned} \tag{30}$$

In order to extract the parameters of the cloud some algebraic manipulations with complex variables must be performed. It does not represent any difficulty and can be done using any symbolic manipulation software. We omit the details of this step and present only the result.

B. Parameters of the 2D-PDF

The cloud, geometry of which we are describing here, is a two-dimensional distribution of a complex Gaussian variable. Therefore, there is no surprise that its geometrical parameters are some functions of means and variances of two initial random processes, i.e. of τ, τ^2 and σ^2. To present these functions in a readable compact form we introduce four simple expressions of τ^2:

$$\begin{aligned} &f_1 = \left(1+\tau^2\right)^2,\ f_2 = 1+4\tau^2, \\ &f_3 = 1-\tau^2,\ f_4 = 1+3\tau^2. \end{aligned} \tag{31}$$

Notice that all f_i equal one when τ=0, i.e. when the exponentially distributed second process is absent and phase is a Gaussian variable with zero mean.

For the position of cloud center, we have:

$$\begin{aligned} &\rho = \sqrt{nV^2} = \sqrt{n}\, e^{-\sigma^2/2} f_1^{-1/4}, \\ &\theta_\rho = \theta_V = \arctan\left(\varepsilon\tau\right). \end{aligned} \tag{32}$$

When τ=0 the angular position of the center is zero. It means that the cloud is centered on the real axis at the distance $\rho_0 = \sqrt{n}\, e^{-\sigma^2/2}$ from the center of coordinates. For small τ, when the contribution of the second process is weak, the angular position equals to the phase mean value: $\theta_\rho \approx \varepsilon\tau = \langle\varphi\rangle$.

When phase mean value ($\varepsilon\tau$) changes but σ remains constant, the center of the cloud moves along a circular line, connecting the point (ρ_0,0) and the center of coordinates: in the positive angular direction if the P-PDF is positively skewed (ε=1) or in the negative angular direction if the P-PDF is negatively skewed (ε=-1). The circular shape of the line can be deduced from Eq.(32): $\rho=\rho_0\cos\theta_\rho$ or $x^2+y^2-x\rho_0=0$. When, on the contrary, σ increases but $\varepsilon\tau$ remains constant, the center of the cloud moves along a ray making angle θ_ρ with Ox inwards (Figure 2).

While moving along a ray or a circle towards the center of coordinates the cloud expands, because increasing τ or σ means that the P-PDF widens approaching a uniform distribution. The spread of the cloud is characterized by the total size which in this example is

$$\begin{aligned} \Sigma &= 1-\frac{\rho^2}{n} = 1-e^{-\sigma^2}\cos^2\theta_\rho \\ &= 1-e^{-\sigma^2} f_1^{-1/2} \approx 1-e^{-\sigma^2}+\tau^2 e^{-\sigma^2}. \end{aligned} \tag{33}$$

The last approximation is given for small τ.

Now we study orientation of the ellipse. First, consider the real and the imaginary parts of $\boldsymbol{W}$:

$$\begin{aligned} \mathrm{Re}\boldsymbol{W} &= \frac{e^{-\sigma^2}}{f_1 f_2}\cdot\left[f_1 e^{-\sigma^2} - f_2 f_3\right], \\ \mathrm{Im}\boldsymbol{W} &= 2\varepsilon\tau\,\frac{e^{-\sigma^2}}{f_1 f_2}\cdot\left[f_1 e^{-\sigma^2} - f_2\right]. \end{aligned} \tag{34}$$

When τ=0 the imaginary part is zero, but the real part is negative and therefore the angle θ_W is π. If ε=1 and τ is sufficiently small both, Re$\boldsymbol{W}$ and Im$\boldsymbol{W}$, are small negative values. In order to be consistent with the definition of angles presented in Figure 1 and to avoid uncertainty of arctan(.) we define a piecewise function:

$$\theta_W = \begin{cases} \pi - \arctan\, T_W,\tau \le \tau_0, \\ -\arctan\, T_W, \tau > \tau_0, \varepsilon = 1, \\ -\arctan\, T_W + 2\pi, \tau > \tau_0, \varepsilon = -1, \end{cases} \tag{35}$$

where

$$T_W = \frac{\mathrm{Im}\boldsymbol{W}}{\mathrm{Re}\boldsymbol{W}} = 2\varepsilon\tau\,\frac{f_1 e^{-\sigma^2} - f_2}{f_1 e^{-\sigma^2} - f_2 f_3} \tag{36}$$

and τ_0 is a point when Re$\boldsymbol{W}$=0, i.e. a positive root of quadratic with respect to τ^2 equation: $f_1 e^{-\sigma^2} - f_2 f_3 = 0$.

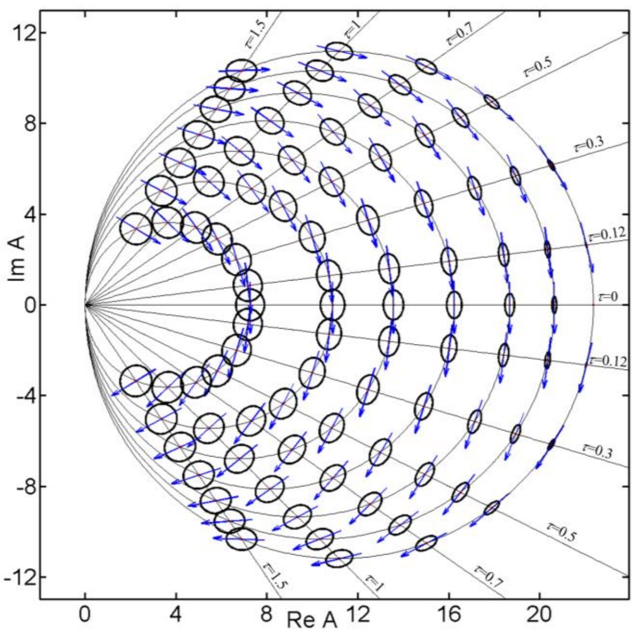

Fig. 2. Family of ellipses showing the geometry of the 2D-PDF for exponentially modified normal phase distribution. Each ellipse corresponds to certain combination (τ,σ). Rays are the lines of constant τ=0, 0.12, 0.3, 0.5, 0.7, 1, 1.5. Circles are the lines of constant σ=0, 0.4, 0.6, 0.8, 1, 1.2, 1.5 (from the outer circle inwards). Upper half-plane: positively skewed P-PDF (ε=1), lower half-plane: negatively skewed P-PDF (ε=– 1). Arrows indicate orientation of ellipse. The angle between an arrow and a ray is inclination angle θ_T. Scaling parameter n=500.

In Figure 2 an arrow indicated the orientation of an ellipse. There exists certain combination (τ,σ) for which the cloud is oriented horizontally. This combination answers to the condition θ_W=0 (θ_T= θ_V) for the upper half-plane and θ_W=2π (θ_T=π–θ_V) for the lower half-plane. The combination answers condition Im$\boldsymbol{W}$=0, and can be found as a positive root of quadratic with respect to τ^2 equation: $f_1 e^{-\sigma^2} - f_2 = 0$.

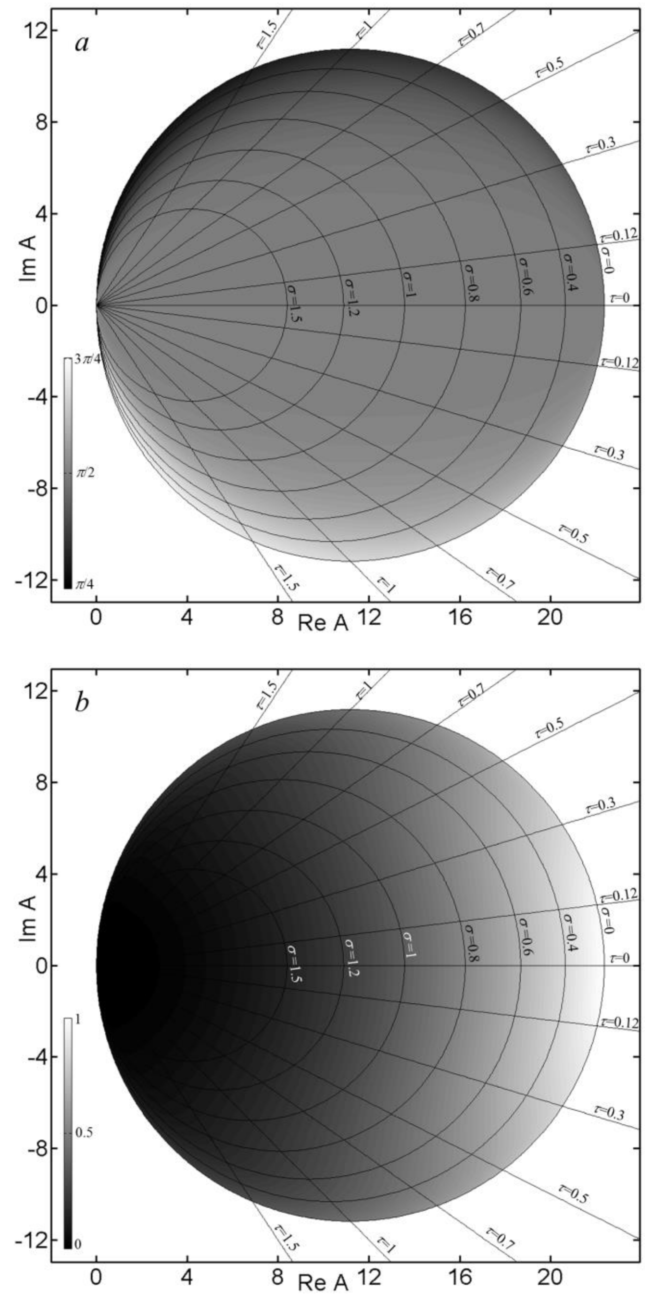

Fig. 3. Inclination angle (a) and eccentricity (b) as functions of ellipse center: (τ, σ) – mapping. Scaling parameter n=500.

The inclination angle θ_T is calculated according to Eq. (20). When the contribution of the second process is weak (τ is small) the inclination angle can be estimated by the following approximation:

$$\theta_T \approx \pi/2 + \tau^3 \varepsilon \left(e^{\sigma^2} - 1\right)^{-1}. \quad \mathbf{(37)}$$

The deviation of the inclination angle from $\pi/2$ is proportional to $\tau^3\varepsilon$. In this term we recognize the third moment of φ (Eq. (28)). Although the equation above cannot serve as a mathematical proof of connection between the deviation of the inclination angle from $\pi/2$ and an asymmetry of the P-PDF, it gives a feeling of this existing connection. In Figure 2 θ_T is the angle between an arrow and a ray.

The eccentricity from Eq.(18) after some algebraic manipulations becomes:

$$e = e^{-\sigma^2} \frac{\sqrt{f_1 e^{-2\sigma^2} - 2 f_4 e^{-\sigma^2} + f_2}}{\sqrt{f_2}\left(\sqrt{f_1} - e^{-\sigma^2}\right)}. \quad \mathbf{(38)}$$

In absence of the second process, i.e. when τ=0 and θ_ρ =0 the eccentricity equals $e^{-\sigma^2} = 1 - \Sigma$: the vertical cloud has maximal elongation with minimal spread. For small τ the following approximation is valid: $e \approx e^{-\sigma^2} - \tau^2 e^{-\sigma^2}$, and the relation $e = 1 - \Sigma$ is preserved. With increase of τ it is not anymore the case.

Figure 2 shows the set of 2D-PDFs plotted with use of Eq.(25) on a grid of constant τ (rays) versus constant σ (circles). Crossing point of a ray and a circle is a center of an ellipse. The center can be located only within the largest circle with diameter $\sqrt{n}$ (boundary circle, σ=0). Every point on the ($Re\boldsymbol{A}$, $Im\boldsymbol{A}$)-plane within the boundary circle is a center of an ellipse with certain value of a given geometrical parameter (eccentricity, for example), which allows plotting this parameter on the ($Re\boldsymbol{A}$, $Im\boldsymbol{A}$)-plane as a two-dimensional function. Mathematically it is done substituting $e^{-\sigma^2} = \rho_x^2/n$, $\tau = \left|\rho_y/\rho_x\right|$ and $\varepsilon = signum(\rho_y)$ into expressions for ellipse parameters (ρ_x and ρ_y being Cartesian coordinates of ellipse center). So we speak about (τ, σ)-mapping when present parameter of ellipse as function of position of its center: two-dimensional function $e(\rho_x, \rho_y)$ maps the (τ, σ)-space. Figure 3 shows the inclinations angle and the eccentricity in this representation.

Not only the geometrical parameters of ellipse but also all values derived from them can be presented on ($Re\boldsymbol{A}$, $Im\boldsymbol{A}$)-plane as (τ, σ)-mapping. Figure 5 in section 4 illustrates it on example of intensity standard deviation.

C. Numerical simulation

We simulate the random phasor sum with the MATLAB R2014. A phase value is generated as a sum of two random numbers: the first one with the normal distribution $N(0,\sigma)$ and the second one with the exponential distribution $E(\tau)$. The phase is used to generate a complex phasor with the unit length, and the random phasor sum sums 500 independent phasors. The case (ε=-1) is simulated by taking the phase value as a difference between $N(0,\sigma)$-distributed number and $E(\tau)$-distributed number. Every outcome is plotted as a point on the complex plane ($Re\boldsymbol{A}$, $Im\boldsymbol{A}$). For every combination (τ, σ) up to 600 points are generated to create a cloud. The smaller clouds are created with the smaller number of points. We have chosen this simple way to generate the complex amplitude and not the full Fourier optics propagation to avoid addition complications related to generation of rough surface with a given statistics. In other words, we deal here only with the second part of the optical task exposed in Section 2A.

Figure 4 shows the simulated clouds when σ is fixed to 0.2 (the statistics of the first random value does not change), while τ increases from 0 to 10. To compare with the theory we plotted an analytical ellipse given by Eq.(25) for every τ and the circle ρ=21.9cosθ_ρ. The behavior is as it is described above: when τ increases the ellipse moves along the circle towards zero, widens, rounds and rotates. It starts from the vertical orientation and rotates slower than the tangential to the arch, therefore it assumes the horizontal orientation not at τ=1, but later at τ_{h} = 1.48.

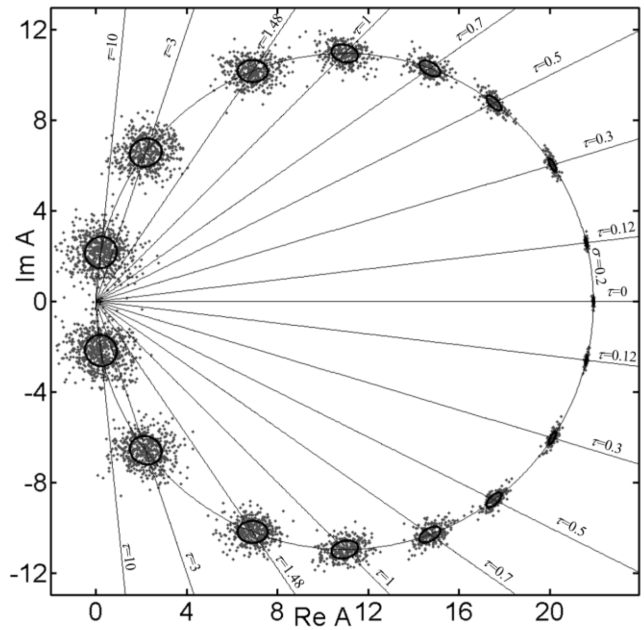

Fig. 4. Two-dimensional PDF of complex amplitude for exponentially modified normal phase distribution. The number of phasors equals 500; σ equals 0.2 *rad*. Solid ellipses show the theoretical one-sigma contours and the points – the simulated distribution for the same set of parameters.

4. STATISTICS OF INTENSITY

The main subject of this paper is joint distribution of the real and the imaginary parts of the complex amplitude, which we refer to as probability cloud or 2D-PDF. Although we wish to keep the focus on the geometry of the cloud itself, it is important to show how the knowledge of geometry helps to predict the statistical behavior of the measurable quantity, i.e. intensity. The standard procedure is as following. First step is finding the joint distribution of modulus and angle of the complex amplitude by moving to the polar coordinates (A, θ_A) from the mutual distribution of the real and the imaginary parts. Second step is passing to a marginal statistics $p_A(A)$ by integrating over θ_A. The final step is obtaining a distribution of intensity by substituting $A = \sqrt{I}$. Except for a small number of cases, the integration over θ_A is either insolvable analytically or the result is complicated and inapplicable. For example, Equation 9 from [4] gives the probability of intensity through an infinite sum of a product of modified Bessel functions; and the sum in its turn is multiplied by an exponential function. It is a very neat formula, but, unfortunately, no mathematical software can digest it properly.

The representation of the cloud through its geometrical parameters allows obtaining the statistics of intensity without passing through the painful integration over the angle. To avoid overcharging this paper, we give only some clues of the procedure leading to the result. We plan to present more details in the next paper.

The key is Eq.(25) giving the one-sigma curve of the 2D-PDF. The remarkable feature of this equation is that the left part of it is a sum of squares of two values. On the other hand, this left part is the double argument of the exponential in the 2D-PDF through Eq.(14). Passing from the distribution $p_A(x, y)$ to the distribution $p(A, \theta')$ (Jacobian of the transformation equals A) and then to the distribution $p(I, \theta')$ (Jacobian of the transformation equals $\left(2\sqrt{I}\right)^{-1}$) we obtain:

$$p(I,\theta') = \frac{1}{\sqrt{2\pi\Sigma(1+e)}}\exp\left[-\frac{\left(\sqrt{2I}\cos\theta' - \rho\sqrt{2}\cos\theta_T\right)^2}{\Sigma(1+e)}\right] \times \frac{1}{\sqrt{2\pi\Sigma(1-e)}}\exp\left[-\frac{\left(\sqrt{2I}\sin\theta' - \rho\sqrt{2}\sin\theta_T\right)^2}{\Sigma(1-e)}\right]. \tag{39}$$

The expression is factorized into a product of two normal distributions. That prompts us to introduce two normally distributed random variables: $x_1 = \sqrt{2I}\cos\theta'$ and $x_2 = \sqrt{2I}\sin\theta'$ with the means and the variances $\mu_1 = \rho\sqrt{2}\cos\theta_T$, $\sigma_1^2 = \Sigma(1+e)$ and $\mu_2 = \rho\sqrt{2}\sin\theta_T$, $\sigma_2^2 = \Sigma(1-e)$ correspondingly. Afterwards, passing from the distribution $p(I, \theta')$ to the distribution $p(x_1, x_2)$ (Jacobian of the transformation equals one) we conclude that x_1 and x_2 are independent because $p(x_1, x_2)=p(x_1)p(x_2)$. The intensity is therefore the half of a sum of squares of two independent normally distributed random variables with known means and variances: $I = \left(x_1^2 + x_2^2\right)/2$. We do not need to integrate over θ'.

Now we can follow the standard methods of statistics to obtain a moment generating function of intensity. We postpone the details of this calculation for the next paper and present only the result:

$$\Xi(t) = \frac{1}{\sqrt{1-2t+t^2\left(1-e^2\right)}} \times \exp\left[\frac{\rho^2 t}{\Sigma}\cdot\frac{1-t\left(1-e\cos 2\theta_T\right)}{1-2t+t^2\left(1-e^2\right)}\right]. \tag{40}$$

Naturally, the angular position of ellipse θ_ρ is not present in this equation. The moments of intensity are the partial derivatives of this function:

$$\left\langle I^n\right\rangle = \Sigma^n \left.\frac{\partial^n \Xi(t)}{\partial t^n}\right|_{t=0} \tag{41}$$

and yield in polynomial expressions of ρ, Σ, e and $\cos 2\theta_T$. These values, as we have studied in the paper, are linked to statistics of phase.

The central moments $M_I^n = \left\langle\left(I - \langle I\rangle\right)^n\right\rangle$ are therefore also some polynomials of the same values. We present here the expressions for the first three:

$$\begin{aligned} M_I^1 &= \rho^2 + \Sigma, \\ M_I^2 &= \Sigma\left[2\rho^2\left(1+e\cos 2\theta_T\right) + \Sigma\left(1+e^2\right)\right], \\ M_I^3 &= 2\Sigma^2\left[3\rho^2\left(1+2e\cos 2\theta_T + e^2\right) + \Sigma\left(1+3e^2\right)\right]. \end{aligned} \tag{42}$$

Figure 5 shows the (τ, σ)-mapping of intensity standard deviation $\sqrt{M_I^2}$ for exponentially modified normal P-PDF from the previous section. This map has two local minima: the first one at the center of coordinates (ρ_x=0, ρ_y=0) and the second one at the opposite extreme of the boundary circle ($\rho_x = \sqrt{n}$, ρ_y=0). The first minimum corresponds to the case studied by Lord Rayleigh of well-developed speckles. The intensity standard deviation equals one. The second minimum corresponds to the situation (also discussed in his paper) when there is no randomness whatsoever, all phasors are identical and the standard deviation of intensity equals zero. Between these two minima there is a ridge of maximal intensity standard deviation varying between 12.2 (τ=0, σ =1.05) and 14.4 (τ=1.16, σ =0). The values depend on the number of phasors and are given for n=500.

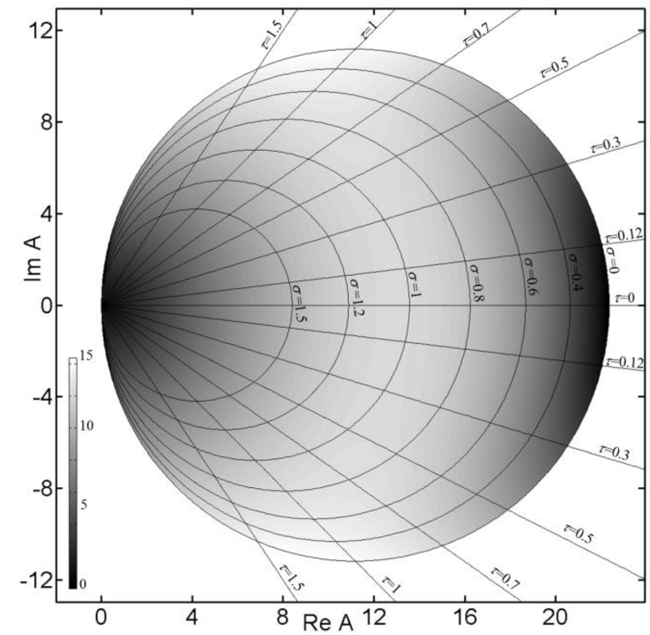

Fig. 5. (τ,σ)-mapping of intensity standard deviation. Scaling parameter n=500.

5. CONCLUSION

Mutual distribution of the real and the imaginary parts of complex amplitude – the resultant of a large number of vibrations, is the best to visualize as probability cloud. The cloud is a powerful tool to describe statistics of light with phase fluctuation. The geometrical parameters of the cloud: its position, extension, elongation and orientation, – are directly related to statistics of phase and involve only two values of phase characteristic function (which in general are complex values). There are common rules applied to the behavior of these parameters independently on particular phase distribution: the cloud always widens when its center approaches the center of coordinates; deviation of the inclination angle from $\pi/2$ indicates certain asymmetry in initial phase distribution; eccentricity tends to one when the cloud shrinks into a point. The demonstration of the last rule (not done in the paper) involves expanding $\boldsymbol{V}$ in series at $\Sigma \rightarrow 0$.

We illustrate the method of geometrical description on example of exponentially modified normal distribution and introduce the principle of (τ,σ)-mapping. This principle takes advantage of the fact that there is one to one correspondence between a vector (τ,σ) and a position on a complex plane marking ellipse center. If for another phase distribution there is no unique relation between parameters of the P-PDF and a point on the complex plane, it nevertheless can be found by combining the parameters. In the considered example of exponentially modified normal distribution we assumed that the mean of normally distributed component is zero. If it is not zero but equals to certain μ, the mapping is still possible through a combined parameter τ' = $\varepsilon\tau+\mu$.

The geometrical representation of two dimensional probability density function of complex amplitude allows obtaining the moment generating function of intensity in closed form and hence expressions for all statistical moments of intensity through the parameters of ellipse. All central moments of intensity are positive regardless statistics of phase. Nevertheless, simulations and experimental data [16] give negative values for some odd moments of intensity or the values based on these moments (skewness, for example). This discrepancy is not due to an error in our calculation, but to limitations of the postulate lying in its base: assuming a large number of phasor, applying the central limit theorem and writing the 2D-PDF as two-dimensional Gaussian function we enforce an elliptical shape of the cloud. The odd moments of intensity are sensitive to cloud non-ellipticity, i.e. to the number of phasors, and therefore cannot be estimated within the frame of the central limit theorem.

Continuation of this work has three directions. First task is to relate the number of phasors (n) to the second-order statistics (autocorrelation function) of phase. Our preliminary calculations show that assigning diameter of a zone to a coherence length yields in unrealistically small n. However, this result requires a more thorough investigation. Second task is to unfold Section 4 of the present paper: having obtained the general expression for moment generating function we must study this expression in details. In particular, we shall see if it gives the correct result for the known intensity statistics included in it: exponential decay, Rician and modified Rayleigh distribution. Third task is to intrude into domain of non-Gaussian complex random variables and try to find not only expressions for intensity moments by means of combinatorics (these expressions have been obtained in [17]), but the shape of the 2D-PDF. A heavy but mathematically correct way to do it is to regard n to be large but not infinite and retain the second term in Euler's theorem.

Acknowledgement.

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