Hierarchy of Universal Relations for Neutron Stars in Terms of Multipole Moments

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Recent studies of the analytical and numerical models of neutron stars strongly suggest that their exterior field can be described by only four arbitrary parameters of the 2-soliton solution of Einstein's equations. Assuming that this is the case, we show that there exists an infinite hierarchy of the universal relations for neutron stars in terms of multipole moments that arises as a series of the degeneration conditions for generic soliton solutions. The simplest of these relations yields a correct expression for the mass-hexadecapole moment as a function of the angular momentum, revealing a need for a more precise definition of this and higher multipole moments in the numerical models of neutron stars.

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Introduction.—In recent years much attention was paid to the study of the universal properties of neutron stars (NSs) with the aid of both the numerical and analytical approaches. A remarkable I-Love-Q relation between the NS's moment of inertia, the tidal Love number and the quadrupole moment was first dicovered by Yagi and Yunes [1] via a numerical analysis of the Hartle-Thorne slow-rotation approximation [2] and then extended to arbitrary rotation and some new universal properties by Pappas and Apostolatos [3] and by Chakrabarti *et al.* [4]. The exact solutions approach to the analysis of various phenomena around NSs was introduced by Sibgatullin and Sunyaev [5] who demonstrated that a 3-parameter quadrupole solution [6] fitted very well the extensive numerical data of the well-known Cook *et al.* paper [7]; they also observed that in terms of the dimensionless multipole moments the properties of NSs independent of the equations of state (EoSs) can be better seen. Comparison of the analytical and numerical models of NSs was performed by Berti and Sterligioulas [8] with the aid of the RNS code [9, 10], and this subsequently led, via Ryan's method [11], to the revision of multipole moments in numerical solutions [12]. A better understanding of the multipole structure of NSs made it possible, on the one hand, to put the universal relations for NS into the language of multipole moments and, on the other hand, to establish [3] that the above structure is generically determined by only four multipole moments, thus being universal for all the physically realistic EoSs known in the literature. Furthermore, in the paper [13] Yagi *et al.* suggested that, similar to black holes, NSs are likely to have their own "no-hair" theorem according to which the higher multipoles could be inferred from the form of the previous four multipoles, and they discussed the numerical evaluation of the NS's mass-hexadecapole moment in the light of their conjecture. In [13] it was observed in particular that the 4-parameter 2-soliton solution of Einstein's equations [14–16], regarded by Pappas and Apostolatos as a possible analytical model describing the geometry around a universal NS, possesses a hexadecapole moment whose spin dependence starts at quadratic order, whereas, according to Yagi *et al.*, this moment should be strictly quartic in angular momentum.

The objective of the present letter is to demonstrate that the Yagi *et al.* no-hair hypothesis, combined with the aforementioned 2-soliton solution (henceforth referred to as the MMR solution), gives rise to a hierarchy of the universal relations for NSs in terms of multipole moments, the simplest relation among which yields the correct expression for the hexadecapole moment M_4 of the MMR solution and rigorously proves that, for a generic NS model, this moment should necessarily contain terms quadratic in spin. As an application, in the case of two known EoSs, we shall compare the values of M_4 computed by means of our analytical formula and by the empirical formulas of Ref. [13].

Multipole moments and the universal relations.—The multipole structure of stationary axially symmetric vacuum spacetimes is well known thanks to the fundamental papers of Geroch [17], Hansen [18] and Thorne [19]. The technical calculation of the moments, describing the distributions of mass and angular momentum, is facilitated by the Fodor-Hoenselaers-Perjés (FHP) procedure [20] which makes use of the Ernst complex potential formalism [21] in order to find the coefficients m_n arising in the expansion of the function

$$X(z) \equiv z\frac{1-e(z)}{1+e(z)} = \sum_{n=0}^{\infty} m_n z^{-n} \tag{1}$$

when $z \to \infty$. The above $e(z)$ denotes the axis ($\rho = 0$) value of the Ernst complex potential $\mathcal{E}(\rho, z)$ of a particular solution, ρ and z being the Weyl-Papapetrou coordinates. The first four quantities m_n coincide with the Geroch-Hansen (GH) complexified multipole moments P_n, $n = \overline{0,3}$, while other m_n, $n \geq 4$, are equal to P_n only up to certain combinations of the lower-order m_l, $l < n$ (see Ref. [20] for the explicit form of those combinations). It is important to note that the equilibrium models of NSs are described by the solutions which, in addition to being stationary and axisymmetric, are also symmetric about the equatorial plane [10], and the latter symme-

try imposes restrictions on the form of the corresponding axis data $e(z)$ and coefficients m_n in (1). As was shown in [22, 23], the function $e(z)$ of an equatorially symmetric spacetime satisfies the condition $e(z)e^*(-z) = 1$ (the star symbol denotes complex conjugation); consequently, all even quantities m_n of such a spacetime are real, and all odd m_n are pure imaginary [22]. The same is true for the corresponding multipoles P_n: in the reflection-symmetric case we have $P_{2k} = M_{2k}$ and $P_{2k+1} = iJ_{2k+1}$, $k = 0, 1, \ldots$, where M_{2k} and J_{2k+1} are, respectively, the mass and angular momentum GH multipole moments.

Noteworthily, the general class of the extended vacuum soliton solutions admits parametrization exclusively in terms of the "multipoles" m_n [24]:

$$e(z) = \frac{e_-}{e_+}, \quad e_\pm = (L_N)^{-1} \tag{2}$$
$$\times \begin{vmatrix} z^N \pm \sum_{n=0}^{N-1} m_n z^{N-1-n} & m_N & \ldots & m_{2N-1} \\ z^{N-1} \pm \sum_{n=0}^{N-2} m_n z^{N-2-n} & m_{N-1} & \ldots & m_{2N-2} \\ \vdots & \vdots & \ddots & \vdots \\ z \pm m_0 & m_1 & \ldots & m_N \\ 1 & m_0 & \ldots & m_{N-1} \end{vmatrix},$$

where the $n \times n$ determinant L_n has the form

$$L_n = \begin{vmatrix} m_{n-1} & m_n & \ldots & m_{2n-2} \\ m_{n-2} & m_{n-1} & \ldots & m_{2n-3} \\ \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & \ldots & m_n \\ m_0 & m_1 & \ldots & m_{n-1} \end{vmatrix}. \tag{3}$$

Restricting ourselves to the equatorially symmetric configurations, we see that the Kerr solution [25] is contained in (2) as the $N = 1$ case, with $m_0 = M$ and $m_1 = iJ$, M being the total mass and J the total angular momentum [26]. The next, $N = 2$ specialization of formulas (2), determines the MMR solution that was recently regarded and advocated as describing the exterior field of a universal NS [3, 15, 27]; it has four arbitrary parameters corresponding to four arbitrary multipole moments:

$$\begin{aligned} m_0 = M_0 \equiv M, \quad m_1 = iJ_1 \equiv iJ, \\ m_2 = M_2, \quad m_3 = iJ_3, \end{aligned} \tag{4}$$

where M_2 is the mass quadrupole moment and J_3 is the angular momentum octupole moment (the explicit form of the MMR solution in two different parametrizations can be found in Ref. [16]). If we now assume that the results of Pappas and Apostolatos [3] obtained on the basis of a variety of very convincing arguments are correct and the geometry around NSs is indeed determined by only four multipole moments (4), then, bearing in mind the no-hair hypothesis for NSs put forward by Yagi *et al.* [13], we inevitably arrive at the MMR spacetime as the simplest and hence most suitable model for the exterior of a NS complying with the conditions of papers [3, 13]. The fact that the MMR solution is the simplest one possessing the required four moments is very important in itself because it makes this solution in a sense similar to the Kerr spacetime whose unique property is that it is the simplest possible solution among infinite number of the 2-parameter solutions defined by the parameters of mass and angular momentum. Clearly, the higher GH multiple moments of the MMK solution will then be some well-defined functions of the above four parameters that can be found from the corresponding axis data by means of the FHP procedure.

The explicit expressions of the multipoles M_{2n} and J_{2n+1}, $n \geq 2$, as functions of the moments (4), in the case of the MMR solution would give us the simplest hierarchy of the universal relations for NSs. Obviously, each relation from this hierarchy determines how the higher multipole M_{2n} or J_{2n+1}, with a specific n, depends on the first four lower moments (4); however, since such relations involve only one higher multipole, they do not actually provide any information about possible interrelations between the higher multipole moments themselves. So it is remarkable that there does exist a more sophisticated hierarchy of the universal relations for NSs that directly connect different higher multipoles with each other. This new hierarchy arises in (2) as a series of the degeneration conditions of the solitonic solutions with $N > 2$ to the $N = 2$ case. Indeed, as was shown in [24], the general N-soliton solution degenerates to the $(N-1)$-soliton case when the determinant L_N defined by (3) becomes equal to zero; then further degeneration would require zero values of the determinants L_{N-1}, L_{N-2} and so on, until we finally arrive at the 2-soliton solution by means of the conditions $L_3 = 0$, $L_2 \neq 0$, the latter nonequality being needed to stop the degeneration process. By inverting this reasoning, we can say that the higher multipole moments of the MMR 2-soliton solution must be such that the conditions

$$L_n = 0 \quad \text{for all} \quad n > 2 \tag{5}$$

are satisfied. It is easy to see from (3) that the above conditions (5) establish how the moments m_{2n-2} depend on the moments m_{2n-3}, or, roughly speaking and taking into account the equatorial symmetry of the MMR solution, how the GH mass multipoles M_{2n-2} depend on the spin multipoles J_{2n-3}.

The simplest of the relations (5), accounting for (4), takes the form

$$L_3 = \begin{vmatrix} M_2 & iJ_3 & m_4 \\ iJ & M_2 & iJ_3 \\ M & iJ & M_2 \end{vmatrix} = 0, \tag{6}$$

whence we get

$$m_4 = \frac{M_2^3 + 2JJ_3M_2 - MJ_3^2}{MM_2 + J^2}. \tag{7}$$

This formula is of importance because it permits us to establish the correct dependence of the hexadecapole moment M_4 on the angular momentum J in the solutions for NSs. In the paper [13] it was assumed, by analogy

with the Kerr solution, that the multipoles M_2, J_3 and M_4 of NSs are proportional, respectively, to J^2, J^3 and J^4, so that, according to Yagi *et al.*, the hexadecapole moment M_4 should not contain any term proportional to J^2. Supposing that $M_2 \propto J^2$ and $J_3 \propto J^3$, one can readily see that the quantity m_4 in (7) is proportional to J^4. Nonetheless, the relation of m_4 to the GH hexadecapole moment M_4 is defined by the formula [20]

$$m_4 = M_4 + \frac{1}{7}M(J^2 + MM_2), \tag{8}$$

and it is clear that the second term on the right-hand side of (8) is proportional to J^2, so that the general expression of M_4 should necessarily contain terms proportional both to J^2 and J^4. Moreover, the condition for M_4 to be proportional strictly to J^4 implies $M_2 = -J^2/M$, which is exactly the value of the mass-quadrupole moment of the Kerr solution. Mention that the situation will be the same if we opt to use the multipole moments constructed according to Thorne's definition [19], since these are known [29] to be proportional to the GH multipoles. Therefore, the extrapolation of the behavior of Kerr's higher moments to the corresponding multipole moments of the generic NS spacetime made in the paper [13] should be considered as an unfortunate mistake that springs from overlooking the non-vanishing terms proportional to J^2 in the definition of M_4.

As was remarked in [3], the universality of the relations for NSs discovered in recent years consists in particular in their non-dependence on the total mass M when these are rewritten in terms of the rescaled, dimensionless moments. Then, bearing this in mind and introducing the rescaled moments via the formulas

$$J = jM^2, \quad M_2 = qM^3, \quad J_3 = sM^4, \quad M_4 = \mu M^5, \tag{9}$$

it is possible, taking into account (8), to rewrite formula (7) in the 'M free' form

$$\mu = -\frac{1}{7}(j^2 + q) + \frac{q^3 + 2jqs - s^2}{j^2 + q}, \tag{10}$$

thus demonstrating the universal character of the relation $L_3 = 0$.

Note that the next relation from the hierarchy (5), $L_4 = 0$, which involves the multipoles M_6 and J_5, can be also shown to be universal in the sense that, written in terms of the dimensionless moments, it does not contain the mass M; however, here we do not write down this universal relation explicitly because of its rather complicated form. In principle, it is not difficult to show that, after an appropriate rescaling, M always cancels out from the generic relation $L_n = 0$ in the equatorially symmetric case under consideration.

An application.—Let us compare now the values of the hexadecapole moment M_4 calculated, for some available numerical models of NSs, with the aid of the Yagi *et al.* approach [13] based on Ryan's method [11], on the one hand, and our formula (7) after the substitution in it of (8), on the other hand. To obtain the values of the first type, which we denote as $M_4^{(n)}$, we used the hints left in [13] for the evaluation of M_4 in the case of EoSs AU and L; the concrete models were taken from tables II and VI of the Supplement to the paper [3]. For the same models we then estimated the mass-hexadecapole moments $M_4^{(a)}$ by means of our analytical formula. The results are summarized in Tables I and II, which also include the ratios $M_4^{(a)}/M_4^{(n)}$ for convenience. One can see that the correspondence between $M_4^{(n)}$ and $M_4^{(a)}$ for all three sequences of EoS AU from Table I is quite reasonable. We are sure the discrepancy will be further diminished after a correct way of reading off the GH mass-hexadecapole moment from Ryan's expansions will be found. We recall in this respect that for years some not very accurate definitions of the mass-quadrupole and spin-octupole moments were used in the numerical models of NSs, until Pappas and Apostolatos have been able recently to rectify those definitions [12]. The situation with the hexadecapole moment M_4 in the numerical models of NSs is analogous: the present definition introduced by Yagi *et al.* has intrinsic problems since it reproduces (and we do not really know how well) only one of the two parts of the genuine GH expression for M_4. As for Table II, it seems that the disagreement between the values $M_4^{(n)}$ and $M_4^{(a)}$ calculated for the models with the EoS L is rather substantial, whence we draw a tentative conclusion that the latter EoS is not really a good one for modeling the interior of NSs.

It is worth noting in conclusion that the scope of applicability of the universal relations considered in the present letter actually goes beyond the NSs only. Thus, for instance, the whole hierarchy (5) is eligible in the case of the Kerr solution too, and besides should be supplemented with the relation $L_2 = 0 \Leftrightarrow M_2 = -J^2/M$ which shows that a NS collapses to a black hole when its quadruple moment becomes that of the Kerr solution, independently of the value of its spin-octupole moment J_3. Moreover, other stellar objects with a richer structure than that of NSs and hence requiring more than four arbitrary real parameters for its description, could be analytically approximated by the $N = 3, 4, \ldots$ extended soliton solutions, in which case the inequality in (5) starts, respectively, from $3, 4, \ldots$ The degeneration conditions then would reflect in particular the evolution of stars from one type to another.

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TABLE I: Multipole moments for three sequences of numerical models constructed with EoS AU from table II of [12]. The procedure for evaluating $M_4^{(n)}$ is given in Ref. [13], while the values of $M_4^{(a)}$ have been obtained with the aid of formula (7) of the present paper.

M	J/M^2	M_2	J_3	$M_4^{(n)}$	$M_4^{(a)}$	$M_4^{(a)}/M_4^{(n)}$
2.072	0.201	-1.45	-1.14	2.217	1.947	0.878
2.087	0.414	-6.08	-10.0	40.21	25.51	0.634
2.097	0.529	-9.96	-21.2	107.7	71.38	0.662
2.108	0.616	-13.6	-34.1	199.1	121.0	0.608
2.112	0.661	-15.7	-42.7	264.4	161.4	0.610
3.164	0.194	-1.68	-1.37	2.202	1.825	0.829
3.207	0.406	-8.08	-14.5	41.97	30.23	0.720
3.253	0.550	-16.1	-41.4	140.3	116.2	0.828
3.291	0.645	-23.9	-75.1	263.7	253.1	0.960
3.318	0.706	-30.3	-107.	376.7	403.0	1.07
3.388	0.510	-12.9	-28.1	101.2	65.81	0.650
3.393	0.520	-13.7	-31.1	109.2	75.78	0.694
3.422	0.587	-19.1	-51.6	176.3	148.6	0.843
3.458	0.659	-26.0	-82.7	277.9	277.6	0.999
3.487	0.713	-32.5	-115.	378.4	428.8	1.133

TABLE II: Multipole moments for three sequences of numerical models constructed with EoS L from table VI of [12].

M	J/M^2	M_2	J_3	$M_4^{(n)}$	$M_4^{(a)}$	$M_4^{(a)}/M_4^{(n)}$
2.071	0.194	-2.76	-2.28	7.405	5.730	0.774
2.080	0.417	-12.2	-22.0	159.3	90.39	0.567
2.087	0.543	-20.3	-47.9	460.9	243.6	0.529
2.095	0.650	-28.6	-81.3	952.9	478.5	0.502
2.097	0.698	-32.9	-100.	1269.	628.6	0.495
4.012	0.178	-3.80	-4.23	16.8	8.943	0.532
4.051	0.375	-18.5	-45.4	338.5	138.6	0.410
4.098	0.528	-40.3	-144.	1367.	599.8	0.439
4.139	0.635	-62.6	-279.	2930.	1415.	0.483
4.167	0.700	-79.8	-401.	4399.	2276.	0.518
4.321	0.479	-29.5	-90.2	1059.	308.0	0.291
4.325	0.489	-31.9	-101.	1153.	358.4	0.311
4.355	0.555	-45.2	-170.	1950.	709.4	0.364
4.396	0.641	-66.0	-299.	3560.	1488.	0.418
4.420	0.686	-79.4	-394.	4742	2140.	0.451