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Generalized uncertainty principles, effective Newton constant and regular black holes

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Abstract

In this paper, we explore the quantum spacetimes that are potentially connected with the generalized uncertainty principles. By analyzing the gravity-induced quantum interference pattern and the Gedanken for weighting photon, we find that the generalized uncertainty principles inspire the effective Newton constant as same as our previous proposal. A characteristic momentum associated with the tidal effect is suggested, which incorporates the quantum effect with the geometric nature of gravity. When the simplest generalized uncertainty principle is considered, the minimal model of the regular black holes is reproduced by the effective Newton constant. The black hole's tunneling probability, accurate to the second order correction, is carefully analyzed. We find that the tunneling probability is regularized by the size of the black hole remnant. Moreover, the black hole remnant is the final state of a tunneling process that the probability is minimized. A theory of modified gravity is suggested, by substituting the effective Newton constant into the Hilbert-Einstein action.

Keywords: generalized uncertainty principle, effective Newton constant, characteristic momentum, regular black hole, quantum tunneling, WKB approximation.

1 Introduction

On the ground of dimensional analysis[1], the Planck length is defined as $\ell_{\rm P} \equiv \sqrt{\hbar G/c^3}$, where c is the speed of light, G Newton constant, and $\hbar$ Planck constant. This unit of length should appear in any theory reconciling general relativity and quantum theory. It is generally believed that $\ell_{\rm P}$ is the shortest measurable length, and quantum gravity effects(or quantum fluctuations in spacetimes) become crucial to understanding the physics on this length scale. As a classical theory, general relativity involves only c and G, and the minimal length cannot be predicted naturally by the theory itself. There are some problems are somewhat related to the defect that the Planck length is absent in the classical spacetimes.

One of problems is the spacetime singularity[2]. Following from Penrose and Hawking's theorems, the spacetime singularity is inevitable in the framework of classical general relativity. In a certain extent, the singularity is characterized by the divergence of Kretschmann scalars ($K^2 = R_{\rho\lambda\mu\nu}R^{\rho\lambda\mu\nu}$). For a Schwarzschild black hole, $K^2 \sim M^2/r^6$ become divergent, as $r \to 0$.

Another problem is the fate of black hole evaporation[3]. In the case of Hawking's temperature expression($T_H \sim M^{-1}$), the negative capacity($C \sim -M^{-2}$) makes the black hole evaporation faster and faster. Both the temperature and the mass loss rate($\dot{M} \sim -M^{-2}$) become divergent, if the black hole vanishes.

The third problem is related to the tunneling picture of the black hole radiation[4, 5, 6, 7, 8, 9]. The tunneling probability accurate to the first order correction becomes explosive[7, 9], if the final size of the black hole is allowed to be arbitrary small. When the second order correction to the tunneling probability is considered, the situation becomes worse[9]. We are confronted with an unacceptable picture that a black hole of any mass could vanish in an instant. This difficulty is associated with the absence of the minimal length, and then it is not necessarily overcome by improving the WKB method.

These problems are expected to be solvable at the presence of quantum gravity effects. The generalized uncertainty principle(GUP[10]), as one of methods of quantum gravity phenomenology, has been applied to the black hole thermodynamics in some heuristic manners[11, 12, 13]. GUP imposes a lower bound on the size of the black hole, and modifies the black hole's thermodynamics. However, it is hard to understand that the temperature approaches a value of order of Planck temperature, while the heat capacity of the minimal black hole vanishes. In other words, GUP predicts the existence of black hole remnant, but the temperature puzzle has not been solved completely.

This dilemma may be associated with such a working hypothesis in the literature that the matter is dominated by GUP, while the spacetimes are classical. This hypothesis can be named as the GUP-revised semiclassical theory. However, the corrections to classical spacetimes should be considered seriously, especially on the Planck scale. After all, a spacetime dominated by quantum gravity effects may be essentially different from the classical and smooth background. The quantum spacetime should reflect the existence of the minimal length, when the GUP is considered in an appropriate manner.

Some regular black holes with finite Kretschmann scalars have been suggested in the literature[14, 15, 16, 17, 18, 19, 20, 21, 22], and they give rise to the zero temperature remnants. As a tentative attempt, we suggest a regular black hole which is connected with the GUP by an effective Newton constant[19]. This suggestion is based on an observation upon the role that the GUP plays in the relation between the gravitational acceleration and Newton potential, in the context of operators. The effective Newton constant is motivated by substituting the GUP for Heisenberg commutator. However, this direct substitution of commutators is in the shortage of a clear physical picture. Moreover, it not consistent with the GUP-modified Hamiton equation[23], and its reliability should be checked by other methods. We expect that the effective Newton constant can be inspired by the GUP in some concrete physical processes. Another limitation of our previous work is that the simplest GUP(as presented in the next section) doesn't give rise to a regular black hole, although it means the existence of the minimal length. This shortage is related to such a characteristic momentum as $\Delta p \sim \hbar/r$. Although this momentum scale is motivated by Heisenberg's principle, it is irrelevant to the quantum fluctuations in spacetime, because it doesn't involve the Newton constant and the mass of source of gravitational field.

The aim of this paper is to gain a better understanding of the GUP-inspired effective Newton constant and quantum spacetimes, at the level of quantum gravity phenomenology. Firstly, the effective Newton constant inspired by the GUP will be reexamined in two Gedankens involving gravitation and quantum theory, i.e. COW phase shift and Einstein-Bohr's box. Secondly, the momentum scale will be reconsidered. In our opinion, an appropriate scale should reflect the amplitudes of the quantum fluctuations in the curved spacetime, and may be related to the geometric character of gravity. It would be different from that suggested in Ref.[19]. Thirdly, we will consider a regular black hole, which is inspired by the simplest GUP, i.e. the most popular version. Finally, quantum radiation from this black hole will be discussed seriously in this work. For a mini black hole, the temperature may lose its usual meaning in the thermodynamics. It is more reasonable to consider the quantum tunneling from this black hole.

We are interested in the role that the minimal length plays in the tunneling process, and in the question of whether the explosion of tunneling probability occurs in the quantum spacetime.

2 Effective Newton constant inspired by the generalized uncertainty principles

Based on some theoretical considerations and gedanken experiments for incorporating gravitation with quantum theory, Heisenberg's uncertainty principle is likely to suffer a modification as follows[10]

$$\Delta x \sim \frac{\hbar}{\Delta p} + \frac{\alpha \ell_{\rm P}^2}{\hbar} \Delta p, \tag{1}$$

Where α is a dimensionless number of order of unity. Since GUP means the minimal length of order of $\ell_{\rm P}$, it should be crucial for the Planck scale physics. Corresponding to (1), Heisenberg's commutator is extended to[26, 27]

$$[\hat{x}, \hat{p}] = i\hbar \left(1 + \frac{\alpha \ell_{\rm P}^2}{\hbar^2} \hat{p}^2 \right), \tag{2}$$

which will be considered seriously in this paper. However, there are some other types of the generalized uncertainty principles[24][25], and the commutation relations are not necessarily the same as (2). So we begin with a more general commutator as follows[19]

$$[\hat{x}, \hat{p}] = i\hbar z, \tag{3}$$

where $z = z(\hat{p})$ is a function of momentum. The average value of z, should ensure a lower bound on the measurable distance,

$$\Delta x \geq \frac{z\hbar}{\Delta p} \geq \ell_{\rm P}, \tag{4}$$

which suggests the discreteness of the spacetime[28]. Considering the relation between Heisenberg's principle and the wave-particle duality, let us derive the modified de-Broglie formula from the generalized commutator (3).

2.1 Modified wave-particle duality

It is well known that the proposal of uncertainty principle is closely related to the wave particle duality. Uncertainty relation can be derived from de Broglie formula, by analyzing the Heisenberg's microscope gedanken experiment or the single slit diffraction of light. However,

once the framework of quantum mechanics is established and Heisenberg's uncertainty relation is regarded as a fundamental principle, de Broglie formula becomes a deduction[29]. Concretely speaking, the momentum eigenstate $\psi_p = \exp(ipx/\hbar)$ can be derived from the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$, and de Broglie formula is obtained by comparing the momentum eigenstate with a plane wave function $\exp(2\pi ix/\lambda)$. It is expectable that de Broglie formula should suffer a modification, when the Heisenberg's commutator is changed. Corresponding to the generalized commutator (2), the modified de Broglie relation is given by[26, 27]

$$\lambda = \frac{2\pi\ell_{\rm P}\sqrt{\alpha}}{\arctan(\ell_{\rm P}\sqrt{\alpha}p/\hbar)}. \tag{5}$$

It is easy to check that Eq.(5) satisfies the following relation

$$\frac{d}{dp}\left(\frac{2\pi}{\lambda}\right) = \hbar^{-1}\left(1 + \frac{\alpha\ell_{\rm P}^2}{\hbar^2}p^2\right)^{-1}. \tag{6}$$

As argued in the following, a more general formula associated with the commutator (3) is given by

$$\frac{d}{dp}\left(\frac{2\pi}{\lambda}\right) = \hbar^{-1}z^{-1}. \tag{7}$$

In order to explain this formula, we first construct a commutator as follows

$$[\hat{x}, \hat{k}] = i, \tag{8}$$

where $\hat{k} = k(\hat{p})$ is a function of momentum operator. Following from the law of operator algebra, we obtain

$$[\hat{x}, \hat{p}] = [\hat{x}, k]\frac{d\hat{p}}{dk} = i\frac{d\hat{p}}{dk}. \tag{9}$$

Comparing (3) with (9), we have

$$\frac{dk}{d\hat{p}} = \hbar^{-1}z^{-1},$$

and then

$$k(\hat{p}) = \hbar^{-1}\int z^{-1}d\hat{p}. \tag{10}$$

Obviously, $[\hat{p}, k(\hat{p})] = 0$, this means that there is a common eigenstate ψ_p of eigenvalue p, which satisfies

$$\begin{aligned} \hat{p}\psi_p &= p\psi_p, \\ k(\hat{p})\psi_p &= k(p)\psi_p, \end{aligned} \tag{11}$$

where

$$k(p) = \hbar^{-1} \int z^{-1}(p) dp, \tag{12}$$

is the eigenvalue of the operator $k(\hat{p})$. Since $[\hat{x}, \hat{p}] \neq i\hbar$, the momentum operator is no longer represented by $\hat{p} = -i\hbar\nabla$. However, comparing (8) with Heisenberg's commutator, we obtain $\hat{k} = -i\nabla$. For one dimensional case, the second equation of (11) becomes

$$-i\frac{d\psi_p}{dx} = k(p)\psi_p. \tag{13}$$

So the momentum eigenstate is given by

$$\psi_p = \exp(ikx), \tag{14}$$

which describes a plane wave of wavelength $\lambda = 2\pi/k$. Thus $k(\hat{p})$ introduced in (8) can be viewed as the wave-vector operator, and (12) is just the wave-number. The modified wave-particle duality is characterized by (7) or (12), which is the basis for the following discussions.

2.2 COW phase shift

In 1975, Colella, Overhauser, and Werner(COW) observed the gravity-induced quantum interference pattern of two neutron beams[30]. When the plane of two beams is vertical to the horizontal plane, the phase shift is given by[31, 32]

$$\Delta\varphi = \frac{mgA}{\hbar v}, \tag{15}$$

where g denotes the earth's gravitational acceleration, v the average speed of neutrons, and A the area enclosed by two interfering neutron beams that propagate on two paths on a plane.

This famous experiment may be regarded as a test of the property of gravity in the microscopic world[33]. It is naturally expected to shed light on the quantum structure of spacetime, by attaching the GUP's significance to the gravity-induced phase shift. In the following discussions, COW experiment will be revisited in a heuristic manner[32]. Let us consider two interfering neutron beams. For simplicity, the plane of two beams is set to be vertical to the horizontal plane. The first (upper) beam propagates on a horizontal path and a vertical downward path in sequence. The second (lower) beam propagates on a vertical downward path and a horizontal path in sequence. The area enclosed by two beams is $A = yl$, where l is the length of each horizontal path, and y is the height of the upper horizontal path with respect to the lower horizontal one. It is shown by simple analysis that the change in the phase of one vertical beam

cancel out that of another vertical beam, and then the phase shift is completely attributed to the gravity-induced difference in the wavelength of two horizontal beams,

$$\begin{aligned}\Delta\varphi' &= 2\pi\left(\frac{l}{\lambda_2}-\frac{l}{\lambda_1}\right)\\ &= l(k_2-k_1)\\ &= l\Delta k = l\frac{\Delta k}{\Delta p}\Delta p,\end{aligned} \tag{16}$$

where $\Delta p = p_2 - p_1$ is the difference in momentum of two horizontal beams. Since Δp is a small quantity, Eq.(16) can be expressed as

$$\begin{aligned}\Delta\varphi' &\approx l\frac{dk}{dp}\Delta p\\ &= \hbar^{-1}z^{-1}l\Delta p,\end{aligned} \tag{17}$$

where Eq.(7) has been considered. The neutron beams propagate in the earth's gravitational field, and obey energy conservation law, so we have

$$\begin{aligned}mgy &= \frac{p_2^2-p_1^2}{2m}\\ &= v\Delta p,\end{aligned} \tag{18}$$

where the earth's rotation is neglected, and $v = (p_1 + p_2)/2m$. Substituting (18) into (17), we obtain

$$\Delta\varphi' = \frac{mgyl}{z\hbar v} = \frac{mg'A}{\hbar v}, \tag{19}$$

where $g' = g/z, A = yl$. When $z = 1$, Eq.(19) returns to (15), which is just the earlier result predicted by usual quantum theory.

As shown by Eqs.(15) and (19), the expression for the corrected phase shift is almost the same as the usual result, except a momentum-dependent factor z. The latter can be obtained from the former by replacing g with g/z. The GUP's significance to COW experiment is equivalent to the situation that two neutron beams propagate in a modified gravitational field characterized by the effective field strength $g' = g/z$.

2.3 Weighting photon

In 1930, Einstein devised a subtle gedanken experiment for weighting photon[34, 35], and tried to demonstrate the inconsistency of quantum mechanics. Einstein considered a box that contains photon gas and hangs from a spring scale. An ideal clock mechanism in the box can open a

shutter. Einstein assumed that the ideal clock could determine the emission time exactly (i.e. $\Delta t \to 0$), when a photon was emitted from the box. On the other hand, the energy of the emitted photon can be obtained by measuring the difference in the box's mass. This seemed to result in $\Delta E \Delta t \to 0$, and violate the uncertainty relation for energy and time.

However, as pointed out by Bohr[34, 35], Einstein neglected the time-dilation effect, and then his deduction was incorrect. Following from general relativity, the time-dilation is attributed to the difference in the gravitational potential. Two clocks tick at different rates if they are at different heights. For the clock in the box, the time uncertainty due to the vertical position uncertainty Δx is given by[34, 35]

$$\Delta t = \frac{g\Delta x}{c^2} t, \tag{20}$$

where t denotes a period of weighting the photon. Bohr argue that the accuracy of the energy of the photon is restricted as

$$\Delta E \geq \frac{c^2 \hbar}{g t \Delta x}. \tag{21}$$

When Eq. (20) is considered, the inequality (21) gives rise to the uncertainty relation $\Delta E \Delta t \geq \hbar$, and the consistency of quantum theory is still maintained. Obviously, gravity plays an important role in the Bohr's argument.

In the following, GUP will be considered along the line of Bohr's argument, and its significance to gravitation will be analyzed in the gedaken experiment for weighting the photon. We first read the original position of the pointer on the box before the shutter opens. After the photon is released, the pointer moves higher than its original position. In order to lower the pointer to its original position, we hang some little weights on the box. The pointer returns to its original position after a period t. The photon's weight $g\Delta m$ equals the total weight that hangs on the box. Obviously, the accuracy of weighting the photon is determined by the minimum of the added weight. The measurement becomes meaningless, if the added weight is too small to be observable. The weight $g\Delta m$ should be restricted by quantum theory. Let Δx denote the accuracy of measuring the position of the pointer(or of the clock), the minimum of the momentum uncertainty is given by

$$\Delta p_{min} = \frac{z\hbar}{\Delta x}, \tag{22}$$

where (4) has been considered. Over a period t, the smallest weight is $\Delta p_{min}/t$, which is the quantum limit of weighting the photon. Thus we obtain

$$\frac{z\hbar}{t\Delta x} = \frac{\Delta p_{min}}{t} \leq g\Delta m,$$

or

$$z\hbar = \Delta x \Delta p_{min} \leq \Delta m (g\Delta x) t. \tag{23}$$

Substituting (20) into (23), the latter becomes

$$z\hbar \leq c^2 \Delta m \Delta t = \Delta E \Delta t. \tag{24}$$

Such a modified uncertainty relation means the shortest interval of the time. It can be explained as follows. Since GUP is required to ensure the shortest observable length, we have (4), and then $z\hbar \geq \ell_{\rm P}/\Delta p$. The time uncertainty is restricted by $\Delta t \geq z\hbar/\Delta E \geq \ell_{\rm P}(\Delta p/\Delta E) \approx \ell_{\rm P}(dp/dE)$. dE/dp is the speed of the box, so we obtain $\Delta t \geq \ell_{\rm P}/v \geq \ell_{\rm P}/c = t_{\rm P} = \sqrt{G\hbar/c^5}$.

Now we turn our attention to the inequality (23), from which the accuracy of the energy of the emitted photon is restricted as

$$\Delta E \geq \frac{zc^2\hbar}{gt\Delta x} = \frac{c^2\hbar}{g' t \Delta x}, \tag{25}$$

Comparing it with (21), the difference is only that g is replaced by $g' = g/z$. Let us define $p' = \hbar k$, which denotes the canonical momentum and satisfies Heisenberg's commutation relation as follows

$$[\hat{x}, \hat{p}'] = i\hbar. \tag{26}$$

The corresponding uncertainty $\Delta p'$ can be expressed as

$$\Delta p' = \hbar \Delta k \approx \hbar \frac{dk}{dp} \Delta p = \Delta p/z, \tag{27}$$

where (12) has been considered. On the other hand, the inequality (23) can be rewritten as

$$\hbar = \Delta x \Delta p_{min}/z \leq \Delta m (g\Delta x) t/z. \tag{28}$$

Considering (27) and (23), we have

$$\begin{aligned} \hbar = \Delta x \Delta p'_{min} &= \Delta x \Delta p_{min}/z \\ &\leq \Delta m (g\Delta x) t/z = \Delta m (g' \Delta x) t. \end{aligned} \tag{29}$$

We find that g' is reproduced in the inequality (29), accompany with the return of Heisenberg principle. The means that g' should be understood in the context of usual quantum theory. According to the formula (20), when $g \to g'$, the time uncertainty becomes

$$\Delta t' = \frac{g' \Delta x}{c^2} t = \Delta t/z. \tag{30}$$

Substituting it into (29), we obtain

$$\hbar = \Delta x \Delta p'_{min} \leq \Delta E \Delta t', \tag{31}$$

as required by usual quantum theory. The consistency of theory is maintained by the transformation: $g \to g'$, when the GUP's significance is explained in the context of usual quantum mechanics.

In summary, we suggest two pictures for understanding the COW phase shift and the gedanken experiment of weighting the photon. One picture is that GUP is considered directly in a classical gravitational field. In another picture, the usual quantum theory is retained by introducing the effective gravitational field strength g'. Two pictures are equivalent, since an observer cannot distinguish the effect of GUP from the effective field strength. This equivalence inspires an effective Newton constant G', since the effective field strength can be expressed as

$$g' = g/z = \frac{GM}{zR^2} = \frac{G'M}{R^2},$$

where $G' = G/z$ is just the same as the suggestion in Ref.[19].

In view of the above analysis, we introduce two working hypotheses: (i) the matters obey Heisenberg's uncertainty principle; (ii) quantum spacetimes are characterized by the GUP-inspired effective Newton constant. They are the basis for the following discussions. When G is replaced by G', we obtain a modified Schwarzschild metric as follows

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2G'M}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2G'M}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega, \\ G' &= G/z. \end{aligned} \tag{32}$$

This metric describes a family of spacetimes that depend on different scales of momentum. In the following section, we will propose a characteristic momentum to incorporate quantum effect with geometric character of gravity.

3 Gravitational tidal force and the characteristic momentum

Now we consider a black hole described by (32), with $z = 1 + \alpha \ell_{\rm P}^2 p^2/\hbar^2$. Let T denote the black hole temperature, and the characteristic momentum is identified with $k_B T/c$[36, 37], the metric (32) becomes

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2G\widetilde{M}}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2G\widetilde{M}}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega, \\ \widetilde{M} &= \frac{M}{1 + \alpha \ell_{\rm P}^2 k_B^2 T^2/c^2\hbar^2}. \end{aligned} \tag{33}$$

The horizon is located by $r_T = 2G\widetilde{M}/c^2$. The temperature, proportional to the surface gravity, is determined by

$$T = \frac{m_{\rm P}^2 c^2}{8\pi k_B \widetilde{M}} = \frac{m_{\rm P}^2 c^2}{8\pi k_B M}\left(1 + \frac{\alpha \ell_{\rm P}^2 k_B^2 T^2}{c^2 \hbar^2}\right), \tag{34}$$

where $m_{\rm P} = \sqrt{\hbar c/G}$ is the Planck mass. So we obtain

$$T = \frac{4\pi M - \sqrt{(4\pi M)^2 - \alpha m_{\rm P}^2}}{\alpha k_B/c^2}, \tag{35}$$

which returns to the usual formula $T_H = m_{\rm P}^2 c^2/(8\pi k_B M)$, as $\alpha \to 0$. Except an inessential factor, the modified temperature (35) is consistent with the previous work in the literarure[11, 12, 13, 36, 37]. Certainly, those old problems have yet not been solved. The expression (35) gives rise to the maximum temperature when the mass approaches the minimal value $\sqrt{\alpha} m_{\rm P}/4\pi$. Furthermore, the metric (33) indicates that there is still a singularity at $r = 0$. However, the minimal mass means a lower bound on the size of black hole, $r_T \geq \sqrt{\alpha}\ell_{\rm P}/4\pi$. It is of order of the shortest observable distance derived from GUP. This make us believe that quantum spacetime is still characterized by the effective Newton constant $G' = G/z$, if an appropriate characteristic scale is taken into account. The shortage of the metric (33) may be attributed to the fact that the black hole temperature is not an universal scale of meaning, since it can't describe an ordinary star. Moreover, the black hole temperature is position independent, and doesn't reflect the difference between strong gravitational field and weak field. It is necessary to reinvest the momentum scale with new physical meaning, if we expect for something beyond the previous efforts. As an observable quantity, $p^2 \geq \Delta p^2$. In view of the limitation of the black hole temperature, we require the quantum fluctuation Δp to satisfy some reasonable expectations.

Firstly, Δp should be associated with the gravity, and should increase with the strength of the gravitational field, i.e. $\Delta p \sim r^{-s}$, $s > 0$.

Secondly, Δp should play a crucial role that improves the spacetime singularity, and make the black hole regular. As argued in Refs.[15, 17, 19], the asymptotic behavior of the regular potential must satisfy $\phi \to r^{2+\delta}$, as $r \to 0$. This demands $\Delta p \sim r^{-3/2-\delta}$, and $\delta \geq 0$.

Thirdly, the minimal value of Δp is intrinsic, and reflect the universal property of the gravitational fields of black holes and ordinary stars. According to general relativity, the gravitational field is regarded as the curved spacetime characterized by Riemann tensor $R_{\rho\lambda\mu\nu}$. This suggests Δp be associated with those quantities constructed by Riemann tensor. Such a characteristic momentum may be estimated by combining the gravitational tidal force with quantum theory, since the tidal force is associated with the curvature of spacetime[1, 38].

Let us consider a pair of virtual particles with energy ΔE. When the virtual particles are separated by a distance Δx, according to the geodesic deviation equation, the tidal force reads[1, 38, 39]

$$F = \frac{2GM}{r^3}\left(\frac{\Delta E}{c^2}\right)\Delta x. \tag{36}$$

Let Δt denote the life-time of the virtual particles, the momentum uncertainty due to the tidal force is given by

$$\Delta p = F\Delta t = \frac{2GM}{r^3}\left(\frac{\Delta E}{c^2}\right)\Delta t\Delta x. \tag{37}$$

The observability requires $\Delta p\Delta x \geq \hbar,\ \Delta E\Delta t \geq \hbar$, when the virtual particles are subject to the tidal force and become real. Following from (37), we obtain

$$\begin{aligned}(\Delta p)^2 &\geq \frac{\hbar\Delta p}{\Delta x} = \frac{\hbar F}{\Delta x}\Delta t \\ &= \frac{2\hbar}{c^2}\left(\frac{GM}{r^3}\right)\Delta E\Delta t \geq \frac{2\hbar^2}{c^2}\left(\frac{GM}{r^3}\right).\end{aligned} \tag{38}$$

The right hand side of the inequality suggests a characteristic momentum, $\Delta p_m \sim \sqrt{GM\hbar^2/c^2r^3}$. We find that $\Delta p_m \to 0$ as $G \to 0$, which is associated with a free particle traveling in the flat spacetime. This characteristic scale can be understood as the minimal momentum of those particles produced from the quantum fields in the curved spacetime.

We can also analyze a real particle which is detected by a photon with energy ΔE. Let Δx denote the uncertainty in the position of the particle, the momentum uncertainty of the particle is given by

$$\Delta\tilde{p} \geq \frac{\hbar}{\Delta x} + \frac{2GMm}{r^3}\Delta x\Delta t, \tag{39}$$

where m is the mass of the particle, and $\Delta t \geq \hbar/\Delta E$ is the characteristic time in the process of the photon-particle collision. On the right hand side of the inequality (39), the first and the second terms belong different stories respectively. The second term is attributed to the tidal effect of gravity, and it vanishes in a flat spacetime. Following from (39), we obtain

$$\Delta\tilde{p} \geq 2\sqrt{\frac{2GM\hbar m\Delta t}{r^3}} \geq 2\sqrt{\frac{2GM\hbar^2}{r^3}\left(\frac{m}{\Delta E}\right)}, \tag{40}$$

where the time-energy uncertainty relation is considered. In order to avoid the production of new particles, the energy should be restricted as $\Delta E < mc^2$, otherwise it becomes meaningless to measure the position of the particle[40]. So we obtain

$$\Delta\tilde{p} > 2\sqrt{\frac{2GM\hbar^2}{c^2r^3}}. \tag{41}$$

The difference between (38) and (41) is only a constant coefficient. This inessential difference is caused by a rough estimate of the amplitude of ΔE. It will vanish, if we take a different estimate, such as $\Delta E < mc^2/4$.

In the above discussion, we don't discriminate the black hole from an ordinary star, so the characteristic momentum appeared in (38) is suitable for both of the two spacetimes. A similar scale has also been suggested by a different way[15], but its physical meaning is different from our understanding. As one of two scales suggested in Ref.[15], it is identified with the inverse of the proper time of an observer falling into the Schwarzschild black hole. Obviously, it is different from that scale of an ordinary star.

Identifying the characteristic scale with the right hand side of (38), we obtain an effective Newton constant as follows

$$G' = \frac{G}{1 + 2\alpha \ell_{\mathrm{P}}^2 \Delta p_m^2/\hbar^2} = \frac{G}{1 + 2\alpha \ell_{\mathrm{P}}^2 GM/r^3 c^2}. \tag{42}$$

Different from Ref.[15], the scale $\Delta p \sim 1/r$ is no longer considered in the following discussions. This is because it is motivated by Heisenberg principle, while Heisenberg principle has been incorporated with the tidal effect in the inequalities (38) and (39). We will explore a black hole characterized by the effective Newton constant as presented by (42).

4 Quantum tunneling from the regular black hole

In this section, we take the Planck units, $G = \hbar = c = k_B = 1$. Substituting (42) into (32), we obtain a modified Schwarzschild black hole as follows

$$\begin{aligned} ds^2 &= -\left(1 + 2\Phi\right) dt^2 + \left(1 + 2\Phi\right)^{-1} dr^2 + r^2 d\Omega, \\ \Phi &= -\frac{Mr^2}{r^3 + 2\alpha M}. \end{aligned} \tag{43}$$

It is just the minimal model of the regular black hole suggested in Ref.[17]. Let ρ denote the radius of this black hole and satisfy $g^{11}(\rho) = 0$, we have

$$1 - \frac{2M\rho^2}{\rho^3 + 2\alpha M} = 0. \tag{44}$$

The horizon is located by

$$\rho = \frac{2M}{3} + \frac{4M}{3} \cos\left[\frac{1}{3} \arccos\left(1 - \frac{27\alpha}{8M^2}\right)\right], \tag{45}$$

provided $M \geq M_c = 27\alpha/16$. When $M \leq M_c$, the metric (43) doesn't describe a black hole, since the equation (44) has no positive solution and then the horizon is absent in this spacetime[17].

The critical mass is the lower bond on the mass of an object that forms a black hole. Corresponding to this critical mass, there is a minimal radius of the black hole, $\rho_{min} = 4M_c/3 = \sqrt{3\alpha}$. In the following, we will investigate the quantum tunneling of this regular black hole, and focus on the question of whether the tunneling probability is regularized by the minimal length.

In the tunneling picture of black hole radiation[4, 5], the tunneling probability is determined by the imaginary part of the action for a particle which tunnels through the horizon along a classically forbidden trajectory. At the zeroth order WKB approximation, the tunneling probability is suppressed by the change in the Bekenstein-Hawking entropy. This is consistent with the unitarity of quantum theory. When the second order correction is considered, the tunneling probability is given by[9]

$$\Gamma \sim \frac{\rho_i^2}{\rho_f^2} \exp[-2\mathbf{Im}(S_0 - S_2)], \tag{46}$$

where ρ_i denotes the initial radius of black hole in the tunneling process, and ρ_f the final radius. $S_0 - S_2$ is the action for a particle crossing the horizon from ρ_i to ρ_f. Concretely speaking, S_0 is associated with the zeroth order term of WKB wave function, and S_2 is related to the second order correction. The first order term S_1 doesn't appear in the imaginary part of the action, since it is real. In order to evaluate the emission rate of the regular black hole, we first introduce the Painleve type coordinate[4, 6]

$$\tilde{t} = t + \int \frac{\sqrt{-2\Phi}}{1+2\Phi} dr.$$

The metric (43) is therefore rewritten as

$$ds^2 = -(1+2\Phi)d\tilde{t}^2 + 2\sqrt{-2\Phi}d\tilde{t}dr + dr^2 + r^2 d\Omega. \tag{47}$$

It is appropriate for describing the particle which tunnels through the horizon, since the coordinate singularity has been removed. Setting $ds^2 = 0 = d\Omega$, we obtain the equation of the radial null geodesics as follows

$$\dot{r} = \frac{dr}{d\tilde{t}} = 1 - \sqrt{-2\Phi}, \tag{48}$$

where the ingoing geodesics is neglected. When a particle is emitted from the black hole and the energy conservation is considered, a shell with energy ω' travals in a spacetime of mass $M' = M - \omega'$. So we have

$$\begin{aligned} 1 + 2\Phi &= \frac{r^3 - 2M'r^2 + 2\alpha M'}{r^3 + 2\alpha M'} \\ &= -\frac{r^2 - \alpha}{r^3 + 2\alpha M'} \left(2M' - \frac{r^3}{r^2 - \alpha} \right). \end{aligned} \tag{49}$$

The zero order action is given by

$$S_0 = \int_{\rho_i}^{\rho_f} S_0' dr = \int_{\rho_i}^{\rho_f} p_r dr, \tag{50}$$

where

$$S_0' = p_r = \int \frac{dM'}{\dot{r}} = \int_M^{M-\omega} \frac{dM'}{1-\sqrt{-2\Phi}}. \tag{51}$$

Substituting (49) into (51), we obtain

$$\begin{aligned} p_r &= \int_M^{M-\omega} \frac{1+\sqrt{-2\Phi}}{1+2\Phi} dM' \\ &= -\int_M^{M-\omega} \frac{(1+\sqrt{-2\Phi})(r^3+2\alpha M')}{(r^2-\alpha)[2M'-r^3/(r^2-\alpha)]} dM'. \end{aligned} \tag{52}$$

There exists a singularity at $2M' = r^3/(r^2-\alpha)$. In order for the positive frequency modes to decay with time[4], we deform the contour into the lower half ω' plane, or into the upper half M' plane. By residue theorem, we obtain

$$\begin{aligned} p_r &= \left(\frac{-i\pi}{2}\right) \frac{2\times[r^3+\alpha r^3/(r^2-\alpha)]}{r^2-\alpha} \\ &= (-i\pi)\frac{r^5}{(r^2-\alpha)^2}. \end{aligned} \tag{53}$$

Substituting it into (50), we get the imaginary part of the action as follows

$$\mathbf{Im} S_0 = -\frac{\pi}{2}\left[r^2 + 2\alpha\ln(r^2-\alpha) - \frac{\alpha^2}{r^2-\alpha}\right]\Bigg|_{\rho_i}^{\rho_f}. \tag{54}$$

Following the procedure of WKB method applied in Ref.[9], we can also evaluate the higher order terms of the action. The first order term is determined by the following equation

$$S_1' = -\frac{S_0''}{2S_0'} = -\frac{r^2-5\alpha}{2r(r^2-\alpha)}.$$

So we have

$$S_1'' = \frac{r^4-14\alpha r^2+5\alpha^2}{2r^2(r^2-\alpha)^2},$$

and then

$$S_2' = -\frac{S_1'^2+S_1''}{2S_0'} = (-i)\times\frac{3r^4-38\alpha r^2+35\alpha^2}{8\pi r^7}. \tag{55}$$

The imaginary part of the second order term is given by

$$\begin{aligned} \mathbf{Im} S_2 &= \mathbf{Im}\int_{\rho_i}^{\rho_f} S_2' dr \\ &= \frac{1}{96\pi}\left(\frac{18}{r^2} - \frac{114\alpha}{r^4} + \frac{70\alpha^2}{r^6}\right)\Bigg|_{\rho_i}^{\rho_f}. \end{aligned} \tag{56}$$

Substituting (54) and (56) into (46), and considering $\rho_i^2/\rho_f^2 = \exp\ln(\rho_i^2/\rho_f^2)$, we obtain the tunneling probability accurate to the second order correction, $\Gamma \sim e^{\Delta S}$, where

$$\Delta S = \left[\pi r^2 - \ln r^2 + \frac{3}{8\pi r^2} + 2\alpha\pi\ln(r^2-\alpha) - \frac{\alpha^2\pi}{r^2-\alpha} - \frac{19\alpha}{8\pi r^4} + \frac{35\alpha^2}{24\pi r^6}\right]\Bigg|_{\rho_i}^{\rho_f}. \tag{57}$$

In the consideration of the unitarity of quantum theory, ΔS should be understood as the change in the entropy of the regular black hole. The entropy, including the first and the second order corrections, reads

$$S = \pi\rho^2 - \ln\rho^2 + \frac{3}{8\pi\rho^2} + 2\alpha\pi\ln(\rho^2-\alpha) - \frac{\alpha^2\pi}{\rho^2-\alpha} - \frac{19\alpha}{8\pi\rho^4} + \frac{35\alpha^2}{24\pi\rho^6}, \tag{58}$$

where ρ is the radius of the black hole. The first three terms are similar to the expression for the Schwarzschild black hole[9], while the last four terms are new. New corrections are relevant to the parameter α, and denote the difference between the regular black hole and the Schwarzschild black hole.

Let us make some remarks on the expressions (57) and (58). Let us consider the thermodynamical entropy of the regular black hole. As the inverse period of the imaginary time of the regular spacetime (43), the black hole temperature is given by

$$\begin{aligned} T &= \frac{1}{2\pi}\left(\frac{d\Phi}{dr}\right)_{r=\rho} \\ &= \frac{1}{8\pi M} - \frac{\alpha}{2\pi\rho^3} = \frac{\rho^2-3\alpha}{4\pi\rho^3}, \end{aligned} \tag{59}$$

where (44) has been considered. The thermodynamical entropy is defined as

$$\begin{aligned} S^{(0)} &= \int\frac{dM}{T} = \int T^{-1}\left(\frac{dM}{d\rho}\right)d\rho \\ &= 2\pi\int\frac{\rho^5 d\rho}{(\rho^2-\alpha)^2} \\ &= \pi\left[\rho^2 + 2\alpha\ln(\rho^2-\alpha) - \frac{\alpha^2}{\rho^2-\alpha}\right], \end{aligned} \tag{60}$$

which is different from (57). However, it is consistent with the entropy derived from the zero order action of WKB method, as shown by (54). This is similar to the Schwarzschild black hole: the zero order action for the tunneling particle is related to the change in the Bekenstein-Hawking entropy[4, 9].

For the Schwarzschild black hole, the emission rate accurate to the second order approximation, is determined by the first three terms in (57). Since classical general relativity doesn't restrict the size of black hole, ΔS and Γ become divergent as $\rho_f \to 0$. However, the tunneling

probability of the regular black hole is finite, because it is regularized by the minimal radius of the horizon. This conclusion is nontrivial, in view of the subtle relation between the entropy expression (58) and the minimal radius ρ_{min}. If ρ_{min} is allowed to be less than $\sqrt{\alpha}$, the entropy would be ill defined for the black hole of radius $\rho = \sqrt{\alpha}$, because of the divergence of the fourth and the fifth terms in (58). It is gratifying that $\rho_{min} = \sqrt{3\alpha} > \sqrt{\alpha}$, and those dangerous terms such as $(\rho^2 - 3\alpha)^{-1}$ don't appear in the entropy expression.

According to the third law of thermodynamics, the entropy vanishes when a system of matter is in the ground state and its temperature approaches zero. For a given excited state, the probability of the transition to the ground state should be minimal, because it is greatly suppressed by the change in the entropy. The regular black hole has similar property, if the parameter α is not too small. This is because the entropy expression (58) is a monotonic increasing function of the horizon area.[1] For an initial black hole with radius ρ_i, the minimal value of the tunneling probability is given by

$$\Gamma \sim \exp\left[-\pi\rho_i^2 + \ln\rho_i^2 - \frac{3}{8\pi\rho_i^2} - 2\alpha\pi\ln(\rho_i^2 - \alpha) + \frac{\alpha^2\pi}{\rho_i^2 - \alpha} + \frac{19\alpha}{8\pi\rho_i^4} - \frac{35\alpha^2}{24\pi\rho_i^6}\right], \tag{61}$$

which points to the black hole remnant with final radius $\rho_f = \sqrt{3\alpha}$.

In the expression (58), the fourth and the fifth terms is a part of the thermodynamical entropy of the regular black hole. It is interesting and confusing that they tend to cancel out the similar corrections to the entropy of the Schwarzschild black hole, such as the second and the third terms in (58). This fact might indicate a subtle correlation between quantum spacetimes and the quantum matters, but we don't know how to explain it. We also notice that the regular black hole gives rise to the higher order corrections to the entropy, such as the last two terms in (58). We predict that the similar and opposite corrections might appears in the tunneling probability of a Schwarzschild black hole, when the fourth order WKB approximation is considered.

5 Summary and outlook

This work involves two parts. The first part is devoted to the question of what is the significance of the GUP for the quantum spacetime. The answer may point to the a scale-dependent Newton constant, which is motivated by analyzing the role that the GUP plays in the COW phase shift and Einstein-Bohr's Gedanken for weighting photon. It is consistent with our previous suggestion

[1] It can be shown by numerical method that $dS/d\rho > 0$, when the parameter satisfies $\alpha > 0.024$.

in Ref.[19]. The minimal model of the regular black hole can be reproduced by considering the simplest GUP and a momentum scale associated with the tidal force. The second part is to calculate the tunneling probability accurate to the second order WKB approximation. The tunneling probability is regular, because the black hole has a nonzero minimal radius. Not only this, the tunneling probability of an initial black hole is minimized by the black hole remnant, if the parameter α is of order of the unity. In other words, the tunneling probability is minimal, if the final state of the black hole is a remnant.

Let us consider the matter source of the regular black hole. In this paper, the quantum spacetime is understood by connecting the GUP with the running of Newton constant. It reflects the quantum gravitational effects on the classical spacetime. According to the general theory of relativity, the effective stress-energy tensor can be derived from Einstein's field equation, when the regular black hole is regarded as an input. The quantum gravitational effects are simulated by a matter fluid described by the effective stress-energy tensor[15, 17]. We hope that this matter fluid can be reproduced from the GUP dominated vacuum fluctuations. This problem will be investigated in the future.

Besides constructing the above regular black hole, we also explore a theory of modified gravity. This alternative theory is based on a generalization of the effective Newton constant, and it may be characterized by a modified Hilbert-Einstein action as follows

$$I' = \int \frac{R - 2\Lambda}{16\pi G'} \sqrt{-g} d^4x, \tag{62}$$

where $G = c = 1$, $G' = z^{-1}(p)$, and Λ is the cosmological constant. In order for the Lagrangian to be an invariant, the momentum scale p is restricted to be a scalar. For a Schwarzshild spacetime, $p^2 \sim M/r^3$, as argued in the section 3. This suggests that the characteristic momentum should be identified as $p \sim \sqrt{K}$, and then $z = z(K)$, where K is the square root of the Kretschmann scalar. Considering the simplest GUP[as given by (2)], the gravitational action can be expressed as

$$I = \frac{1}{16\pi G} \int (1 + \gamma K)(R - 2\Lambda) \sqrt{-g} d^4x, \tag{63}$$

where γ is a parameter, which is not necessarily the same as that in the metric (43). The action (63) belongs to a class of more general theories of modified gravity[41, 42, 43], and then the field equation is given by

$$(1 + \gamma K)(G_{\mu\nu} + \Lambda g_{\mu\nu}) + \gamma H_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{64}$$

where $G_{\mu\nu}$ is the Einstein tensor, and

$$H_{\mu\nu} = \frac{R-2\Lambda}{K} R_{\rho\lambda\sigma\mu} R^{\rho\lambda\sigma}{}_{\nu} + (g_{\mu\nu}\nabla_\sigma\nabla^\sigma - \nabla_\mu\nabla_\nu)K - 2\nabla_\rho\nabla_\sigma \left[\frac{R-2\Lambda}{K} R^{\rho}{}_{(\mu\nu)}{}^{\sigma} \right]. \tag{65}$$

Direct calculation shows that the metric (43) is not a solution for the field equation (64). It is not strange, since the metric (43) and the equation (64) are suggested along different lines of argument, even though they are motivated by the effective Newton constant. However, the regular spacetime (43) has a de Sitter core near $r = 0$, which satisfies the field equation (64). This implies that the field equation (64) permits the existence of the regular black holes. We also take notice of those terms associated with the cosmological constant Λ in (64), i.e. $(1 + \gamma K)\Lambda g_{\mu\nu}$. Usually, the first term $\Lambda g_{\mu\nu}$ is utilized to cancel out the huge contribution from the vacuum energy on the right hand side of the field equation, where the bare Λ must be of order of unity. Thus the second term of $\gamma K\Lambda$ play the role of the effective cosmological constant. It is interesting that this term is of order of the observed value. The modified gravity with the square root of Kretschmann scalar seems to be ignored in the literature. The field equation (64) and relevant problems will be discussed in detail elsewhere. We hope that the spacetime singularities and the cosmological constant problem can be improved in this alternative theory.

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