SINGULAR MASAS IN TYPE III FACTORS AND CONNES’ BICENTRALIZER PROPERTY

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Abstract. We show that any type III\(_1\) factor with separable predual satisfying Connes’ Bicentralizer Property (CBP) has a singular maximal abelian \(\ast\)-subalgebra that is the range of a normal conditional expectation. We also investigate stability properties of CBP under finite index extensions/restrictions of type III\(_1\) factors.

1. Introduction

Let \(M\) be any von Neumann algebra and \(A \subset M\) any maximal abelian \(\ast\)-subalgebra (abbreviated MASA). Denote by \(N_M(A) = \{u \in U(M) : uAu^* = A\}\) the group of unitaries in \(M\) that normalize \(A\). We say that \(A \subset M\) is singular if \(N_M(A) = U(A)\), that is, the only unitaries in \(M\) that normalize \(A\) are in \(A\). It has been shown in \([Po82]\) that any diffuse semifinite von Neumann algebra with separable predual and any type III\(_\lambda\) factor (\(0 \leq \lambda < 1\)) with separable predual has a singular MASA. A new approach to this result has been recently given in \([Po16]\).

But while many examples of type III\(_1\) factors are known to have singular MASAs, the problem of whether any type III\(_1\) factor has a singular MASA remained open.

Following \([Co80]\), if \(M\) is a type III\(_1\) factor with a normal faithful state \(\varphi\), then the bicentralizer of \((M, \varphi)\) is defined by

\[
B(M, \varphi) = \left\{ a \in M : \lim_n \|x_n a - ax_n\|_\varphi = 0, \forall (x_n)_n \in AC(M, \varphi) \right\}
\]

where \(AC(M, \varphi) = \{(x_n)_n \in \ell^\infty(N, M) : \lim_n \|x_n \varphi - \varphi x_n\| = 0\}\). It is known that \(B(M, \varphi)\subset M\) is a von Neumann subalgebra that is globally invariant under the modular flow \(\sigma^\varphi\). By Connes–Størmer transitivity theorem (\([CS76]\)), it follows that if \(B(M, \varphi) = C_1\) for some normal faithful state \(\varphi\) on \(M\), then \(B(M, \psi) = C_1\) for any normal faithful state \(\psi\) on \(M\) (cf. \([Ha85, Corollary 1.5]\)). We say that \(M\) satisfies Connes’ Bicentralizer Property (abbreviated CBP) if \(B(M, \varphi) = C_1\) for some (equivalently, for any) normal faithful state \(\varphi\) on \(M\).

Haagerup showed in \([Ha85]\) that any amenable type III\(_1\) factor with separable predual satisfies CBP. Together with the work of Connes \([Co85]\), this showed the uniqueness of the amenable factor of type III\(_1\) with separable predual. Haagerup also obtained in \([Ha85, Theorem 3.1]\) the following characterization of CBP: a type III\(_1\) factor \(M\) with separable predual satisfies CBP if and only if it has a normal faithful state \(\varphi\) such that \((M_\varphi)' \cap M = C_1\). Several classes of nonamenable type III\(_1\) factors have been shown to satisfy CBP: free Araki–Woods factors \([Ho08]\); free product factors \([HU15]\); nonamenable factors satisfying Ozawa’s condition (AO) \([HI15]\). However, Connes’ Bicentralizer Problem is still open for arbitrary type III\(_1\) factors.

In this note, we prove that every factor \(M\) with separable predual that has a normal faithful state \(\varphi\) satisfying the condition \((M_\varphi)' \cap M = C_1\), contains an abelian \(\ast\)-subalgebra \(A \subset M_\varphi\).
that’s maximal abelian and singular in \( M \) (see Theorem 3). We prove this result by adapting the argument used in [Po16] Theorem 2.1 for the type II\(_1\) case. By combining our result with Haagerup’s characterization of CBP [Ha85] explained above, we derive that any type III\(_1\) factor with separable predual satisfying CBP has a singular MASA that is the range of a normal conditional expectation. We end the paper with some results and comments about the stability of Connes’ Bicentralizer Property for inclusions of type II\(_1\) factors with normal conditional expectation, under the assumption that the inclusion has finite index (see Theorem 5).

Acknowledgments. The first named author is grateful to Yusuke Isono for useful discussions.

2. Singular MASAs in type III\(_1\) factors

We recall from [Po81] two results that will be used in the proof of Theorem 3.

**Lemma 1** ([Po81] Theorem 2.5). Let \( M \) be any \( \sigma \)-finite von Neumann algebra, \( \varphi \) any normal faithful state on \( M \) and \( N \subset M \) be any von Neumann subalgebra such that \( N' \cap M \subset N \).

For any finite dimensional abelian \(*\)-subalgebra \( D \subset N \), any \( x_1, \ldots, x_n \in M \) and any \( \varepsilon > 0 \), there exists a finite dimensional abelian \(*\)-subalgebra \( A \subset N \) that contains \( D \) and for which we have

\[
\forall 1 \leq i \leq n, \quad \|E^\varphi_{A \cap N}(x_i) - E^\varphi_A(x_i)\|_\varphi \leq \varepsilon.
\]

**Proof.** Write \( D = \bigoplus_{j \in J} C e_j \) where \( J \) is a nonempty finite set and \( (e_j)_{j \in J} \) are the nonzero minimal projections of \( D \). For every \( j \in J \), we have \( e_j Ne_j \subset (e_j Me_j)\varphi_{e_j} \) and \( (e_j Ne_j)' \cap e_j Me_j \subset e_j Ne_j \) (see [Po81] Lemma 2.1).

Let \( x_1, \ldots, x_n \in M \) and \( \varepsilon > 0 \). For every \( j \in J \), by [Po81] Theorem 2.5, there exists a finite dimensional abelian \(*\)-subalgebra \( A_j \subset e_j Ne_j \) such that

\[
\forall 1 \leq i \leq n, \quad \|E^\varphi_{A_j \cap e_j Me_j}(e_j x_i e_j) - E^\varphi_{A_j}(e_j x_i e_j)\|_{\varphi_{e_j}} \leq \varepsilon.
\]

Put \( A = \bigoplus_{j \in J} A_j \). Then \( A \subset N \) is a finite dimensional abelian \(*\)-subalgebra that contains \( D \). Moreover, for all \( 1 \leq i \leq n \), we have

\[
\|E^\varphi_{A \cap N}(x_i) - E^\varphi_A(x_i)\|_\varphi^2 = \sum_{j \in J} \varphi(e_j)\|E^\varphi_{A_j \cap e_j Me_j}(e_j x_i e_j) - E^\varphi_{A_j}(e_j x_i e_j)\|_{\varphi_{e_j}}^2
\]

\[
\leq \sum_{j \in J} \varphi(e_j)\varepsilon^2 = \varepsilon^2.
\]

\( \square \)
Lemma 2 ([Po81, Theorem 3.2]). Let $M$ be any factor with separable predual, $\varphi$ any normal faithful state on $M$ and $N \subset M_{\varphi}$ be any von Neumann subalgebra such that $N' \cap M = C_1$.

For any finite dimensional abelian $\ast$-subalgebra $D \subset N$, there exists an abelian $\ast$-subalgebra $A \subset N$ that contains $D$ and that is maximal abelian in $M$.

Proof. Write $D = \bigoplus_{j \in J} C e_j$ where $J$ is a nonempty finite set and $(e_j)_{j \in J}$ are the nonzero minimal projections of $D$. For every $j \in J$, we have $e_j Ne_j \subset (e_j Me_j)_{\varphi e_j}$ and $(e_j Ne_j)' \cap e_j Me_j = C e_j$ (see [Po81, Lemma 2.1]). For every $j \in J$, by [Po81] Theorem 3.2, there exists an abelian $\ast$-subalgebra $A_j \subset e_j Ne_j$ that is maximal abelian in $e_j Me_j$. Put $A = \bigoplus_{j \in J} A_j$. Then $A \subset N$ is an abelian $\ast$-subalgebra that contains $D$ and that is maximal abelian in $M$. □

Theorem 3. Let $M$ be any non-type I factor with separable predual, $\varphi$ any normal faithful state on $M$ and $N \subset M_{\varphi}$ any subalgebra such that $N' \cap M = C_1$.

Then there exists an abelian $\ast$-subalgebra $A \subset N$ that is maximal abelian and singular in $M$.

Proof. We follow the lines of the proof of [Po16, Theorem 2.1]. Choose a sequence $x_n \in \text{Ball}(M)$ that is $\ast$-strongly dense in $\text{Ball}(M)$ and a sequence of projections $e_n \in N$ that is strongly dense in the set of all projections of $N$ with $e_0 = 1$. We may further assume that each projection $e_n$ appears infinitely many times in the sequence $(e_m)_{m \in \mathbb{N}}$.

We construct inductively an increasing sequence $A_n$ of finite dimensional abelian $\ast$-subalgebras of $N$ together with a sequence of projections $f_n \in A_n$ and a sequence of unitaries $v_n \in \mathcal{U}(A_n f_n)$ satisfying the following properties:

(P1) $\|f_n - e_n\|_{\varphi} \leq 7\|e_n - E_{A_{n-1}}^{\varphi} N(e_n)\|_{\varphi}$;

(P2) $\|E_{A_{n-1}}^{\varphi} M(x_i^* v_n x_j) f_n\|_{\varphi} \leq 2^{-n}$ for all $0 \leq i, j \leq n$;

(P3) $\|E_{A_{n-1}}^{\varphi} M(x_j) - E_{A_{n-1}}^{\varphi} (x_j)\|_{\varphi} \leq 2^{-n}$ for all $0 \leq j \leq n$.

We put $A_{-1} = A_0 = \mathcal{C} 1$, $f_0 = v_0 = 1$. Assume that we have constructed $(A_k, f_k, v_k)$ for all $0 \leq k \leq n$. Put $f_{n+1} := 1_{[1/2]}(E_{A_n}^{\varphi} N(e_{n+1}))$. Then $f_{n+1} \in A_n \cap N$ is a projection that satisfies $\|f_{n+1} - e_{n+1}\|_{\varphi} \leq 7\|e_{n+1} - E_{A_n}^{\varphi} N(e_{n+1})\|_{\varphi}$ by [Po82] Lemma 1.4. Then (P1) holds true for $f_{n+1}$.

Assume that $f_{n+1} \notin \{0, 1\}$. Then (P2) holds true for any choice of $A_{n+1}$. By Lemma 1 we may find a finite dimensional abelian $\ast$-subalgebra $A_{n+1} \subset N$ that contains $A_n$ and that satisfies

$$\forall 0 \leq i \leq n + 1, \quad \|E_{A_{n+1}}^{\varphi} M(x_i) - E_{A_{n+1}}^{\varphi} (x_i)\|_{\varphi} \leq 2^{-(n+1)}.$$ 

Thus, (P3) holds true for $A_{n+1}$.

Assume that $f_{n+1} \notin \{0, 1\}$. By Lemma 2 there exists an abelian $\ast$-subalgebra $B \subset N$ that contains $A_n \vee C f_{n+1}$ and that is maximal abelian in $M$. Since $(A_n f_{n+1})' \cap f_{n+1} N f_{n+1}$ is a type II$_1$ von Neumann algebra and $B f_{n+1} \subset f_{n+1} N f_{n+1}$ is an abelian subalgebra with normal expectation, [HV12, Theorem 2.3] (see also [Po83] Theorem 2.1 and Corollary 2.3)) implies that there exists $v_{n+1} \in \mathcal{U}((A_n f_{n+1})' \cap f_{n+1} N f_{n+1})$ for which we have

$$\forall 0 \leq i, j \leq n + 1, \quad \left\| E_{B f_{n+1}}^{\varphi} \left( f_{n+1}^* x_i^* f_{n+1} v_{n+1} f_{n+1} x_j f_{n+1}^* \right) \right\|_{\varphi} < 2^{-(n+2)}.$$ 

Using the spectral theorem, we may further assume that $v_{n+1} \in \mathcal{U}((A_n f_{n+1})' \cap f_{n+1} N f_{n+1})$ has finite spectrum and still satisfies (2.2). We may then choose a finite dimensional abelian $\ast$-subalgebra $D_1 \subset B f_{n+1}$ that contains $A_n f_{n+1}$ and $v_{n+1}$. Moreover, using [Po81] Lemma 1.2
and (2.2), we may choose a finite dimensional abelian \(\ast\)-subalgebra \(D_2 \subset B f^\perp_{n+1}\) that contains \(A_n f^\perp_{n+1}\) and for which we have

\[
\forall 0 \leq i, j \leq n + 1, \quad \| E^\varphi_{D_2 \cap f^\perp_{n+1}}(f^\perp_{n+1} x_i^* f_{n+1} v_{n+1} f_{n+1} x_j f^\perp_{n+1}) \|_\varphi < 2^{-(n+1)}.
\]

Letting \(D := D_1 \oplus D_2\), we can then rewrite (2.3) as

\[
\forall 0 \leq i, j \leq n + 1, \quad \| E^\varphi_D (f^\perp_{n+1} x_i^* f_{n+1} v_{n+1} f_{n+1} x_j f^\perp_{n+1}) \|_\varphi < 2^{-(n+1)}.
\]

By Lemma 1, we may find a finite dimensional abelian \(\ast\)-subalgebra \(A_{n+1} \subset N\) that contains \(D\) (and hence that contains \(A_n\)) and that satisfies (2.4). Thus, (2.4) shows that (P3) holds true for \(A_{n+1}\). Since \(D \subset A_{n+1}\), we have \(A_{n+1} \cap M \subset D' \cap M\) and (2.4) implies that (P2) holds true for \(f_{n+1}\) and \(A_{n+1}\). Thus, we have constructed \((A_{n+1}, f_{n+1}, v_{n+1})\).

Put \(A = \bigvee_{n \in \mathbb{N}} A_n\). Property (P3) and [Po81] Lemma 1.2 imply that \(A' \cap M = A\) and hence \(A\) is maximal abelian in \(M\). It remains to prove that \(A\) is singular in \(M\). By contradiction, assume that \(A \neq N_{M}(A)''\). Choose \(u \in N_{M}(A) \setminus U(A)\). We can then find a nonzero projection \(z \in A\) such that \(uzu^* \perp z\). Denote by \(h\) the unique nonsingular (possibly) unbounded positive selfadjoint operator affiliated with \(A\) such that \(\varphi \circ \text{Ad}(u)|_A = \varphi(h \cdot)|_A\). By [Co72] Lemme 1.4.5(2), we have \(\sigma^\varphi(u) = uh^t\) for every \(t \in \mathbb{R}\). Since \(h\) is nonsingular, there exists a projection \(p \in A\) large enough so that \(zp \neq 0\) and \(\delta > 0\) so that \(\delta p \leq hp \leq \delta^{-1} p\). It follows that the nonzero partial isometry \(up \in M\) (resp. \((up)^* \in M\)) is entire analytic with respect to the modular automorphism group \(\sigma^\varphi\). Hence, there exists \(\kappa_1 \geq 1\) (resp. \(\kappa_2 \geq 1\)) such that \(\|x up\|_\varphi \leq \kappa_1 \|x\|_\varphi\) (resp. \(\|x (up)^*\|_\varphi \leq \kappa_2 \|x\|_\varphi\)) for every \(x \in M\).

Put \(q := pz \in A\). For every \(n \in \mathbb{N}\), we have

\[
\|q f_n\|_\varphi = \|z v_n p\|_\varphi = \|u^* uz v_n pu^* up\|_\varphi \\
\leq \kappa_1 \|uz v_n pu^*\|_\varphi = \kappa_1 \|E^\varphi_{A_n' \cap M}(uz v_n pu^*)\|_\varphi (\text{since } uz v_n pu^* \in z^\perp(A_n' \cap M)z^\perp).
\]

Since \(z, v_n \in A \subset M_p\) and since \((up)^*\) is entire analytic with respect to the modular automorphism group \(\sigma^\varphi\), for all \(0 \leq i, j \leq n\), we further have

\[
\|E^\varphi_{A_n' \cap M}(uz v_n pu^*)\|_\varphi \leq \|E^\varphi_{A_n' \cap M}(x_i^* v_n pu^*)z^\perp\|_\varphi + \kappa_2 \|x_i^* - u z\|_\varphi \\
\leq \|E^\varphi_{A_n' \cap M}(x_i^* v_n x_j)\|_\varphi + \|x_j - pu^*\|_\varphi + \kappa_2 \|x_i^* - u z\|_\varphi \\
\leq \|E^\varphi_{A_n' \cap M}(x_i^* x_j)\|_\varphi + \|e_n - z\|_\varphi + \|x_j - pu^*\|_\varphi + \kappa_2 \|x_i^* - u z\|_\varphi.
\]

Since \(z \in A \subset A_{n-1}' \cap N\), for every \(n \in \mathbb{N}\), using (P1) we have

\[
\|e_n - f_n\|_\varphi \leq 7\|e_n - E^\varphi_{A_{n-1}' \cap N}(e_n)\|_\varphi = 7\|e_n - z - E^\varphi_{A_{n-1}' \cap N}(e_n - z)\|_\varphi \leq 7\|e_n - z\|_\varphi.
\]

Choose now \(n_0 \in \mathbb{N}\) large enough so that \(2^{-n_0} \leq \|q\|_\varphi/(100 \kappa_1)\). By density, we may then choose \(i, j \geq 0\) and \(n \geq \max(i, j, n_0)\) so that \(\|e_n - z\|_\varphi + \|x_j - pu^*\|_\varphi + \kappa_2 \|x_i^* - u z\|_\varphi \leq \|q\|_\varphi/(100 \kappa_1)\).
Using (2.6) and (P2), we then have

\[
\|E_{A'' \cap M}(uzv_n pu^*)z\|_\varphi \leq \|E_{A'' \cap M}(x_n^* v_n x_j)e_n^*\|_\varphi + \|q\|_\varphi/(100\kappa_1) \\
\leq \|E_{A'' \cap M}(x_n^* v_n x_j)f_n^*\|_\varphi + \|f_n - e_n\|_\varphi + \|q\|_\varphi/(100\kappa_1) \\
\leq \|E_{A'' \cap M}(x_n^* v_n x_j)f_n^*\|_\varphi + 7\|e_n - z\|_\varphi + \|q\|_\varphi/(100\kappa_1) \\
\leq 2^{-n} + 7\|q\|_\varphi/(100\kappa_1) + \|q\|_\varphi/(100\kappa_1) \\
\leq 9\|q\|_\varphi/(100\kappa_1) \\
\leq \|q\|_\varphi/(2\kappa_1).
\]

Combining (2.6) and (2.7), we obtain

\[
\|qf_n\|_\varphi \leq \|q\|_\varphi/2.
\]

On the other hand, we have

\[
\|qf_n - q\|_\varphi \leq \|q(f_n - z)\|_\varphi \leq \|f_n - z\|_\varphi \\
\leq \|f_n - e_n\|_\varphi + \|e_n - z\|_\varphi \\
\leq 8\|e_n - z\|_\varphi \\
\leq 8\|q\|_\varphi/(100\kappa_1) \\
\leq \|q\|_\varphi/4.
\]

Combining (2.8) and (2.9), we finally obtain

\[
\|q\|_\varphi/2 \geq \|qf_n\|_\varphi \geq \|q\|_\varphi - \|qf_n - q\|_\varphi \geq 3\|q\|_\varphi/4.
\]

Since \(\|q\|_\varphi \neq 0\), this is contradiction. Therefore \(A = N^\prime M(A)''\) and hence \(A \subset M\) is singular. \(\Box\)

**Corollary 4.** Every type III_1 factor M with separable predual satisfying CBP has a singular maximal abelian \(\ast\)-subalgebra \(A \subset M\) with normal expectation.

**Proof.** By [Ha85, Theorem 3.1], there exists a faithful state \(\varphi \in M_\ast\) such that \((M_\varphi)' \cap M = C1\). By Theorem 3, there exists an abelian \(\ast\)-subalgebra \(A \subset M_\varphi\) that is maximal abelian and singular in \(M\). Moreover, \(A \subset M\) is the range of a normal faithful conditional expectation. \(\Box\)

### 3. Stability of CBP under finite index extensions/restrictions

In this section we investigate the stability properties of CBP for inclusions of type III_1 factors \(N \subset M\) with normal faithful conditional expectation \(E_N : M \to N\). Fix a normal faithful state \(\varphi\) on \(M\) such that \(\varphi = \varphi \circ E_N\). The bicentralizer algebras \(B(N, \varphi)\) and \(B(M, \varphi)\) are not related in any obvious way and so it is hopeless to try to prove in general that CBP passes to subalgebras or overalgebras. In fact, any type III_1 factor \(N\) embeds in an irreducible way and with NCE into a type III_1 factor \(M\) that satisfies CBP. Indeed, choose any normal faithful state \(\psi_N\) on \(N\) and put \((M, \psi_M) = (N, \psi_N) \ast (L(\mathbb{Z}_2), \tau_{\mathbb{Z}_2})\). It follows from [HU15, Theorem A.1] that \(M\) is a type III_1 factor that satisfies CBP, \(N \subset M\) is with NCE and \(N' \cap M = C1\).

However, when the inclusion \(N \subset M\) has finite index, we show here that \(N\) satisfies CBP if and only if \(M\) satisfies CBP.

**Theorem 5.** Let \(N \subset M\) be any inclusion of type III_1 factors with separable predual such that \(N\) is the range of a normal faithful conditional expectation and \(N\) has finite index in \(M\).

Then \(N\) satisfies CBP if and only if \(M\) satisfies CBP.
Proof. If \( N \) satisfies CBP then \( M \) satisfies CBP. Denote by \( (M, L^2(M), J, L^2(M)_+) \) the standard form of \( M \). Fix a normal faithful conditional expectation \( E_N : M \to N \) with finite index. Denote by \( (M, N) := (JN)_N \cap B(L^2(M)) \) the Jones basic construction and by \( e_N : L^2(M) \to L^2(N) : x_\xi \mapsto E_N(x_\xi) \) the Jones projection. We denote by \( \Phi : (M, N) \to M \) the canonical normal faithful conditional expectation (see [GoM]).

Since \( N \) satisfies CBP, by [Ha85, Theorem 3.1], there exists a faithful state \( \varphi \in M \) such that \( \varphi = \xi \circ E_N \) and \( (N_\varphi)_N \cap N = C_1 \). Put \( P = (N_\varphi)_N \cap M \) and observe that \( P \subset M \) is globally invariant under \( \sigma^\varphi \). Let \( e : L^2(P) \to C_\xi \varphi \) be the rank-one orthogonal projection. Since \( (N_\varphi)_N \cap N = C_1 \), we have \( E_N(x) = \varphi(x)1 \) for every \( x \in P \). Let \( \langle P, e \rangle = (P \cup \{e\})'' \) and observe that \( \langle P, e \rangle = B(L^2(P)) \). Denote by \( \psi(P, e) \) (resp. \( \psi(N_\varphi, M, N) \)) the normal faithful semifinite state defined on \( (P, e) \) (resp. \( (M, N) \)). Then the map

\[ V : L^2((P, e), \psi(P, e)) \to L^2((M, N), \psi(N_\varphi, M, N)) : \Lambda_{\psi(P, e)}(xey) \mapsto \Lambda_{\psi(N_\varphi, M, N)}(xe_Ny), \quad x, y \in P \]

is an isometry. Denote by \( P \subset (M, N) \) the weak closure of the (possibly) nonunital \(*\)-subalgebra \( \text{span}\{xe_Ny : x, y \in P\} \). Observe that \( \sigma^{\psi(N_\varphi, M, N)}(xey) = \sigma^{\psi(N_\varphi, M, N)}(y) \) for all \( t \in R \) and all \( x, y \in P \). It follows that \( \text{span}\{xe_Ny : x, y \in P\} \) is globally invariant under \( \sigma^{\psi(N_\varphi, M, N)} \). Thus, we have \( \sigma^{\psi(N_\varphi, M, N)}(1_P) = 1_P \) for every \( t \in R \) and \( P \subset 1_P(M, N)1_P \) is globally invariant under \( \sigma^{\psi(N_\varphi, M, N)} \). Observe that \( e_N \leq 1_P \). Since \( \psi(N_\varphi, M, N)(1_P \cdot 1_P) \) is semifinite on \( P \), there exists a \( \psi(N_\varphi, M, N)(1_P \cdot 1_P) \)-preserving conditional expectation \( E_P : 1_P(M, N)1_P \to P \) (see [Ha85, Theorem IX.4.2]). Then the projection \( V^*V \) is nothing but the orthogonal projection \( L^2((M, N), \psi(N_\varphi, M, N)) \to L^2(P, \psi(N_\varphi, M, N)(1_P \cdot 1_P)) \). Denote by \( \Theta : (P, e) \to B(L^2(P)) \) defined by \( \Theta(a)V = Va \) for \( a \in (P, e) \) a unital normal \(*\)-embedding that satisfies \( \Theta(a)1 = a1 \) for every \( a \in (P, e) \) and \( \Theta : (P, e) \to P \) is a \( P \)-bimodule normal faithful unital completely positive map.

Regard \( (P, e) = B(L^2(P)) \) and define \( \Psi : B(L^2(P)) \to P \) by the composition \( \Psi = E_P \circ \Phi \circ \Theta \). Then \( \Psi \) is a \( P \)-\( P \)-bimodule normal faithful completely positive map. Since \( \psi(1) \in Z(P)_+ \), we may choose a nonzero element \( c \in Z(P)_+ \) so that \( z := c^{1/2}\psi(1)c^{1/2} \) is a nonzeroprojection in \( Z(P) \). Then the map \( \Psi_z : B(L^2(P)) \to P \) defined by \( \Psi_z := c^{1/2}\psi(z \cdot z)c^{1/2} \) is a \( P \)-\( P \)-bimodule normal unital completely positive map and hence a normal conditional expectation. Therefore \( P \) is a discrete von Neumann algebra. By [HI15, Theorem 3.5], the bicanonicalizer algebra \( B(M, \varphi) \) satisfies the following dichotomy: either \( B(M, \varphi) = C_1 \) or \( B(M, \varphi) \) is a type \( II_1 \) factor. Since the inclusions \( B(M, \varphi) \subset (M_\varphi)_N \subset \cap (N_\varphi)_N \cap M = P \) are all globally invariant under \( \sigma^\varphi \) and since \( P \) has a minimal projection, it follows that \( B(M, \varphi) = C_1 \). Therefore, \( M \) satisfies CBP.

If \( M \) satisfies CBP then \( N \) satisfies CBP. Since the inclusion \( N \subset M \) has finite index, we may choose a normal faithful conditional expectation \( E_N : M \to N \) for which there exists \( \kappa > 0 \) such that \( E_N(x) \geq \kappa x \) for every \( x \in M_+ \) (see [PPS4, Po95]). Let \( \varphi \) be a normal faithful state on \( M \) such that \( \varphi = \varphi \circ E_N \). Fix a nonprincipal ultrafilter \( \omega \in B(N) \setminus N \) and denote by \( M^\omega \) (resp. \( N^\omega \)) the Ocneanu ultraproduct of \( M \) (resp. \( N \)) with respect to \( \omega \) (see [Oc85, AH12]). Following [Po95, Section 1.3], define the normal faithful conditional expectation \( E_{N^\omega} : M^\omega \to N^\omega \) by the formula \( E_{N^\omega}(x_n)^\omega = (E_N(x_n))^{\omega} \) for every \( (x_n)^\omega \in M^\omega \). Then we have \( E_{N^\omega}(x) \geq \kappa x \) for every \( x \in (M^\omega)_+ \). Put \( \mathcal{N} = (N^\omega)_\omega \) and \( M = (M^\omega)_\omega \) and observe that both \( \mathcal{N} \) and \( M \) are type \( II_1 \) factors by [AH12, Proposition 4.24]. Since \( \varphi^\omega = \varphi \circ E_{N^\omega} \) and since \( E_{N^\omega}(x) \in N \) for every \( x \in M \), we have \( E_N(x) = E_{N^\omega}(x) \geq \kappa x \) for every \( x \in M_+ \), where \( E_N : \mathcal{N} \to \mathcal{N} \) denotes the unique trace preserving conditional expectation. Thus, the inclusion \( \mathcal{N} \subset M \) has finite index by [PPS4, Theorem 2.2].

We first prove that \( \mathcal{M}' \cap M^\omega = C_1 \). Since \( M \) satisfies CBP, by [Ha85, Theorem 3.1], there exists a normal faithful state \( \psi \) on \( M \) such that \( (M_\psi)' \cap M = C_1 \). Then [Po95, Lemma 2.3] implies that \( (M^\omega)_\omega)' \cap M^\omega = C_1 \). Since \( (M^\omega)_\omega \subset (M^\omega)_\omega \), we have \( (M^\omega)_\omega)' \cap M^\omega = C_1 \).
By Connes–Størmer transitivity theorem \( [CS76] \) (see also \[AH12\] Theorem 4.20), there exists \( u \in \mathcal{U}(M^\omega) \) such that \( \psi^\omega = u\varphi^\omega u^* \). Thus, we obtain
\[
M' \cap M^\omega = ((M^\omega)_\varphi^\omega)' \cap M^\omega = u^*(((M^\omega)_{\psi^\omega})' \cap M^\omega)u = C1.
\]

We next prove that \( \mathcal{N}' \cap M^\omega \) has a nonzero minimal projection following the lines of \[Po09\] Lemma 3.3. Since \( \mathcal{N} \subset M \) is a finite index inclusion of type II\(_1\) factors, we may choose a projection \( e \in \mathcal{M} \) such that \( E_{\mathcal{N}}(e) = [\mathcal{M} : \mathcal{N}]^{-1}1 \). Put \( \mathcal{P} = \{e\}' \cap \mathcal{N} \) so that \( \mathcal{P} \subset \mathcal{N} \) is a finite index inclusion of type II\(_1\) factors and \( \mathcal{M} = \langle \mathcal{N}, e \rangle \) (see \[PP84\] Corollary 1.8).

By \[PP84\] Proposition 1.3, choose a finite basis \( (X_j)_{j \in J} \) of \( \mathcal{N} \) over \( \mathcal{P} \). Recall that we have \( \sum_{j \in J} X_j e X_j^* = 1 \), \( \sum_{j \in J} X_j X_j^* = [\mathcal{M} : \mathcal{N}] \) and for every \( j \in J, p_j := E_\mathcal{P}(X_j^*X_j) \) is a projection in \( \mathcal{P} \). Since \( \sum_{j \in J} X_j e X_j^* \in \mathcal{M}' \cap M^\omega = C1 \) for every \( x \in \mathcal{N}' \cap M^\omega \), we may define the state \( \Psi \in (\mathcal{N}' \cap M^\omega)_* \) by the formula \( \sum_{j \in J} X_j e X_j^* = \Psi(x)1 \). Moreover, we have \( exe = \Psi(x)e \) for every \( x \in \mathcal{N}' \cap M^\omega \). Following \[Po09\] Lemma 3.3, put \( b = [\mathcal{M} : \mathcal{N}] \cdot E_{\mathcal{N} \cap M}(e) = [\mathcal{M} : \mathcal{N}] \cdot E_{\mathcal{N}' \cap M^\omega}(e) \) in \( \mathcal{N}' \cap \mathcal{M} \). For every \( x \in \mathcal{N}' \cap M^\omega \), we have
\[
\varphi^\omega \left( \sum_{j \in J} X_j eb^{-1/2} x b^{-1/2} e X_j^* \right) = \sum_{j \in J} \varphi^\omega(x b^{-1/2} e X_j^* X_j eb^{-1/2}) = \sum_{j \in J} \varphi^\omega(x b^{-1/2} p_j eb^{-1/2}) = \sum_{j \in J} \varphi^\omega \circ E_\mathcal{P}(x b^{-1/2} pb^{-1/2}) = [\mathcal{M} : \mathcal{N}] \cdot \varphi^\omega(x b^{-1/2} eb^{-1/2}) = \varphi^\omega(x).
\]

Moreover, \( f = b^{-1/2} eb^{-1/2} \in \mathcal{M} \) is a projection such that \( fxf = \varphi^\omega(x)f \) for every \( x \in \mathcal{N}' \cap M^\omega \). This implies that the von Neumann algebra \( (\mathcal{N}' \cap M^\omega, f) = ((\mathcal{N}' \cap M^\omega) \cup \{f\})' \) has a minimal projection, namely \( f \). Since \( f \in \mathcal{M} \), the von Neumann subalgebra \( (\mathcal{N}' \cap M^\omega, f) \subset M^\omega \) is globally invariant under \( \sigma^\omega \). Since the inclusions \( \mathcal{N}' \cap M^\omega \subset (\mathcal{N}' \cap M^\omega, f) \subset M^\omega \) are all globally invariant under \( \sigma^\omega \), we obtain that \( \mathcal{N}' \cap M^\omega \) has a minimal projection as well.

By \[HI15\] Proposition 3.3, we have \( B(N, \varphi) = ((N^\omega)_{\varphi^\omega})' \cap N = \mathcal{N}' \cap N \). Since the inclusion \( B(N, \varphi) = \mathcal{N}' \cap N \subset \mathcal{N}' \cap M^\omega \) is globally invariant under \( \sigma^\omega \) and since \( \mathcal{N}' \cap M^\omega \) has a minimal projection, \[HI15\] Theorem 3.5 implies that \( B(N, \varphi) = C1 \). Therefore, \( N \) satisfies CBP.

4. Open problems

Formulated some forty years ago and still open, Connes’s Bicentralizer Problem remains one of the most famous unsolved problems in von Neumann algebras. It is certainly the central, most important open problem in the theory of type III\(_1\) factors. The fundamental role it plays in unraveling the structure of type III\(_1\) factors comes from its equivalent form as existence of a normal faithful state with large centralizer (due to \[Ha85\]). In turn, this latter form of CBP (often accompanied by Connes–Størmer’s theorem) allows adapting arguments from III\(_1\) factors to the “III\(_1\) factor world”. For instance, it has been a key feature in developing a type III\(_1\) version of the second named author’s deformation-rigidity theory, which has been initially developed in III\(_1\) factor framework (see e.g. \[HI13\] \[HI15\]).

It is somewhat notorious that Connes and Haagerup strongly believed CBP had an affirmative answer. But since all efforts to prove it have failed, during the last decade there have been attempts to produce counterexamples as well, in fact some of the papers involving the first named author have been motivated by such attempts (see e.g. \[Ho08\]).
However, at this moment, both authors of this paper believe CBP has a positive answer. The purpose of the previous section was to offer some supporting evidence in this respect, with its partial results bound to become redundant if CBP is proven in its full generality. We’ll formulate in this section several related problems, including some stronger versions of the CBP conjecture.

Let us first recall that Connes’ Bicentralizer Property for a type III$_1$ factor with separable predual $M$ was shown in [Ha85] to be equivalent to the weak relative Dixmier property of the inclusion $M_\Phi \subset M$, where $\Phi$ is any normal faithful dominant weight on $M$ and $M_\Phi$ denotes the fixed point algebra of its automorphism group.

The terminology weak relative Dixmier property for an inclusion of (arbitrary) von Neumann algebras $N \subset M$ is in the sense of [Po98], and it means that the convex set $K_N(x) = \overline{\text{co}} \{ uxu^* \mid u \in U(N) \}$ has non-empty intersection with $N' \cap M$, for any $x \in M$. Note that if $M = B(H)$ then this condition for $N \subset M$ is equivalent to $N$ being amenable (cf. [Sc63]). Note also that in the case $N \subset M$ is an irreducible inclusion of factors (i.e., if $N' \cap M = C_1$), then this condition to hold true it is sufficient to have an amenable (equivalently approximately finite dimensional, by [Co75]) von Neumann subalgebra $B \subset N$ such that $B' \cap M \subset B$ (thus $B' \cap M = Z(B)$). Indeed, because then by [Sc63] we have $K_B(x) \cap B' \cap B \neq \emptyset$ for all $x \in M$, and by applying in the factor $N$ the Dixmier averaging theorem [Di57] to an element in this intersection set, one gets $K_N(x) \cap C_1 \neq \emptyset$ (see [Ha85] Remark 3.9).

In particular, if $N \subset M$ is an irreducible inclusion of factors that satisfies Kadison’s property, i.e., $N$ contains an abelian von Neumann subalgebra that’s a MASA in $M$, then $N \subset M$ has the weak relative Dixmier property. The existence of a MASA in a subfactor $N \subset M$ clearly implies irreducibility, and one of the well known problems in [Ka67] asks whether the converse is true as well, i.e., if Kadison’s property actually characterizes irreducibility.

It is easy to see that if $N \subset M$ is an inclusion of von Neumann algebras with NCE and $N$ is semifinite, then $N \subset M$ satisfies the weak relative Dixmier property (see [Po81], [Po98]). It has in fact been shown in [Po81] that if in addition $N' \cap M \subset N$, then $N$ contains a MASA of $M$ with NCE (so such $N \subset M$ do satisfy Kadison’s property), and that if $N \subset M$ is irreducible then $N$ contains a hyperfinite subfactor $R \subset N$ with NCE such that $R' \cap M = C_1$.

**Problem 6.** Let $N \subset M$ be an irreducible inclusion with NCE of type III$_1$ factors with separable predual such that $N$ satisfies CBP. Does $N$ contain an amenable (or even abelian) von Neumann subalgebra $B \subset N$ with NCE and such that $B' \cap M \subset B'$? Is this at least true if $N$ is the hyperfinite type III$_1$ factor?

Note that by a result in [GP96], there do exist examples of irreducible inclusions of factors $N \subset M$ with $N$ of type II$_1$, $M$ of type II$_\infty$ such that $N$ contains no amenable von Neumann subalgebra $B$ with the property that $B' \cap M \subset B$. But the examples of irreducible inclusions in [GP96] that do not satisfy Kadison’s property are not with NCE. Thus, the problem of whether Kadison’s criterion characterizes irreducibility for an inclusion of factors seems quite subtle, in its full generality.

In turn, the weak relative Dixmier property may still be true for arbitrary irreducible inclusions.

**Problem 7.** Let $N \subset M$ be an arbitrary irreducible inclusion of factors with separable predual. Does $N \subset M$ have the weak relative Dixmier property? Is this at least true when the NCE condition is satisfied?

As we mentioned before, if this is true in the case $M$ is type III$_1$ and $N$ is its type II$_\infty$ core then CBP holds true. Note that by [Po81], if $N$ is any non-Gamma type II$_1$ factor (e.g., if $N$ is the free group factor $L(F_n)$, cf. [MVN43] and $M$ is the ultrapower factor $N^\omega$, for some nonprincipal ultrafilter on $N$, then $N \subset M$ is irreducible, yet $N$ contains no MASAs of $M$. 

But in these examples the larger factor is non-separable. However, such inclusions $N \subset M$ do satisfy the weak relative Dixmier property.

References


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