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INTEGRAL EQUATIONS, KERR–SCHILD FIELDS AND GRAVITATIONAL SOURCES

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Abstract

Kerr–Schild solutions to the vacuum Einstein equations are considered from the viewpoint of integral equations. We show that, for a class of Kerr–Schild fields, the stress-energy tensor can be regarded as a total divergence in Minkowski spacetime. If one assumes that Minkowski coordinates cover the entire manifold (no maximal extension), then Gauss' theorem can be used to reveal the nature of any sources present. For the Schwarzschild and Vaidya solutions the fields are shown to result from a δ-function point source. For the Reissner–Nordstrom solution we find that inclusion of the gravitational fields removes the divergent self-energy familiar from classical electromagnetism. For more general solutions a complex structure is seen to arise in a natural, geometric manner with the role of the unit imaginary fulfilled by the spacetime pseudoscalar. The Kerr solution is analysed leading to a novel picture of its global properties. Gauss' theorem reveals the presence of a disk of tension surrounded by the matter ring singularity. Remarkably, the tension profile over this disk has a simple classical interpretation. It is also shown that the matter in the ring follows a light-like path, as one expects for the endpoint of rotating, collapsing matter. Some implications of these results for physically-realistic black holes are discussed.

1 Introduction

Many of the important solutions to the Einstein field equations can be expressed in Kerr–Schild form (see, for example, the discussion in [1]). These include all black hole solutions, and a range of solutions representing radiation. Here we analyse solutions of Kerr–Schild type from the viewpoint of the gauge theory approach to gravity [2, 3, 4]. In this approach the gravitational fields are gauge fields defined over a flat Minkowski spacetime. These fields ensure that all relations between physical quantities are independent of the position and orientation of the matter fields — a scheme that ensures that the background spacetime plays no dynamic role in the physics and has no measurable properties. Kerr–Schild metrics are constructed from a null vector field in the background Minkowski spacetime, so are particularly well-suited to analysis via this gauge-theoretic approach. In this paper we show that, for all fields of Kerr–Schild type, the Einstein tensor is a total divergence in the background Minkowski spacetime. Various consequences of this result are explored. Gauss' theorem is used to convert volume integrals of the Einstein tensor to surface

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integrals, enabling us to probe the nature of the matter singularities generating the gravitational fields.

The gauge-theory viewpoint always produces a metric that satisfies the Einstein equations (or their generalisation to include torsion). But working with fields defined over a Minkowski background does place additional restrictions on the form of the solutions. For example, general relativity admits two possibilities when dealing with the Kerr solution [5]

1. The complete Kerr manifold can be covered by a single set of Minkowski coordinates. This implies a discontinuity in the fields over the entire disk region bounded by the matter singularity.

2. The fields are smooth everywhere away from the ring, but an observer passing through the ring emerges in a new, asymptotically flat, region. This is achieved by extending the radial coordinate r to negative values, producing the maximally-extended Kerr spacetime.

By adopting a flat-spacetime, gauge-theory formulation we restrict ourselves to considering case 1 only. This can be justified on the grounds that the full, maximally-extended Kerr solution is not thought to be a feasible endpoint for any collapse process. Similar comments apply to the maximally-extended Schwarzschild and Reissner–Nordstrom solutions, neither of which are considered here.

The first applications we consider are to spherically-symmetric fields, concentrating on the Schwarzschild, Reissner–Nordstrom and Vaidya solutions. In all cases the integrals provide sensible results for the total energy contained in the fields, with the mass contribution to the energy residing in a point-source δ-function. For the Reissner–Nordstrom solution the inclusion of gravitational fields removes the infinite electromagnetic self-energy for a point charge familiar from classical electromagnetism [6]. This result is achieved without requiring any form of regularisation procedure, and ensures that the total electromagnetic self-energy is zero.

We next turn to more general fields following the work of Schiffer *et al.* [7]. These authors showed that stationary Kerr–Schild vacuum solutions are generated by a single, complex generating function. This complex structure underlies the 'trick' by which the Kerr solution is obtained from the Schwarzschild solution via a complex 'coordinate transformation' [8]. (This is a trick because there is no *a priori* justification for expecting the complex transformation to result in a new vacuum solution.) The complex structure associated with vacuum Kerr–Schild fields is shown here to have a simple geometric origin, with the role of the unit imaginary fulfilled by the spacetime pseudoscalar — the same entity that is responsible for duality transformations of the Riemann tensor.

The remainder of this paper deals with a detailed analysis of the Kerr solution. For this we require a careful choice of branch cut in the complex square route in the generating function. Once this is made, Gauss' theorem reveals the detailed structure of the singular region, confirming that the matter is concentrated in a ring that circulates on a lightlike trajectory. This is as one would expect, since the Riemann tensor only diverges on a ring, and special relativity alone is sufficient to predict that rotating collapsing matter will fall inwards until its velocity becomes lightlike. A more surprising result is obtained from

considering integrals inside the ring, which reveal the presence of a disc of planar tension [9]. This tension is isotropic over the disk and has a simple radial dependence, rising to ∞ at the ring. Remarkably, the functional form of the tension has a simple non-gravitational interpretation. In non-relativistic dynamics a membrane holding together a rotating ring of disconnected particles would be under a constant tension. When special-relativistic effects are included the picture is altered by the fact that tension can act as a source of inertia. This introduces a radial dependence into the tension, the functional form of which is precisely that which lies at the heart of the Kerr solution. These conclusions are gauge invariant and are not artifacts of the use of the background spacetime. This is demonstrated by eigen-decompositions of the stress-energy and Riemann tensors, from which we extract the gauge-invariant information.

There has been considerable debate over many years surrounding the nature of sources for the Kerr metric. Many physicists have attempted to construct extended sources for which the Kerr metric could represent the external geometry (see Krasinski for an early review [10]). More recently, a series of authors have constructed disk sources for the Kerr metric [11, 12, 13]. These solutions represent extended sources and do not have horizons present. The present work is of a different nature, dealing solely with the structure of the singular region — the endpoint of a collapse process. The first authors to consider this were Newman & Janis [8] and Isreal [5]. We disagree with Isreal's result for the energy distribution over the disk, agreeing instead with Hamity's later result [14]. Our techniques enable us to go some way beyond Hamity's description, both in revealing the physical properties of the disk and in understanding the nature of the singularity around the ring. The simple physics of the disk was first pointed out in [9].

Many of the calculations here are simplified by using the language of 'spacetime algebra' [3, 15, 16]. This is crucial to understanding the geometric nature of the complex structure at the heart of Kerr–Schild solutions. The algebraic structure of the spacetime algebra is that of the Dirac γ-matrices. Using this algebraic structure one can develop a mathematical language that is adept at describing many aspects of relativistic physics. This language includes a calculus that is somewhat more powerful than any available in alternative languages. The gauge theory of gravity developed in [2] takes on its most natural and compelling form when expressed in the spacetime algebra. We start with an introduction to the spacetime algebra, giving the necessary conventions and notations. Further details can be found in [2, 3, 16, 17] and references contained therein. Reference [2] includes an appendix describing how to convert between spacetime algebra and more conventional tensor calculus. Natural units ($G = c = \epsilon_0 = 1$) are employed throughout this paper.

1.1 Spacetime algebra

The basic algebraic structure behind the spacetime algebra will be familiar to most physicists in the guise of the algebra of the Dirac γ-matrices. The geometric interpretation the spacetime algebra attaches to this algebra may be less familiar, though it is remarkably well-suited to most problems in relativistic physics [3, 15, 17]. The spacetime algebra is generated by four vectors $\{\gamma_\mu\}$, $\mu = 0 \ldots 3$, equipped with an associative (Clifford) product denoted by juxtaposition. The symmetric and antisymmetric parts of this product define the inner and

outer products, and are denoted with a dot and a wedge respectively, so

$$\gamma_\mu \cdot \gamma_\nu = \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \eta_{\mu\nu} = \mathrm{diag}(+\,-\,-\,-) \tag{1.1}$$

and

$$\gamma_\mu \wedge \gamma_\nu = \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{1.2}$$

The outer product of two vectors defines a bivector — a directed plane segment representing the plane defined by the two vectors. A full basis for the spacetime algebra is provided by the set

$$\begin{array}{ccccc} 1 & \{\gamma_\mu\} & \{\boldsymbol{\sigma}_k, I\boldsymbol{\sigma}_k\} & \{I\gamma_\mu\} & I \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar,} \\ \text{grade 0} & \text{grade 1} & \text{grade 2} & \text{grade 3} & \text{grade 4} \end{array} \tag{1.3}$$

where

$$\boldsymbol{\sigma}_k = \gamma_k\gamma_0, \quad k = 1\ldots 3 \tag{1.4}$$

and

$$I = \gamma_0\gamma_1\gamma_2\gamma_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3. \tag{1.5}$$

The pseudoscalar I squares to -1, anticommutes with all odd-grade elements and commutes with even grade elements. Both the $\{\boldsymbol{\sigma}_k\}$ and $\{\gamma_\mu\}$ are algebraic entities with clear geometric significance. They should not be thought of as matrices acting on an internal spin space. (The same symbols as employed in quantum theory are used here simply because the algebraic relations are the same.)

An arbitrary real superposition of the basis elements (1.3) is called a 'multivector' and these inherit the associative Clifford product of the $\{\gamma_\mu\}$ generators. The inner and outer products with a vector a are of particular importance. For these we write

$$a\cdot A_r = \tfrac{1}{2}(aA_r - (-1)^r A_r a), \qquad a\wedge A_r = \tfrac{1}{2}(aA_r + (-1)^r A_r a). \tag{1.6}$$

The outer and geometric products are associative, but the inner product is not. We also employ the commutator product,

$$A\times B = \tfrac{1}{2}(AB - BA). \tag{1.7}$$

Vectors are usually denoted in lower case Latin, $a = a^\mu\gamma_\mu$, or Greek for basis frame vectors. In the absence of brackets the inner, outer and commutator products take precedence over geometric products.

An inertial system is picked out by a future-pointing timelike (unit) vector. If this is chosen to be the γ_0 direction then the γ_0-vector determines a map between spacetime vectors $a = a^\mu\gamma_\mu$ and the even subalgebra of the full spacetime algebra via

$$a\gamma_0 = a_0 + \boldsymbol{a}, \tag{1.8}$$

where

$$a_0 = a\cdot\gamma_0, \quad \text{and} \quad \boldsymbol{a} = a\wedge\gamma_0. \tag{1.9}$$

The 'relative vector' $\boldsymbol{a}$ can be decomposed in the $\{\boldsymbol{\sigma}_k\}$ frame and represents a spatial vector as seen by an observer in the γ_0-frame. Relative (or spatial) vectors in the γ_0-system are written in bold type to record the fact that they

are in fact spacetime bivectors. This distinguishes them from spacetime vectors, which are left in normal type. The $\{\boldsymbol{\sigma}_k\}$ generate the (Pauli) algebra of three-dimensional space, and we occasionally require that the dot and wedge symbols define the three-dimensional inner and outer products. The convention we adopt is that, if both arguments of a dot or wedge product are written in bold, then the product takes its three-dimensional meaning. For example, $\boldsymbol{a}\wedge\boldsymbol{b}$ is a relative bivector, and so also a spacetime bivector, and not a spacetime four-vector.

The vector derivative, ∇, is defined by

$$\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu} \tag{1.10}$$

where the $\{x^\mu\}$ are a set of Cartesian coordinates and the $\{\gamma^\mu\}$ are the reciprocal frame to the associated coordinate frame $\{\gamma_\mu\}$, *i.e.* $\gamma^\mu\cdot\gamma_\nu = \delta^\mu_\nu$. The spacetime split of the vector derivative ∇ goes through slightly differently, since we require that the $\boldsymbol{\nabla}$ symbol agrees with its conventional three-dimensional meaning. This is achieved by writing

$$\gamma_0\nabla = \partial_t + \boldsymbol{\nabla}, \tag{1.11}$$

so that $\boldsymbol{\nabla} = \boldsymbol{\sigma}_i\partial_{x^i}$. The ∇ operator has the algebraic properties of a vector, and often acts on objects to which it is not adjacent. The 'overdot' notation is a convenient way to encode this:

$$\dot{\nabla}A\dot{B} = \gamma^\mu A\frac{\partial B}{\partial x^\mu}. \tag{1.12}$$

The ∇ operator acts on the object to its immediate right unless brackets or overdots are present. If brackets are present then ∇ operates on everything in the bracket, so that, for example, $\nabla(AB) = \nabla AB + \dot{\nabla}A\dot{B}$. The same rules apply to $\boldsymbol{\nabla}$.

One of the two gravitational gauge fields is the (position-dependent) linear function $\mathsf{h}(a)$, which maps vectors to vectors (where a is the vector argument). Linear functions of this type have their action extended to general multivectors via the rule

$$\mathsf{h}(a\wedge b\cdots\wedge c) = \mathsf{h}(a)\wedge\mathsf{h}(b)\wedge\cdots\wedge\mathsf{h}(c), \tag{1.13}$$

which defines a grade-preserving linear operation. The pseudoscalar is unique up to a scale factor, and the determinant is defined by

$$\mathsf{h}(I) = \det(\mathsf{h})I. \tag{1.14}$$

The adjoint is denoted with an overbar, $\bar{\mathsf{h}}(a)$. The function $\mathsf{h}(a)$ and its adjoint are related by [16]

$$\begin{aligned} A_r\cdot\bar{\mathsf{h}}(B_s) &= \bar{\mathsf{h}}(\mathsf{h}(A_r)\cdot B_s) \qquad r\leq s \\ \mathsf{h}(A_r)\cdot B_s &= \mathsf{h}(A_r\cdot\bar{\mathsf{h}}(B_s)) \qquad r\geq s. \end{aligned} \tag{1.15}$$

A number of manipulations in linear algebra are simplified by using the vector derivative in place of frame contractions. For example, the trace of $\mathsf{h}(a)$ can be written as

$$\mathrm{Tr}(\mathsf{h}) = \gamma^\mu\cdot\mathsf{h}(\gamma_\mu) = \partial_a\cdot\mathsf{h}(a), \tag{1.16}$$

where ∂_a is the vector derivative with respect to a. The following results are also useful:

$$\partial_a\, a\cdot A_r = rA_r \tag{1.17}$$

$$\partial_a\, a\wedge A_r = (n-r)A_r \tag{1.18}$$

$$\partial_a A_r a = (-1)^r(n-2r)A_r, \tag{1.19}$$

where A_r is a multivector of grade r and n is the dimension of the space.

1.2 The field equations

The gravitational gauge fields are a linear function $\bar{\mathsf{h}}(a)$ mapping vectors to vectors and a linear function $\Omega(a)$ mapping vectors to bivectors. Both of these gauge fields have an arbitrary position dependence. The gauge-theoretic origin of these fields is described in [2, 3]. The gauge fields are related by the equation

$$2\Omega(a) = -\bar{\mathsf{h}}(\nabla\wedge\mathsf{g}(a)) + \mathsf{h}^{-1}(\partial_b)\wedge(a\cdot\nabla\bar{\mathsf{h}}(b)), \tag{1.20}$$

where

$$\mathsf{g}(a) = \bar{\mathsf{h}}^{-1}\mathsf{h}^{-1}(a). \tag{1.21}$$

The argument of the linear function, usually denoted by a vector a or b, is always assumed to be independent of position. To recover the more conventional representation of general relativity we introduce an arbitrary set of coordinates x^μ, with e_μ the associated coordinate frame vectors,

$$e_\mu = \frac{\partial x}{\partial x^\mu}. \tag{1.22}$$

With e^μ denoting the reciprocal frame vectors we then define the vectors

$$g_\mu = \mathsf{h}^{-1}(e_\mu), \qquad g^\mu = \bar{\mathsf{h}}(e^\mu). \tag{1.23}$$

In terms of these the metric tensor is defined by

$$g_{\mu\nu} = g_\mu\cdot g_\nu. \tag{1.24}$$

The $\bar{\mathsf{h}}(a)$ field ensures that one only ever has to make 'flat-space' contractions, which is an attractive feature of the gauge-theory approach.

The field strength corresponding to the $\Omega(a)$ gauge field is defined by

$$\mathsf{R}(a\wedge b) = a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega(a)\times\Omega(b) \tag{1.25}$$

and is a linear function mapping bivectors to bivectors. From this the covariant Riemann tensor is defined by

$$\mathcal{R}(a\wedge b) = \mathsf{R}(\mathsf{h}(a\wedge b)). \tag{1.26}$$

We often write this in the form $\mathcal{R}(B)$, where B is an arbitrary (constant) bivector argument. The tensor components of the Riemann tensor are recovered by writing

$$R^\mu{}_{\nu\rho\sigma} = (g^\mu\wedge g_\nu)\cdot\mathcal{R}(g_\sigma\wedge g_\rho). \tag{1.27}$$

The Ricci and Einstein tensors are defined from the Riemann tensor in the obvious way,

$$\text{Ricci Tensor:} \quad \mathcal{R}(b) = \partial_a \cdot \mathcal{R}(a \wedge b) \tag{1.28}$$

$$\text{Ricci Scalar:} \quad \mathcal{R} = \partial_a \cdot \mathcal{R}(a) \tag{1.29}$$

$$\text{Einstein Tensor:} \quad \mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2} a \mathcal{R}. \tag{1.30}$$

Again, the tensor components of the Ricci and Einstein tensors are easily recovered.

1.3 Kerr–Schild fields

We are interested in fields of the form

$$\bar{\mathsf{h}}(a) = a + a \cdot l \, l \tag{1.31}$$

where l is a (flat-space) null vector, $l^2 = 0$. This is the gauge theory analogue of the Kerr–Schild ansatz. The function (1.31) extends to act on multivectors as

$$\bar{\mathsf{h}}(A) = \mathsf{h}(A) = A + A \cdot l \, l, \tag{1.32}$$

and we see immediately that $\det(\bar{\mathsf{h}}) = 1$. The following results are also useful:

$$\mathsf{h}^{-1}(A) = \bar{\mathsf{h}}^{-1}(A) = A - A \cdot l \, l \tag{1.33}$$

$$\mathsf{g}(A) = A - 2A \cdot l \, l \tag{1.34}$$

$$\bar{\mathsf{h}}(l) = l. \tag{1.35}$$

In terms of an orthonormal coordinate frame γ_μ we can write

$$g_\mu = \gamma_\mu - l_\mu l \tag{1.36}$$

which confirms that the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} - 2 l_\mu l_\nu, \tag{1.37}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric tensor.

The $\Omega(a)$ field defined by (1.31) has the simple form

$$\begin{aligned} \Omega(a) &= \bar{\mathsf{h}}\big(\nabla \wedge (a \cdot l \, l)\big) \\ &= \nabla \wedge (a \cdot l \, l) - a \cdot l \, v \wedge l \end{aligned} \tag{1.38}$$

where

$$v = l \cdot \nabla l. \tag{1.39}$$

It follows from the fact that l is null that

$$l \cdot v = 0 \tag{1.40}$$

and

$$\Omega(l) = 0. \tag{1.41}$$

Following the route adopted by Chandrasekhar [18, Section 57], we next form the quantity

$$\begin{aligned} l\cdot\mathcal{R}(l) &= l\cdot(\partial_a\cdot\mathcal{R}(a\wedge l)) \\ &= (l\wedge\partial_a)\cdot\mathsf{R}(a\wedge l) \\ &= (l\wedge\partial_a)\cdot\big(a\cdot\dot{\nabla}\dot{\Omega}(l) - l\cdot\nabla\Omega(a)\big). \end{aligned} \tag{1.42}$$

Substituting equation (1.38) into the above we find that

$$\begin{aligned} l\cdot\mathcal{R}(l) &= (l\wedge\partial_a)\cdot\big(-\dot{\nabla}((a\cdot\nabla l)\cdot\dot{l})\wedge l - l\cdot\nabla\nabla\wedge(a\cdot l\, l)\big) \\ &= \partial_a\cdot l\,(a\cdot\nabla l)\cdot v - l\cdot\nabla(\nabla\cdot l\, l + v)\cdot l \\ &= v^2 - (l\cdot\nabla v)\cdot l \\ &= 2v^2. \end{aligned} \tag{1.43}$$

If we are looking solely for vacuum solutions, then we can conclude from this that v must be null. Since $v\cdot l = 0$, it follows that v must be parallel to l,

$$v = \phi l, \tag{1.44}$$

where ϕ is a scalar field. We will restrict attention to solutions for which this relation does hold, even if matter is present. (This places a restriction on the form of matter distributions that we can consider.) It follows from equation (1.44) that $\Omega(a)$ reduces to the simpler form

$$\Omega(a) = \nabla\wedge(a\cdot l\, l). \tag{1.45}$$

The Riemann tensor now splits into terms that are second-order and fourth-order in l. The fourth-order contribution is

$$\mathcal{R}_4(a\wedge b) = -\dot{\Omega}\big(((a\wedge b)\cdot l\, l)\cdot\dot{\nabla}\big) + \Omega(a)\times\Omega(b). \tag{1.46}$$

After some rearrangement this can be brought to the form

$$\mathcal{R}_4(B) = \tfrac{1}{4}\partial_a\cdot\partial_b\,(a\cdot\nabla l)lBl(b\cdot\nabla l) - \tfrac{1}{4}(a\cdot\nabla l)\cdot(b\cdot\nabla l)\,\partial_a lBl\partial_b. \tag{1.47}$$

Both the contraction, $\partial_a\cdot\mathcal{R}(a\wedge b)$, and the protraction, $\partial_a\wedge\mathcal{R}(a\wedge b)$, of this contribution to the Riemann tensor vanish. This can be seen from the result that

$$\partial_a F_1 a\wedge b F_2 = \partial_a F_1(ab - a\cdot b)F_2 = -bF_1F_2, \tag{1.48}$$

which holds for any two bivectors F_1 and F_2. The presence of the null vector l in the analogous terms in $\mathcal{R}_4(B)$ ensures that

$$\partial_a\mathcal{R}_4(a\wedge b) = 0, \tag{1.49}$$

so that $\mathcal{R}_4(B)$ makes no contribution to the Ricci tensor.

The only part of $\mathcal{R}(B)$ that contributes to the Einstein tensor is therefore the second-order term

$$\mathcal{R}_2(a\wedge b) = a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a). \tag{1.50}$$

Contracting this and setting the result to zero we find that the vacuum Einstein equations reduce to solving the equation

$$\mathcal{R}(a) = \nabla\cdot\Omega(a) - a\cdot\nabla\,\partial_b\cdot\Omega(b) = 0. \tag{1.51}$$

The Ricci scalar and Einstein tensor are now straightforward to calculate:

$$\mathcal{R} = -2\nabla\cdot(\partial_a\cdot\Omega(a)) \tag{1.52}$$

and

$$\mathcal{G}(a) = \nabla\cdot\big(\Omega(a) - a\wedge(\partial_b\cdot\Omega(b))\big). \tag{1.53}$$

The formulae for $\Omega(a)$ (1.45) and $\mathcal{G}(a)$ are valid for any Kerr–Schild type solution for which $l\cdot\nabla l = \phi l$. For such fields the Einstein tensor (1.53) is a total divergence in Minkowski spacetime. In general, the field equations will be satisfied everywhere except for some singular region over which the fields are discontinuous. This singular region contains the source of the fields. In this paper we assume that the entire solution to the Einstein equations is described by fields defined over a single Minkowski spacetime, so that the manifold has not been subjected to maximal extension. In this case we can use Gauss' theorem straightforwardly to convert volume integrals over the source region to surface integrals and so learn how the source matter is distributed. For the case of static fields, Virbhadra [19] gave a formula which agrees with (1.53) for the timelike component $a = \gamma_0$, but the fact that the expression is a total divergence was not exploited.

2 Spherically-symmetric solutions

As our first application we consider spherically-symmetric solutions. For these it is useful to introduce a set of polar coordinates:

$$\begin{aligned} t &= x\cdot\gamma_0 & \cos\theta &= x\cdot\gamma^3/r \\ r &= \sqrt{(x\wedge\gamma_0)^2} & \tan\phi &= (x\cdot\gamma^2)/(x\cdot\gamma^1). \end{aligned} \tag{2.1}$$

We also define

$$e_r = x\wedge\gamma_0\,\gamma_0/r, \qquad \boldsymbol{\sigma}_r = e_r\gamma_0, \tag{2.2}$$

and

$$e_\pm = \gamma_0 \pm e_r. \tag{2.3}$$

For spherically-symmetric solutions l can be written in the form

$$l = \sqrt{\alpha'}\,e_\pm, \tag{2.4}$$

where $\alpha' = \alpha'(t,r)$. For fields of this type it is a simple matter to demonstrate that the fourth-order contribution to the Riemann tensor (1.47) vanishes. To see this consider the case of e_+, for which we obtain

$$\begin{aligned} \mathcal{R}_4(B) &= \frac{\alpha'^2}{4}\big(-\partial_a\cdot\partial_b\,a\cdot\nabla\boldsymbol{\sigma}_r(1-\boldsymbol{\sigma}_r)B(1+\boldsymbol{\sigma}_r)b\cdot\nabla\boldsymbol{\sigma}_r \\ &\quad + (a\cdot\nabla\boldsymbol{\sigma}_r)\cdot(b\cdot\nabla\boldsymbol{\sigma}_r)\,\partial_a e_+ B e_+\partial_b\big) \\ &= \frac{\alpha'^2}{4r}\big(\dot{\boldsymbol{\nabla}}(1-\boldsymbol{\sigma}_r)B(1+\boldsymbol{\sigma}_r)\dot{\boldsymbol{\sigma}}_r - \dot{\boldsymbol{\nabla}}(1-\boldsymbol{\sigma}_r)B(1+\boldsymbol{\sigma}_r)\dot{\boldsymbol{\sigma}}_r\big) \\ &= 0, \end{aligned} \tag{2.5}$$

with the same result holding for e_-. It follows that the Riemann tensor is given entirely by (1.50), which is also a total divergence and so can be analysed using Gauss' theorem. We now turn to three applications of these results.

2.1 The Schwarzschild solution

The simplest solution to the field equations is the Schwarzschild solution, obtained from

$$\alpha' = M/r, \qquad l = \sqrt{\alpha'}\,(\gamma_0 - e_r), \tag{2.6}$$

as we confirm in section 3. The line element generated by this solution is that of the advanced Eddington–Finkelstein form of the Schwarzschild solution. The Riemann tensor for the solution (2.6) can be constructed using equation (1.50), from which we find

$$\begin{aligned}\mathcal{R}(\boldsymbol{a}) &= \boldsymbol{a}\cdot\boldsymbol{\nabla}\Omega(\gamma_0)\\ &= \boldsymbol{a}\cdot\boldsymbol{\nabla}\big(\nabla\wedge(M(\gamma_0 - e_r)/r)\big)\\ &= M\boldsymbol{a}\cdot\boldsymbol{\nabla}\frac{\boldsymbol{x}}{r^3},\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\mathcal{R}(I\boldsymbol{b}) &= \dot{\Omega}\big((I\boldsymbol{b})\cdot\dot{\boldsymbol{\nabla}}\gamma_0\big)\\ &= \nabla\wedge\left(-\frac{M}{r}I\,\boldsymbol{b}\wedge\boldsymbol{\sigma}_r\wedge\boldsymbol{\nabla}\,\boldsymbol{\sigma}_r\gamma_0\right)\\ &= MI\,\boldsymbol{\nabla}\cdot\big(\boldsymbol{b}\wedge\frac{\boldsymbol{x}}{r^3}\big).\end{aligned} \tag{2.8}$$

Away from the origin, these derivatives evaluate to

$$\mathcal{R}(\boldsymbol{a}) = \frac{M}{r^3}(\boldsymbol{a} - 3\boldsymbol{a}\cdot\boldsymbol{\sigma}_r\,\boldsymbol{\sigma}_r), \tag{2.9}$$

and

$$\mathcal{R}(I\boldsymbol{b}) = \frac{IM}{r^3}(\boldsymbol{b} - 3\boldsymbol{b}\cdot\boldsymbol{\sigma}_r\,\boldsymbol{\sigma}_r), \tag{2.10}$$

so we can write the vacuum Riemann tensor in the form

$$\mathcal{R}(B) = -\frac{M}{2r^3}(B + 3\boldsymbol{\sigma}_r B\boldsymbol{\sigma}_r). \tag{2.11}$$

Self duality of the vacuum Riemann tensor has the simple expression $\mathcal{R}(IB) = I\mathcal{R}(B)$ in the spacetime algebra formalism [2]). The form of equation (2.11) shows that the Riemann tensor is manifestly self-dual. This form of the Riemann tensor (2.11) for the Schwarzschild solution was first given in [20] and [21].

The form of the Riemann tensor in equation (2.11) is valid everywhere away from the singularity. To study the form of the singularity, we return to the differential expressions for the Riemann tensor and integrate over a sphere of radius r_0, centered on the origin. Using Gauss' theorem to convert the volume integrals to surface integrals, we obtain

$$\int_{r\le r_o} d^3x\,\mathcal{R}(\boldsymbol{a}) = M\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\,\sin\theta\,\boldsymbol{a}\cdot\boldsymbol{\sigma}_r\,\boldsymbol{\sigma}_r = \frac{4\pi M}{3}\boldsymbol{a}, \tag{2.12}$$

and

$$\int_{r\leq r_o} d^3x\,\mathcal{R}(I\boldsymbol{b}) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\,\sin\theta\, I\boldsymbol{\sigma}_r\cdot(\boldsymbol{b}\wedge\boldsymbol{\sigma}_r) = -\frac{8\pi M}{3} I\boldsymbol{b}. \tag{2.13}$$

These results combine to give

$$\begin{aligned}\int_{r\leq r_o} d^3x\,\mathcal{R}(B) &= \frac{4\pi M}{3}(B\cdot\gamma_0\gamma_0 - 2B\wedge\gamma_0\gamma_0)\\ &= -\frac{2\pi M}{3}(B + 3\gamma_0 B\gamma_0),\end{aligned} \tag{2.14}$$

which contracts to yield

$$\int d^3x\,\mathcal{R}(a) = 4\pi M\,\gamma_0 a\gamma_0 \tag{2.15}$$

$$\int d^3x\,\mathcal{R} = -8\pi M \tag{2.16}$$

$$\int d^3x\,\mathcal{G}(a) = 8\pi M a\cdot\gamma_0\,\gamma_0. \tag{2.17}$$

Since $\mathcal{R}(a) = 0$ everywhere except for the origin, the integrals (2.15)–(2.17) can be taken over any region of space enclosing the origin. It is clear then that the solution represents a point source of matter, with the matter stress-energy tensor given by

$$\mathcal{T}(a) = M\delta(\boldsymbol{x})a\cdot\gamma_0\gamma_0. \tag{2.18}$$

The same conclusion was reached in [2], where the calculations were performed in a different gauge. This result confirms Feynman's speculation in Lecture 15 of [22] that "it will not be possible to demonstrate that $G^\mu{}_\nu = 0$ *everywhere*, but rather that $G^\mu{}_\nu = \delta(\boldsymbol{x})$, or something of the kind". The integrals performed above are not gauge invariant, but gauge-invariant information is extracted from them in the form of the matter stress-energy tensor (2.18). Furthermore, the integral of the Ricci scalar provides a direct measure of the mass of the source, without the need to resort to constructing integrals in an asymptotically flat region of spacetime.

2.2 The Reissner–Nordstrom solution

The Reissner–Nordstrom solution can be written in the form

$$\bar{\mathsf{h}}(a) = a + \eta a\cdot e_-\, e_- \tag{2.19}$$

where q the charge of the source (in natural units) and

$$\eta = \frac{M}{r} - \frac{q^2}{8\pi r^2}. \tag{2.20}$$

The Einstein tensor for this solution is

$$\mathcal{G}(a) = \nabla\cdot\big(\nabla(\eta a\cdot e_-)\wedge e_- - \nabla\cdot(\eta e_-)\,a\wedge e_-\big). \tag{2.21}$$

Away from the origin we know that the mass term can be ignored, which leaves

$$\mathcal{G}(\gamma_0) = -\frac{q^2}{4\pi}\nabla\cdot\left(\frac{\boldsymbol{\sigma}_r}{r^3}\right) = \frac{q^2}{4\pi r^4}\,\gamma_0, \tag{2.22}$$

$$\mathcal{G}(\gamma_i) = \frac{q^2}{8\pi}\nabla\cdot\left(\nabla\left(\frac{\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_r}{r^2}\right)\wedge e_-\right) = -\frac{q^2}{4\pi r^4}\,\boldsymbol{\sigma}_r\gamma_i\boldsymbol{\sigma}_r. \tag{2.23}$$

These combine to give a corresponding matter stress-energy tensor of

$$\mathcal{T}(a) = \frac{1}{8\pi}\mathcal{G}(a) = -\tfrac{1}{2}\mathcal{F}a\mathcal{F} \tag{2.24}$$

where $\mathcal{F} = q\boldsymbol{\sigma}_r/(4\pi r^2)$. This is the expected form for the electromagnetic stress-energy tensor due to a point source of charge q. (See [2] for a detailed explanation of how to handle electromagnetism in gauge-theory gravity.)

To study the behaviour of the fields near the origin we return to the differential form for $\mathcal{G}(a)$ and again construct integrals over a sphere of radius r_0. For this case we find that

$$\begin{aligned}\int_{r\le r_o} d^3x\,\mathcal{G}(\gamma_0) &= \int_{r\le r_o} d^3x\,\boldsymbol{\nabla}\cdot\left(\frac{2M}{r^2}\,\boldsymbol{\sigma}_r - \frac{q^2}{4\pi r^3}\,\boldsymbol{\sigma}_r\right)\gamma_0 \\ &= \left(8\pi M - \frac{q^2}{r^0}\right)\gamma_0,\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}\int_{r\le r_o} d^3x\,\mathcal{G}(\boldsymbol{a}\gamma_0) &= \int_{r\le r_o} d^3x\,\frac{q^2}{8\pi}\gamma_0\boldsymbol{\nabla}\cdot\left(\frac{1}{r^3}\boldsymbol{a}\wedge\boldsymbol{\sigma}_r\right) \\ &= \frac{q^2}{3r_0}\,\boldsymbol{a}\gamma_0,\end{aligned} \tag{2.26}$$

which combine to give

$$\int_{r\le r_o} d^3x\,\mathcal{T}(a) = Ma\cdot\gamma_0\,\gamma_0 + \frac{q^2}{24\pi r_0}(a - 4a\cdot\gamma_0\,\gamma_0). \tag{2.27}$$

The mass term here is precisely as expected and shows again that a point source is located at the origin. The electromagnetic contribution is traceless, as one expects for the electromagnetic stress-energy tensor. Focusing attention on the γ_0-frame energy component of the stress-energy tensor, we see that

$$\int_{r\le r_o} d^3x\,\gamma_0\cdot\mathcal{T}(\gamma_0) = M - \frac{q^2}{8\pi r_0}. \tag{2.28}$$

This result was also obtained by Tod [23], who calculated the quasi-local mass for the Reissner–Nordstrom solution as defined by Penrose [24]. Tod argued that this result implies that a source for the Reissner–Nordstrom solution should have $r > q^2/(8\pi M)$ at the surface in order to meet the dominant energy condition. However, this misses the point that the negative contribution to the integral comes entirely from the origin. Everywhere off the origin the stress-energy tensor satisfies the dominant energy condition. Taking the integrals over the volume defined by $r_0 < r < \infty$ we find that the electromagnetic field energy is

$q^2/(8\pi r_0)$, which agrees with the formula given by Virbhadra [19] and is simply the classical result.

The electromagnetic contribution to (2.28) is negative and finite for all finite r_0, and tends to zero as the integral extends over all space. This is in stark contrast to the standard picture from classical electromagnetism, where the integral of the γ_0-frame energy $\boldsymbol{E}^2/2$ diverges for the interior of any surface enclosing the origin — the classical self-energy problem discussed by many authors (see [6, 25], for example). Inclusion of the gravitational field has removed this divergence, ensuring that the total electromagnetic self-energy is zero. The manner in which this regularisation is achieved is both simple and instructive. The electromagnetic energy density is rewritten as

$$\frac{\boldsymbol{E}^2}{2} = \frac{q^2}{32\pi r^4} = -\frac{q^2}{32\pi}\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{x}}{r^4}\right), \tag{2.29}$$

so that the integral over space of the electromagnetic energy density can be converted to a surface integral, recovering the contribution to (2.28). Since the electromagnetic energy density near a point source is very large, it is unsurprising that the inclusion of gravity has significant consequences, and these clearly have implications for the status of self-energies in classical field theory. However, since only classical fields are employed above, it is not clear whether this result has similar implications for the divergent self-energies encountered in QED.

2.3 The Vaidya solution

As a final example of the use of integral equations for spherically-symmetric Kerr–Schild fields, we consider Vaidya's 'shining star' solution [1]. This is generated by the field

$$\bar{\mathsf{h}}(a) = a + \frac{\mu(t-r)}{r}\, a\cdot e_+\, e_+, \tag{2.30}$$

which is clearly similar to the Schwarzschild solution, except that now the mass $\mu = \mu(t-r)$ is variable and the null geodesics e_+ are outgoing rather than incoming. The solution (2.30) is clearly of Kerr–Schild type, and defining l by

$$l = \sqrt{\mu/r}\, e_+, \tag{2.31}$$

we find that

$$l\cdot\nabla l = \left(\frac{\mu}{r}\right)^{1/2} e_+\cdot\nabla\left(\left(\frac{\mu}{r}\right)^{1/2} e_+\right) = -\frac{1}{2}\left(\frac{\mu}{r^3}\right)^{1/2} l, \tag{2.32}$$

so that equation (1.44) is satisfied. The Einstein tensor for (2.30) is

$$\mathcal{G}(a) = \nabla\cdot\left(\frac{2\mu}{r^2} a\cdot e_+\, \boldsymbol{\sigma}_r\right) \tag{2.33}$$

and away from the origin (where we can set $\nabla\cdot(\boldsymbol{\sigma}_r/r^2) = 0$) this becomes

$$\mathcal{G}(a) = -\frac{2\dot{\mu}}{r^2} a\cdot e_+\, e_+, \tag{2.34}$$

where $\dot{\mu} = \partial_t\mu$. This tensor represents a radially-symmetric flux of outgoing massless particles. Again, the presence of a δ-function point source at the origin

can be inferred from the differential form of the Einstein tensor. By evaluating the integral of $\mathcal{G}(a)$ over a sphere centred on the origin, and shrinking the radius to zero, we find that

$$\mathcal{G}(a) = -\frac{2\dot{\mu}}{r^2} a\cdot e_+ \, e_+ + 8\pi\mu\delta(\boldsymbol{x}) a\cdot\gamma_0 \, \gamma_0. \tag{2.35}$$

The solution (2.30) therefore describes a point mass at rest at the origin which is losing mass at some arbitrary rate. This is borne out by the Riemann tensor,

$$\mathcal{R}(B) = -\frac{\dot{\mu}}{r^2} B\cdot e_+ \, e_+ - \frac{\mu}{2r^3}(B + 3\boldsymbol{\sigma}_r B \boldsymbol{\sigma}_r), \tag{2.36}$$

which exhibits a neat split into a source term describing the energy outflow and a Weyl term due to the point mass at the origin.

The fact that the Einstein tensor is given by the divergence of a bivector implies that

$$\nabla\cdot\mathcal{G}(a) = 0. \tag{2.37}$$

We can therefore define a conserved total energy E by

$$\begin{aligned} 8\pi E &= \int d^3x \, \gamma_0\cdot\mathcal{G}(\gamma_0) \\ &= \int d^3x \, \boldsymbol{\nabla}\cdot\big(2\mu\frac{\boldsymbol{\sigma}_r}{r^2}\big) \\ &= 8\pi\mu(-\infty). \end{aligned} \tag{2.38}$$

The total conserved energy is therefore determined by the mass of the source at $t = -\infty$, before it began radiating, which is clearly a sensible result. A conserved energy of this form will exist for any Kerr–Schild field of the type (1.31), provided that the null vector l satisfies $l\cdot\nabla l = \phi l$.

3 Stationary vacuum solutions

We now turn to a more general analysis, dropping the requirement of spherical symmetry. As we are ultimately interested in the Kerr solution, however, we do restrict to stationary, vacuum solutions. For these we write l in the form

$$l = \sqrt{\alpha'}\, n \tag{3.1}$$

where

$$n = \gamma_0 - \boldsymbol{n}\gamma_0, \tag{3.2}$$

$\boldsymbol{n}^2 = 1$, and α' and $\boldsymbol{n}$ are functions of the spatial position vector $\boldsymbol{x} = x\wedge\gamma_0$ only. This is the most general form for a stationary, Kerr–Schild field. The condition that $l\cdot\nabla l = \phi l$ immediately yields

$$-\boldsymbol{n}\cdot\boldsymbol{\nabla}\big(\sqrt{\alpha'}(\gamma_0 - \boldsymbol{n}\gamma_0)\big) = \phi(\gamma_0 - \boldsymbol{n}\gamma_0) \tag{3.3}$$

hence

$$\phi = -\boldsymbol{n}\cdot\boldsymbol{\nabla}\sqrt{\alpha'}, \tag{3.4}$$

and

$$\boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{n} = 0. \tag{3.5}$$

The final equation shows that the integral curves of $\boldsymbol{n}$ are straight lines. These define possible incoming photon trajectories in space. The fact that these lines a straight in the background space is a gauge-specific statement, and does not correspond to a physically-observable property.

For stationary vacuum fields the equation $\mathcal{R}(a) = 0$ splits into the pair of equations

$$\nabla\cdot\Omega(\gamma_0) = 0 \tag{3.6}$$

and

$$\nabla\cdot\Omega(\gamma_i) + \boldsymbol{\sigma}_i\cdot\boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot(\alpha'\boldsymbol{n})n\big) = 0. \tag{3.7}$$

To simplify equation (3.6) we need the result that

$$\Omega(\gamma_0) = \nabla\wedge(\alpha' n) = -\boldsymbol{\nabla}\alpha' - \boldsymbol{\nabla}\wedge(\alpha'\boldsymbol{n}), \tag{3.8}$$

On splitting into spatial vector and bivector parts equation (3.6) reduces to

$$\boldsymbol{\nabla}^2\alpha' = 0 \tag{3.9}$$

and

$$\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\wedge(\alpha'\boldsymbol{n})\big) = 0. \tag{3.10}$$

3.1 The hidden complex structure

The content of the second field equation (3.7) is summarised neatly in the equation

$$\boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{n} + \boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{n}) = \frac{2\alpha'}{\boldsymbol{\nabla}\cdot(\alpha'\boldsymbol{n})}(\boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{n}))\cdot\boldsymbol{\nabla}\boldsymbol{n}. \tag{3.11}$$

In [7] the authors showed that this equation implies that we can write

$$\boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{n} = \alpha\boldsymbol{a}\wedge\boldsymbol{n}\,\boldsymbol{n} - I\beta\boldsymbol{a}\wedge\boldsymbol{n}, \tag{3.12}$$

where α and β are two new real scalar functions. We see immediately that

$$\boldsymbol{\nabla}\cdot\boldsymbol{n} = 2\alpha, \qquad \boldsymbol{\nabla}\wedge\boldsymbol{n} = -2I\beta\boldsymbol{n}, \tag{3.13}$$

and it follows that

$$\boldsymbol{\nabla}\cdot(\beta\boldsymbol{n}) = 0, \qquad \Longrightarrow \qquad \boldsymbol{n}\cdot\boldsymbol{\nabla}\beta = -2\alpha\beta. \tag{3.14}$$

Now, setting $\boldsymbol{a} = \boldsymbol{\nabla}$ in equation (3.12), we obtain

$$\boldsymbol{\nabla}^2\boldsymbol{n} = \boldsymbol{\nabla}\alpha - \boldsymbol{\nabla}\cdot(\alpha\boldsymbol{n})\boldsymbol{n} - I\,\boldsymbol{\nabla}\wedge(\beta\boldsymbol{n}). \tag{3.15}$$

Similarly, writing equations (3.13) in the form $\boldsymbol{\nabla}\boldsymbol{n} = 2(\alpha - I\beta\boldsymbol{n})$ and differentiating we obtain

$$\boldsymbol{\nabla}^2\boldsymbol{n} = 2\boldsymbol{\nabla}\alpha - 2I\boldsymbol{\nabla}(\beta\boldsymbol{n}). \tag{3.16}$$

On combining these equations we find that

$$\boldsymbol{n}\cdot\boldsymbol{\nabla}\alpha = \beta^2 - \alpha^2, \tag{3.17}$$

and we can therefore write

$$\boldsymbol{\nabla}\alpha - I\boldsymbol{\nabla}\beta\,\boldsymbol{n} = -(\alpha^2 + \beta^2)\boldsymbol{n} + 2I\beta(\alpha - I\beta\boldsymbol{n}). \tag{3.18}$$

This equation is simplified by employing the idempotent element N,

$$N = \tfrac{1}{2} n\gamma_0 = \tfrac{1}{2}(1 - \boldsymbol{n}), \tag{3.19}$$

which satisfies

$$N^2 = N = -\boldsymbol{n}N = -N\boldsymbol{n}. \tag{3.20}$$

(An idempotent is a mixed-grade multivector that squares to give itself. Spacetime idempotents are usually closely related to null vectors.) On postmultiplying by N, equation (3.18) yields

$$\boldsymbol{\nabla}\gamma N = \gamma^2 N, \tag{3.21}$$

where

$$\gamma = \alpha + I\beta. \tag{3.22}$$

It follows that $(\boldsymbol{\nabla}\gamma)^2 = \gamma^4$ so that, if we define ω by

$$\omega = \frac{1}{\gamma}, \tag{3.23}$$

then ω must satisfy

$$(\boldsymbol{\nabla}\omega)^2 = 1. \tag{3.24}$$

This is the first of the pair of complex equations found in [7]. The novel feature of the derivation presented here is that the 'complex' quantity γ is of the form of a scalar + pseudoscalar. This gives a clear geometric origin to the complex structure at the heart of the Kerr solution. This complex structure carries through to the form of the Riemann tensor, and hence to all of the observable quantities associated with the solution.

Use of the idempotent N simplifies many expressions and derivations. For example, differentiating (3.21) and pre- and post-multiplying by N yields

$$\boldsymbol{\nabla}^2\gamma N = 2\gamma^3 N - N\tfrac{1}{2}\big(\gamma^2\boldsymbol{\nabla}\boldsymbol{n} - \dot{\boldsymbol{\nabla}}(\boldsymbol{\nabla}\gamma)\,\dot{\boldsymbol{n}}\big)N \tag{3.25}$$

But we know that $\boldsymbol{n}\cdot\boldsymbol{\nabla}\gamma = -\gamma^2$ and $\boldsymbol{\nabla}\boldsymbol{n}N = 2\gamma N$, so we can rearrange the final term as follows:

$$\begin{aligned} N\dot{\boldsymbol{\nabla}}(\boldsymbol{\nabla}\gamma)\dot{\boldsymbol{n}}N &= -N\boldsymbol{n}\dot{\boldsymbol{\nabla}}(\boldsymbol{\nabla}\gamma)\dot{\boldsymbol{n}}N \\ &= N\dot{\boldsymbol{\nabla}}(\boldsymbol{n}\boldsymbol{\nabla}\gamma)\dot{\boldsymbol{n}}N \\ &= -2N\gamma^2\boldsymbol{\nabla}\boldsymbol{n}N - N\dot{\boldsymbol{\nabla}}\boldsymbol{\nabla}\gamma\dot{\boldsymbol{n}}N \\ &= -2\gamma^3 N. \end{aligned} \tag{3.26}$$

This type of rearrangement is typical of the way that one can take advantage of the properties of idempotents in the spacetime algebra. On substituting this result into equation (3.25) we now find that

$$\boldsymbol{\nabla}^2\gamma\, N = 0, \tag{3.27}$$

and hence

$$\boldsymbol{\nabla}^2\gamma = 0. \tag{3.28}$$

This is the second of the pair of complex equations found in [7]. Solving the field equations now reduces to finding a complex harmonic function γ whose inverse

ω satisfies $(\boldsymbol{\nabla}\omega)^2 = 1$. The above derivation reveals the geometric origin of this complex structure, as well as demonstrating the role of the null vector n through the idempotent N.

To complete the solution we need to find forms for α' and $\boldsymbol{n}$. For the former we note that

$$\frac{\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{n})}{2\alpha} = \frac{\alpha^2+\beta^2}{2\alpha} = \frac{\boldsymbol{\nabla}\cdot(\alpha'\boldsymbol{n})}{2\alpha'} \tag{3.29}$$

and

$$\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\wedge(\alpha\boldsymbol{n})\big) = \boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\wedge(\alpha'\boldsymbol{n})\big) = 0. \tag{3.30}$$

From these it is a simple matter to show that $\alpha' = M\alpha$, where M is some arbitrary constant. To recover $\boldsymbol{n}$ we use equation (3.21) in the form

$$-\boldsymbol{\nabla}\omega(1-\boldsymbol{n}) = (1-\boldsymbol{n}), \qquad \boldsymbol{\nabla}\omega^*(1+\boldsymbol{n}) = (1+\boldsymbol{n}) \tag{3.31}$$

where the $*$ denotes the complex conjugation operation (which can be written as $\omega^* = \gamma_0\omega\gamma_0$). On rearranging we obtain

$$(\boldsymbol{\nabla}\omega + \boldsymbol{\nabla}\omega^*)\boldsymbol{n} = 2 + \boldsymbol{\nabla}\omega - \boldsymbol{\nabla}\omega^* \tag{3.32}$$

$$\Longrightarrow \qquad \boldsymbol{n} = \frac{\boldsymbol{\nabla}\omega + \boldsymbol{\nabla}\omega^* - (\boldsymbol{\nabla}\omega)\times(\boldsymbol{\nabla}\omega^*)}{1+\langle\boldsymbol{\nabla}\omega\boldsymbol{\nabla}\omega^*\rangle}, \tag{3.33}$$

where the angle brackets $\langle\,\rangle$ denote the projection onto the scalar part of a multivector. Recall here that the $\times$ symbol represents half the commutator of the terms on either side, and not the vector cross product.

Some further insight into the nature of the solution and the role of the complex structure is obtained from the form of $\Omega(\gamma_0)$. From equations (3.13) to (3.16) it is straightforward to show that

$$\boldsymbol{\nabla}\wedge(\alpha\boldsymbol{n}) = I\boldsymbol{\nabla}\beta. \tag{3.34}$$

It follows that $\Omega(\gamma_0)$ is now given by

$$\Omega(\gamma_0) = -M(\boldsymbol{\nabla}\alpha + \boldsymbol{\nabla}\wedge(\alpha\boldsymbol{n})) = -M\boldsymbol{\nabla}\gamma. \tag{3.35}$$

This shows how the harmonic function γ generalises the scalar Newtonian potential. This gives rise to many of the novel properties of the Kerr solution. A further result that is useful in later calculations is that

$$\partial_a\cdot\Omega(a) = M(\alpha^2+\beta^2)n. \tag{3.36}$$

3.2 The Riemann tensor

The Riemann tensor would be expected to contain terms of order M and M^2, but it is not hard to see that the latter contribution vanishes. Using equations (1.47) and (3.12) this term can be written in the form

$$\mathcal{R}_4(B) = -\frac{M^2}{2}I\alpha^2\beta(1-\boldsymbol{n})\dot{\boldsymbol{\nabla}}B\dot{\boldsymbol{n}}(1+\boldsymbol{n}). \tag{3.37}$$

But when B is the spatial bivector $\boldsymbol{a}$ we see that

$$\begin{aligned}(1-\boldsymbol{n})\dot{\boldsymbol{\nabla}}\boldsymbol{a}\dot{\boldsymbol{n}}(1+\boldsymbol{n}) &= (1-\boldsymbol{n})(2\boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{n} - \boldsymbol{a}\boldsymbol{\nabla}\boldsymbol{n})(1+\boldsymbol{n})\\ &= (1-\boldsymbol{n})(2\gamma^*\boldsymbol{a}\wedge\boldsymbol{n} - 2\gamma^*\boldsymbol{a})(1+\boldsymbol{n})\\ &= -2\gamma^*\boldsymbol{a}\cdot\boldsymbol{n}(1-\boldsymbol{n})(1+\boldsymbol{n})\\ &= 0,\end{aligned} \tag{3.38}$$

and the same result holds for $B = I\boldsymbol{b}$. The annihilation of orthogonal idempotents in this derivation, $(1+\boldsymbol{n})(1-\boldsymbol{n}) = 0$, is merely a reexpression of the fact that n is a null vector.

The only contribution to $\mathcal{R}(B)$ therefore comes from $\mathcal{R}_2(B)$. For a spatial bivector this contribution can be written as

$$\begin{aligned}\mathcal{R}(\boldsymbol{a}) = \mathcal{R}(a\wedge\gamma_0) &= \boldsymbol{a}\cdot\boldsymbol{\nabla}\Omega(\gamma_0)\\ &= -M\boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}\gamma\\ &= -\tfrac{1}{2}M\dot{\boldsymbol{\nabla}}\boldsymbol{a}\boldsymbol{\nabla}\dot{\gamma}.\end{aligned} \tag{3.39}$$

For a vacuum solution the Riemann tensor only contains a Weyl term, and so satisfies the self-duality property [2] $\mathcal{R}(IB) = I\mathcal{R}(B)$. It follows that we can write

$$\mathcal{R}(B) = -\tfrac{1}{2}M\dot{\boldsymbol{\nabla}}B\boldsymbol{\nabla}\dot{\gamma}, \tag{3.40}$$

for all B. This expression captures all of the terms of the Riemann tensor in a single, highly compact expression. Verifying that we have a vacuum solution now reduces to the identity

$$\partial_a\dot{\boldsymbol{\nabla}}\, a\wedge b\,\boldsymbol{\nabla}\dot{\gamma} = -b\boldsymbol{\nabla}^2\gamma = 0. \tag{3.41}$$

4 The Kerr solution

The simplest solution to the pair of equations $\boldsymbol{\nabla}^2\gamma = 0$ and $(\boldsymbol{\nabla}\omega)^2 = 1$ is $\gamma = 1/r$. This recovers the Schwarzschild solution. To conirm this we first see that equation (3.33) gives

$$\boldsymbol{n} = \boldsymbol{\sigma}_r, \tag{4.1}$$

which is the only possible vector consistent with spherical symmetry. The null vector n is given by $\gamma_0 - e_r$, recovering the form of solution analysed in section 2.1. The Riemann tensor can be found directly from equation (3.40), which yields

$$\begin{aligned}\mathcal{R}(B) &= \tfrac{1}{2}M\boldsymbol{\nabla}(B\boldsymbol{x}/r^3)\\ &= -\frac{M}{2r^3}(B + 3\boldsymbol{\sigma}_r B\boldsymbol{\sigma}_r)\end{aligned} \tag{4.2}$$

recovering equation (2.11).

Since the equations $\boldsymbol{\nabla}^2\gamma = 0$ and $(\boldsymbol{\nabla}\omega)^2 = 1$ are invariant under complex 'coordinate transformations', a new solution is obtained from the Schwarzschild solution by setting

$$\omega = (x^2 + y^2 + (z - IL)^2)^{1/2}. \tag{4.3}$$

This is the most general complex translation that can be applied to the Schwarzschild solution [7] and generates the Kerr solution. The symbol L ($L > 0$) for the angular momentum is preferred here to the more common symbol a as we have already made extensive use of a as a vector variable. As was first shown in [7], this complex transformation justifies the 'trick' first discovered by Newman & Janis [8]. Precisely how the complex square root in (4.3) is defined is discussed further in Section 4.2.

From equation (3.40), we can immediately construct the Riemann tensor as follows:

$$\begin{aligned}\mathcal{R}(B) &= -\frac{M}{2}\boldsymbol{\nabla}(B\boldsymbol{\nabla}\gamma)\\ &= -\frac{3M}{8\omega^5}\boldsymbol{\nabla}(\omega^2)B\boldsymbol{\nabla}(\omega^2)+\frac{M}{4\omega^3}\boldsymbol{\nabla}(B\boldsymbol{\nabla}(\omega^2))\\ &= -\frac{M}{2\omega^3}\left(B+3\frac{\boldsymbol{x}-L\,I\boldsymbol{\sigma}_3}{\omega}\,B\,\frac{\boldsymbol{x}-L\,I\boldsymbol{\sigma}_3}{\omega}\right).\end{aligned} \tag{4.4}$$

The spacetime bivector

$$\boldsymbol{\sigma}_\gamma = \frac{\boldsymbol{x}-L\,I\boldsymbol{\sigma}_3}{\omega} = \boldsymbol{\nabla}\omega \tag{4.5}$$

satisfies ${\boldsymbol{\sigma}_\gamma}^2 = 1$, so the Riemann tensor (4.4) has the same algebraic structure as for the Schwarzschild solution (it is type D). The only difference is that the eigenvalues are now complex, rather than real. The Riemann tensor is only singular when $\omega = 0$, which is over the ring $r = L$, $z = 0$. This is the reason why the solution is conventionally referred to as containing a ring singularity.

The structure of the fields away from the region enclosed by the ring is most easily seen in an oblate spheroidal coordinate system. Such a system is defined by:

$$L\cosh u\,\cos v = (x^2+y^2)^{1/2} = \rho \tag{4.6}$$

$$L\sinh u\,\sin v = z, \tag{4.7}$$

with $0 \le u < \infty$, $-\pi/2 \le v \le \pi/2$. These relations are summarised neatly in the single identity

$$L\cosh(u+Iv) = \rho + Iz. \tag{4.8}$$

The basic identities for oblate spheroidal coordinates, and their relationship to cylindrical polar coordinates, are summarised in Table 1

The point of adopting an oblate spheroidal coordinate system is apparent from the form of ω:

$$\begin{aligned}w &= L(\cosh^2 u\,\cos^2 v + \sinh^2 u\,\sin^2 v - 1 - 2I\sinh u\,\sin v)^{1/2}\\ &= L(\sinh u - I\sin v).\end{aligned} \tag{4.9}$$

This definition of the square root ensures that $\omega \mapsto r$ at large distances. Equation (3.33) yields a unit vector $\boldsymbol{n}$ of

$$\begin{aligned}\boldsymbol{n} &= \frac{2L\cosh u\,\boldsymbol{e}^u - L^2(\cosh u\,\boldsymbol{e}^u + I\cos v\,\boldsymbol{e}^v)\times(\cosh u\,\boldsymbol{e}^u - I\cos v\,\boldsymbol{e}^v)}{1+L^2\langle(\cosh u\,\boldsymbol{e}^u + I\cos v\,\boldsymbol{e}^v)(\cosh u\,\boldsymbol{e}^u - I\cos v\,\boldsymbol{e}^v)\rangle}\\ &= \frac{1}{L\cosh u}(\boldsymbol{e}_u - L\cos v\,\boldsymbol{\sigma}_\phi).\end{aligned} \tag{4.10}$$

As a check,

$$\begin{aligned}\boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{n} &= \left(\frac{1}{L\cosh u}\partial_u - \frac{1}{L\cosh^2 u}\partial_\phi\right)\left(\tanh u\cos v\,\boldsymbol{e}_\rho + \sin v\,\boldsymbol{\sigma}_3 - \frac{\cos v}{\cosh u}\boldsymbol{\sigma}_\phi\right)\\ &= 0\end{aligned} \tag{4.11}$$

Cylindrical Polar Coordinates $\{\rho, \phi, z\}$

$$\rho = (x^2 + y^2)^{1/2}$$
$$\phi = \tan^{-1}(y/x)$$
$$\boldsymbol{e}_\rho = \cos\phi\, \boldsymbol{\sigma}_1 + \sin\phi\, \boldsymbol{\sigma}_2$$
$$\boldsymbol{e}_\phi = \rho \boldsymbol{\sigma}_\phi = \rho(-\sin\phi\, \boldsymbol{\sigma}_1 + \cos\phi\, \boldsymbol{\sigma}_2)$$
$$\boldsymbol{e}_\rho \boldsymbol{e}_\phi \boldsymbol{\sigma}_3 = \rho I$$

Oblate Spheroidal Coordinates $\{u, \phi, v\}$

$$L \cosh u \cos v = \rho$$
$$L \sinh u \sin v = z$$
$$\boldsymbol{e}_u = L(\sinh u \cos v\, \boldsymbol{e}_\rho + \cosh u \sin v\, \boldsymbol{\sigma}_3)$$
$$\boldsymbol{e}_v = L(-\cosh u \sin v\, \boldsymbol{e}_\rho + \sinh u \cos v\, \boldsymbol{\sigma}_3)$$
$$\boldsymbol{e}_u^2 = \boldsymbol{e}_v^2 = L^2(\cosh^2 u - \cos^2 v)$$
$$\boldsymbol{e}_u \boldsymbol{e}_\phi \boldsymbol{e}_v = \rho L^2(\cosh^2 u - \cos^2 v) I$$

Further Relations

$$\boldsymbol{e}_\rho = L(\sinh u \cos v\, \boldsymbol{e}^u - \cosh u \sin v\, \boldsymbol{e}^v)$$
$$\boldsymbol{\sigma}_3 = L(\cosh u \sin v\, \boldsymbol{e}^u + \sinh u \cos v\, \boldsymbol{e}^v)$$
$$\boldsymbol{x} = L^2(\sinh u \cosh u\, \boldsymbol{e}^u - \sin v \cos v\, \boldsymbol{e}^v)$$

Table 1: Some basic relations for oblate spheroidal coordinates.

and

$$\boldsymbol{n}^2 = \frac{1}{\cosh^2 u}(\cosh^2 u - \cos^2 v + \cos^2 v) = 1, \tag{4.12}$$

both as required.

The vector field $\boldsymbol{n}$ satisfies $\boldsymbol{n} \cdot \boldsymbol{\nabla} \boldsymbol{n} = 0$, so its integral curves in flat space are straight lines. These can be parameterised by

$$\boldsymbol{x}(\lambda) = L \cos v_0\, \boldsymbol{e}_\rho(\phi_0) - \lambda(\cos v_0\, \boldsymbol{\sigma}_\phi(\phi_0) - \sin v_0\, \boldsymbol{\sigma}_3), \tag{4.13}$$

where $L \cos v_0$ and ϕ_0 are the polar coordinates for the starting point of the integral curve over the central disk. Plots of these integral curves are shown in Figure 1. As commented on earlier, the fact that the trajectories are straight lines in the background space is a feature of our chosen gauge. The same picture is not produced in alternative gauges, though the fact that the integral curves emerge from the central disk region is gauge invariant.

4.1 Exterior integrals

Oblate spheroidal coordinates are very useful for performing surface integrals in the Kerr solution over regions entirely surrounding the disk. The most conve-

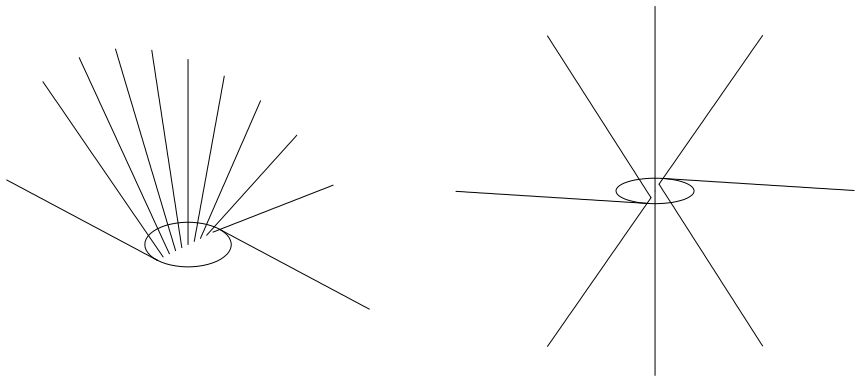

Figure 1: Two views of the integral curves of $\boldsymbol{n}$. The left-hand figure shows the view from above of a set of incoming geodesics that terminate along a diameter of the central disk. This pattern is rotated around the z-axis to give the full set of geodesics. The right-hand figure shows incoming geodesics from above and below the disk. These are the mirror image of each other. The ring is shown for clarity — it is not an integral curve of $\boldsymbol{n}$.

nient surfaces to consider are ellipsoids of constant u, for which the divergence theorem can be given in the form

$$\int_{u'\leq u} d^3x\,(A\overleftrightarrow{\boldsymbol{\nabla}}B) = \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\,\rho A\boldsymbol{e}_u B, \tag{4.14}$$

where A and B are general multivectors. The $\leftrightarrow$ on $\boldsymbol{\nabla}$ indicates that the vector derivative acts both to the left and right,

$$A\overleftrightarrow{\boldsymbol{\nabla}}B = \dot{A}\dot{\boldsymbol{\nabla}}B + A(\boldsymbol{\nabla}B), \tag{4.15}$$

and the measure d^3x is taken as running over u' rather than u. This slightly loose notation ensures that the result of the integral is a function of u. Equation (4.14) accounts for all of the cases that we will encounter.

Our aim is to explore the nature of the matter singularity through the use of integral equations. As a first step, we look at the total mass-energy in the source. Taking the surface as one of constant u we find that

$$\begin{aligned}\int_{u'\leq u} d^3x\,\mathcal{G}(\gamma_0) &= M\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\,\rho(\gamma_0\boldsymbol{e}_u)\cdot\big(-\boldsymbol{\nabla}\gamma + (\alpha^2+\beta^2)\boldsymbol{n}\big)\\ &= M\gamma_0\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\,\rho\left(-\frac{\partial\alpha}{\partial u} - \frac{\partial\beta}{\partial v}\boldsymbol{\sigma}_\phi + (\alpha^2+\beta^2)\frac{\boldsymbol{e}_u^2}{L\cosh u}\right)\\ &= 2\pi M\gamma_0\int_{-\pi/2}^{\pi/2} dv\left(\frac{\cosh^2 u\cos v(\sinh^2 u - \sin^2 v)}{(\sinh^2 u + \sin^2 v)^2} + \cos v\right)\\ &= 8\pi M\gamma_0.\end{aligned} \tag{4.16}$$

So, as the with Schwarzschild case, the total mass-energy in the γ_0-frame is M. For the spatial part we find that

$$\int_{u'\leq u} d^3x\, \mathcal{G}(\boldsymbol{a}\gamma_0) = M\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\, \rho(\gamma_0 \boldsymbol{e}_u)\cdot\big(-\boldsymbol{\nabla}(\alpha\boldsymbol{a}\cdot\boldsymbol{n}) - \boldsymbol{\nabla}\wedge(\alpha\boldsymbol{a}\cdot\boldsymbol{n}\,\boldsymbol{n}) + (\alpha^2+\beta^2)(\boldsymbol{a}+\boldsymbol{a}\wedge\boldsymbol{n})\big). \tag{4.17}$$

Once the angular integral is performed, the γ_0 contribution to this integral becomes

$$2\pi M\gamma_0 \int_{-\pi/2}^{\pi/2} dv\, \rho\big(-\partial_u(\alpha\sin v) + (\alpha^2+\beta^2)L\cosh u \sin v\big) = 0, \tag{4.18}$$

which vanishes as the integrand is odd in v. This is reassuring, as we expect the integrated stress-energy tensor to be symmetric if there is no source of torsion hidden in the singularity. The remaining integral to be performed transforms to

$$\begin{aligned}\int_{u'\leq u} d^3x\, \mathcal{G}(\boldsymbol{a}\gamma_0)\gamma_0 &= M\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\, \rho I\, \boldsymbol{e}_u\wedge\boldsymbol{\nabla}(\beta\boldsymbol{a}\cdot\boldsymbol{n}) \\ &= M\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\, I\big(\rho\boldsymbol{\sigma}_\phi\, \boldsymbol{a}\cdot(\partial_v(\beta\boldsymbol{n})) - \boldsymbol{e}_v\, \boldsymbol{a}\cdot(\partial_\phi(\beta\boldsymbol{n}))\big) \\ &= M\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\, I\beta\, \boldsymbol{a}\cdot\boldsymbol{n}(-\partial_v\rho\,\boldsymbol{\sigma}_\phi + \partial_\phi \boldsymbol{e}_v) \\ &= 0.\end{aligned} \tag{4.19}$$

In terms of the corresponding matter stress-energy tensor $\mathcal{T}(a)$ the above results are summarised by

$$\int d^3x\, \mathcal{T}(a) = Ma\cdot\gamma_0\,\gamma_0, \tag{4.20}$$

where the integral is over any region of space entirely enclosing the central disk.

Integrating the matter stress-energy tensor over the entire disk region averages out any possible angular momentum contribution. To recover the angular momentum we look at

$$x\wedge\mathcal{G}(a) = t\gamma_0\wedge\mathcal{G}(a) + (\boldsymbol{x}\gamma_0)\wedge\Big((\gamma_0\boldsymbol{\nabla})\cdot\big(\Omega(a) - a\wedge(\partial_b\cdot\Omega(b))\big)\Big). \tag{4.21}$$

The first term on the right-hand side will give zero when integrated over a region enclosing the disk. To simplify the remaining term we first write

$$F(a) = \Omega(a) - a\wedge(\partial_b\cdot\Omega(b)). \tag{4.22}$$

We next use the rearrangement

$$(\boldsymbol{x}\gamma_0)\wedge((\gamma_0\boldsymbol{\nabla})\cdot F(a)) = (\boldsymbol{x}\gamma_0)\wedge((\gamma_0\overleftrightarrow{\boldsymbol{\nabla}})\cdot F(a)) - (F(a)\cdot\gamma_0\gamma_0 + 2F(a)\wedge\gamma_0\gamma_0), \tag{4.23}$$

to write the volume integral as

$$\begin{aligned}\int_{u'\leq u} d^3x\, x\wedge\mathcal{G}(a) = &\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\, \rho(\boldsymbol{x}\gamma_0)\wedge((\gamma_0\boldsymbol{e}_u)\cdot F(a)) \\ &- \int_{u'\leq u} d^3x\, (F(a)\cdot\gamma_0\gamma_0 + 2F(a)\wedge\gamma_0\gamma_0).\end{aligned} \tag{4.24}$$

The final volume integral involves

$$\Omega(a) - a\wedge(\partial_b\cdot\Omega(b)) = M\nabla\wedge(\alpha a\cdot nn) - Ma\wedge(\nabla\cdot(\alpha n)n + \alpha n\cdot\nabla n) \tag{4.25}$$

which is also a total divergence and can be converted to a surface integral. For the γ_0 term we now find that

$$\int_{u'\leq u} d^3x\, x\wedge\mathcal{G}(\gamma_0) = M\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\, \rho\Big(\alpha(\boldsymbol{e}_u\wedge\boldsymbol{n}\,\boldsymbol{n} + 2\boldsymbol{e}_u\wedge\boldsymbol{n}) + (\boldsymbol{x}\gamma_0)\wedge\big((\gamma_0\boldsymbol{e}_u)\cdot(-\boldsymbol{\nabla}\gamma + (\alpha^2+\beta^2)\boldsymbol{n})\big)\Big). \tag{4.26}$$

This simplifies down to

$$\begin{aligned} &M\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\, \rho\big(-I\,\boldsymbol{x}\cdot(\boldsymbol{e}_u\wedge\boldsymbol{\nabla}\beta) + 2\alpha\boldsymbol{e}_u\wedge\boldsymbol{n}\big) \\ &= -2\pi MLI\boldsymbol{\sigma}_3 \int_{-\pi/2}^{\pi/2} dv\, \left(\frac{\cosh^2 u\cos^3 v(\sinh^2 u - \sin^2 v)}{(\sinh^2 u + \sin^2 v)^2} + \frac{2\sinh^3 u\cos^3 v}{\sinh^2 u + \sin^2 v}\right) \\ &= -8\pi ML\, I\boldsymbol{\sigma}_3. \end{aligned} \tag{4.27}$$

For the spatial terms we obtain

$$\begin{aligned} &\int_{u\leq u_0} d^3x\, x\wedge\mathcal{G}(\boldsymbol{a}\gamma_0) = M\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\, \rho\Big(-I\,\boldsymbol{x}\cdot\big(\boldsymbol{e}_u\wedge\boldsymbol{\nabla}(\beta\boldsymbol{a}\cdot\boldsymbol{n})\big) \\ &+ \boldsymbol{x}\,\boldsymbol{e}_u\cdot\big(-\boldsymbol{\nabla}(\alpha\boldsymbol{a}\cdot\boldsymbol{n}) + (\alpha^2+\beta^2)\boldsymbol{a}\big) + \alpha\big(2(\boldsymbol{e}_u\wedge\boldsymbol{a})\cdot\boldsymbol{n}\boldsymbol{n} + (\boldsymbol{e}_u\wedge\boldsymbol{a})\cdot\boldsymbol{n}\big)\Big), \end{aligned} \tag{4.28}$$

which reduces down to

$$\begin{aligned} &M\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\, \Big(-2LI\boldsymbol{\sigma}_3\cos v\,\boldsymbol{a}\cdot\boldsymbol{n} - 2\rho\alpha\,\boldsymbol{e}_u\cdot\boldsymbol{n}\,\boldsymbol{a}\wedge\boldsymbol{n} \\ &+ \rho\alpha(\boldsymbol{e}_u\wedge\boldsymbol{a})\cdot\boldsymbol{n} + \rho\boldsymbol{x}\,\boldsymbol{e}_u\cdot\big(-\boldsymbol{\nabla}(\alpha\boldsymbol{a}\cdot\boldsymbol{n}) + (\alpha^2+\beta^2)\boldsymbol{a}\big)\Big) \\ =&M\int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} dv\, \rho\Big(\alpha(\boldsymbol{e}_u\wedge\boldsymbol{a})\cdot\boldsymbol{n} \\ &+ \boldsymbol{x}\,\boldsymbol{e}_u\cdot\big(-\boldsymbol{\nabla}(\alpha\boldsymbol{a}\cdot\boldsymbol{n}) + (\alpha^2+\beta^2)\boldsymbol{a}\big)\Big). \end{aligned} \tag{4.29}$$

The final integral is best performed term by term, yielding

$$\int d^3x\, x\wedge\mathcal{G}(\gamma_1) = 4\pi ML\boldsymbol{\sigma}_2 \tag{4.30}$$

$$\int d^3x\, x\wedge\mathcal{G}(\gamma_2) = -4\pi ML\boldsymbol{\sigma}_1 \tag{4.31}$$

$$\int d^3x\, x\wedge\mathcal{G}(\gamma_3) = 0. \tag{4.32}$$

These results combine to give

$$\int d^3x\, x\wedge\mathcal{T}(a) = ML\big(-a\cdot\gamma_0\, I\boldsymbol{\sigma}_3 + \tfrac{1}{2}(a\wedge\gamma_0)\times I\boldsymbol{\sigma}_3\big), \tag{4.33}$$

where the integral is taken over any region entirely enclosing the central disk. This expression clearly identifies ML as the total angular momentum in the fields, as expected from the long-range behaviour. The expression also has the correct algebraic form for a symmetric stress-energy tensor. A symmetric stress-energy tensor has

$$\partial_a \wedge \mathcal{T}(a) = 0. \tag{4.34}$$

If this relation holds then we expect that

$$\partial_a \wedge \big(-a\cdot\gamma_0\, I\boldsymbol{\sigma}_3 + \tfrac{1}{2}(a\wedge\gamma_0)\times(I\boldsymbol{\sigma}_3)\big) = 0, \tag{4.35}$$

which is easily confirmed. We therefore expect that there are no sources of torsion hidden in the singular region. A discussion of the gauge-invariance of the mass-energy and angular-momentum integrals will be delayed until after we have a more complete understanding of the stress-energy tensor.

4.2 The singularity

In order to fully understand the nature of the source matter for the Kerr solution we must look at the region $\rho < L$. One has to be careful with the application of oblate spheroidal coordinates in this region, and it is safer to return to cylindrical polar coordinates for most calculations. Central to an understanding of this region is the definition of the complex square root in (4.3). If we consider some fixed $\rho > L$, then the complex function ω^2 has a real part > 0 for all values of z and the square root can be defined as a smooth continuous function also with a real part > 0 (see Figure 2.a). This is the definition of the square root implicitly adopted in equation (4.9) with the introduction of oblate spheroidal coordinates. If we now consider a region where $\rho < L$ and z is finite, continuity of ω requires that the square root still be defined to have a positive real part. This means that positive and negative z now correspond to different branches of the square root (see Figure 2.b). As a result, ω is discontinuous across the entire disk $\rho < L$, $z = 0$. This discontinuity is also easily seen in oblate spheroidal coordinates, for which $z = 0$, $\rho < L$ implies that $u = 0$ and $\sin v$ is discontinuous over the disk. The alternative, which is not considered here, is to extend the manifold so that passing through the disk connects an observer to a new spacetime (a new Riemann sheet).

4.3 The Ricci scalar

The simplest gauge-invariant quantity to study over the disk is the Ricci scalar, which is given by the total divergence

$$\mathcal{R} = -2\nabla\cdot(\partial_a\cdot\Omega(a)) = -2M\boldsymbol{\nabla}\cdot((\alpha^2+\beta^2)\boldsymbol{n}). \tag{4.36}$$

We compute the integral of this over an infinite cylinder centred on the z-axis of radius ρ, $\rho < L$. In converting this to a surface integral the contributions from the top and bottom of the cylinder can be ignored, leaving

$$\int_{\rho'\leq\rho} d^3x\,\mathcal{R} = -2M\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\,\rho(\alpha^2+\beta^2)\boldsymbol{e}_\rho\cdot\boldsymbol{n}. \tag{4.37}$$

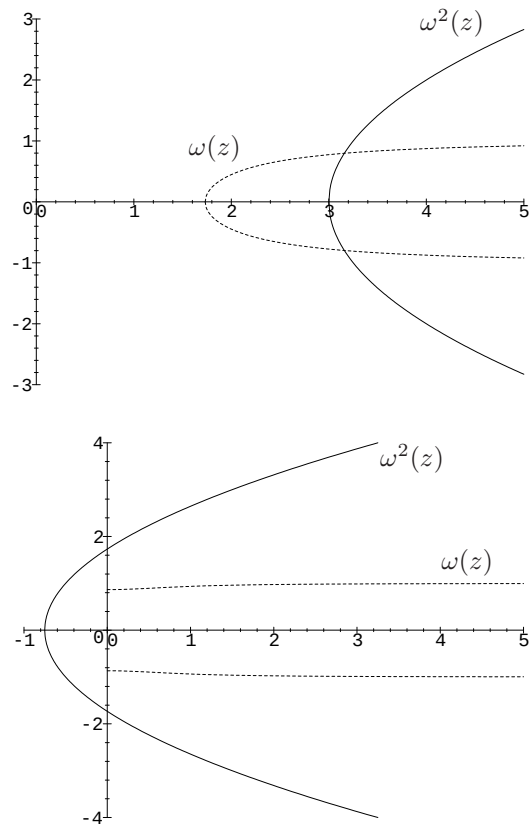

Figure 2: The complex function ω for fixed ρ as a function of z. In both cases $L = 1$. The top figure is for $\rho = 2 > L$ and the lower figure for $\rho = 0.5 < L$. The solid lines are for ω^2 and the broken lines for the square root. Continuity of ω for finite z requires that ω be discontiuous over the central disk.

(As earlier, the dummy radial variable in the measure is taken as ρ', so that the result is a function of ρ.) We therefore define

$$W(\rho) = \int_{\rho' \leq \rho} d^3x\, \partial_a \cdot \mathcal{T}(a)/M = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, \rho\, \gamma\gamma^*\, \boldsymbol{e}_\rho \cdot \boldsymbol{n}. \tag{4.38}$$

Now

$$\boldsymbol{e}_\rho \cdot \boldsymbol{n} = \frac{\sinh u\, \cos v}{\cosh u} = \frac{\rho L \sinh u}{L^2 \cosh^2 u}, \tag{4.39}$$

and we can write

$$L \sinh u = \mathbb{R}\big((\rho^2 + (z - IL)^2)^{1/2}\big) = \mathbb{R}\left(\frac{1}{\gamma}\right). \tag{4.40}$$

We therefore only require an explicit expression for $L^2 \cosh^2 u$ in terms of ρ and

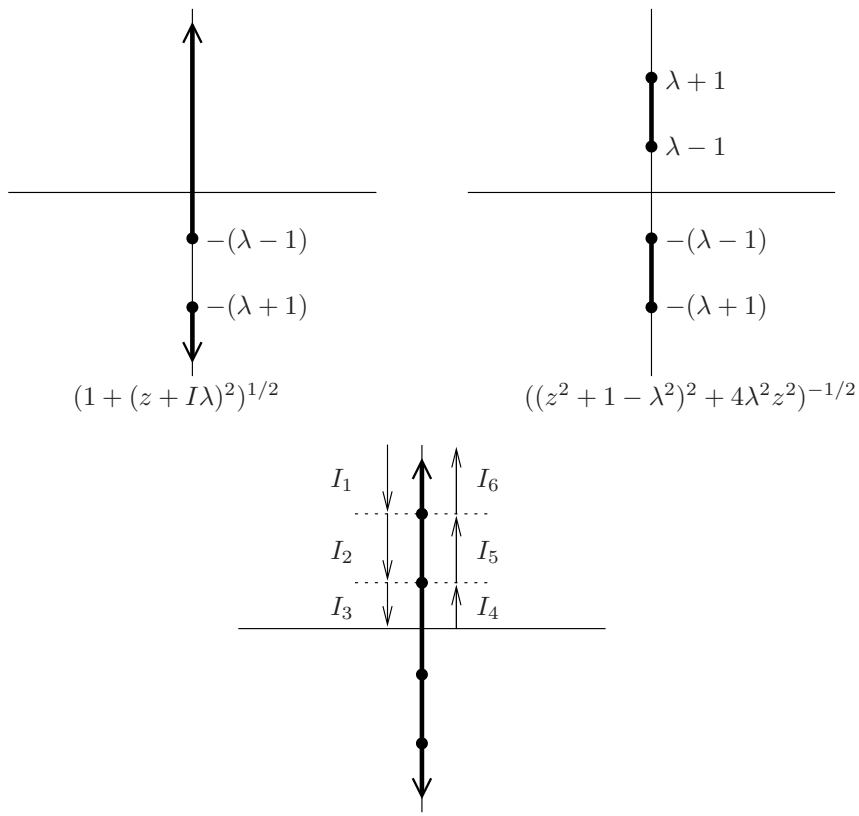

Figure 3: Branch cuts and contours for the integral (4.45). The branch cuts for complex z shown in the top two figures follow from the global definition of ω. The bottom figure shows the six integrals that have to be calculated after the integration contour has been rotated.

z. Such an expression is found from

$$\begin{aligned} \boldsymbol{x}^2 + L^2 &= L^2(\cosh^2 u \cos^2 v + \sinh^2 u \sin^2 v + 1) \\ &= L^2(\cosh^2 u + \cos^2 v) \end{aligned} \tag{4.41}$$

and

$$L^2(\cosh^2 u - \cos^2 v) = \big((\rho^2 + z^2 - L^2)^2 + 4L^2 z^2\big)^{1/2}, \tag{4.42}$$

so that

$$2L^2 \cosh^2 u = \rho^2 + z^2 + L^2 + \big((\rho^2 + z^2 - L^2)^2 + 4L^2 z^2\big)^{1/2}. \tag{4.43}$$

On substituting these results into (4.38) we obtain

$$W(\rho) = \int_{-\infty}^{\infty} dz\, \rho^2 \, \frac{\mathbb{R}\big((\rho^2 + (z+IL)^2)^{-1/2}\big)}{\rho^2 + z^2 + L^2 + ((\rho^2 + z^2 - L^2)^2 + 4L^2 z^2)^{1/2}}, \tag{4.44}$$

where the integrand contains a finite jump at $z = 0$, $(\rho < L)$ and no singularities.

The integral is simplified by rescaling to give

$$W(\rho) = \int_{-\infty}^{\infty} dz \, \frac{\mathbb{R}\big((1+(z+I\lambda)^2)^{-1/2}\big)}{z^2+1+\lambda^2+((z^2+1-\lambda^2)^2+4\lambda^2 z^2)^{1/2}}, \tag{4.45}$$

where $\lambda = L/\rho > 1$. The branch cuts for the complex square roots in this integral follow from the global definition of ω and are shown in Figure 3. The integral is performed by splitting into the two regions $z > 0$ and $z < 0$ and rotating each of the contours to lie on the positive imaginary z axis. This leaves six integrals to compute (shown in Figure 3) which combine as follows:

$$\begin{aligned}
I_1 + I_6 &= -2 \int_{\lambda+1}^{\infty} dy \, \frac{((y+\lambda)^2-1)^{-1/2}}{y^2-\lambda^2-1+((y^2-\lambda^2-1)^2-4\lambda^2)^{1/2}} \\
I_2 + I_5 &= -\frac{1}{2\lambda^2} \int_{\lambda-1}^{\lambda+1} dy \, \frac{y^2-\lambda^2-1}{((y+\lambda)^2-1)^{1/2}} \\
I_3 + I_4 &= +2 \int_{0}^{\lambda-1} dy \, \frac{((y+\lambda)^2-1)^{-1/2}}{\lambda^2+1-y^2+((y^2-\lambda^2-1)^2-4\lambda^2)^{1/2}}.
\end{aligned} \tag{4.46}$$

These combine into the simpler integrals

$$\begin{aligned}
W(\rho) = \lim_{b\to\infty} & -\frac{1}{2\lambda^2} \int_0^b dy \, \frac{y^2-\lambda^2-1}{((y+\lambda)^2-1))^{1/2}} + \frac{1}{2\lambda^2} \int_{\lambda+1}^{b} dy \, ((y-\lambda)^2-1)^{1/2} \\
& - \frac{1}{2\lambda^2} \int_0^{\lambda-1} dy \, ((y-\lambda)^2-1)^{1/2},
\end{aligned} \tag{4.47}$$

which are easily evaluated with cosh substitutions. The cutoff b is introduced since the separate integrals are divergent. On performing the substitutions we find that

$$\begin{aligned}
W(\rho) &= \lim_{b\to\infty} \frac{1}{\lambda} \int_{\cosh^{-1}(\lambda)}^{\cosh^{-1}(b+\lambda)} dw \, \cosh w - \frac{1}{2\lambda^2} \int_{\cosh^{-1}(b-\lambda)}^{\cosh^{-1}(b+\lambda)} dw \, \sinh^2 w \\
&= \lim_{b\to\infty} \frac{1}{4\lambda^2} \big((b-\lambda)((b-\lambda)^2-1)^{1/2} - (b-3\lambda)((b+\lambda)^2-1)^{1/2}\big) \\
&\quad - \frac{1}{\lambda}(\lambda^2-1)^{1/2} \\
&= 1 - \frac{(L^2-\rho^2)^{1/2}}{L},
\end{aligned} \tag{4.48}$$

so that

$$\int_{\rho'\le\rho} d^3x \, \partial_a \cdot \mathcal{T}(a) = M\left(1 - \frac{(L^2-\rho^2)^{1/2}}{L}\right). \tag{4.49}$$

Since the solution is axisymmetric, $\partial_a \cdot \mathcal{T}(a)$ can only depend on ρ and z. We must therefore have, for $\rho < L$,

$$\partial_a \cdot \mathcal{T}(a) = f(\rho)\delta(z), \tag{4.50}$$

where $f(\rho)$ is found from differentiating (4.49):

$$f(\rho) = \frac{M}{2\pi L(L^2-\rho^2)^{1/2}}. \tag{4.51}$$

The function $f(\rho)$ is remarkably simple, given the convoluted route by which it is obtained. However, its true significance is not seen until the remaining gauge-invariant information has been extracted from $\mathcal{G}(a)$. This information resides in the eigenvalues of $\mathcal{G}(a)$, the calculation of which introduces further complexities.

4.4 The Einstein tensor

To calculate the full form of $\mathcal{G}(a)$ over the disk, we start with the most general form that $\mathcal{G}(a)$ can take consistent with the fact that the Kerr solution is axisymmetric. Such a form is defined by, for $\rho < L$,

$$\begin{aligned}
\mathcal{G}(\gamma_0) &= \delta(z)(\alpha_1\gamma_0 + \beta_1\hat{\phi} + \delta_1 e_\rho + \epsilon_1\gamma_3) \\
\mathcal{G}(\hat{\phi}) &= \delta(z)(\alpha_2\hat{\phi} + \beta_2\gamma_0 + \delta_2 e_\rho + \epsilon_2\gamma_3) \\
\mathcal{G}(e_\rho) &= \delta(z)(\alpha_3 e_\rho + \beta_3\gamma_0 + \delta_3\hat{\phi} + \epsilon_3\gamma_3) \\
\mathcal{G}(\gamma_3) &= \delta(z)(\alpha_4\gamma_3 + \beta_4\hat{\phi} + \delta_4 e_\rho + \epsilon_4\gamma_0),
\end{aligned} \tag{4.52}$$

where each of the $\alpha_i \ldots \epsilon_i$ are scalar functions of ρ only. We do not assume that $\mathcal{G}(a)$ is a symmetric linear function so as to allow for the possibility that the matter contains a hidden source of torsion.

Calculation of each of the terms in $\mathcal{G}(a)$ proceeds in the same manner as the calculation of the Ricci scalar. The resulting computations are long and somewhat tedious, and have been relegated to Appendix A. The final results are that, for $\rho < L$,

$$\begin{aligned}
\mathcal{G}(\gamma_0) &= -\delta(z)\frac{2M\rho}{L(L^2-\rho^2)^{3/2}}(\rho\gamma_0 + L\hat{\phi}) \\
\mathcal{G}(\hat{\phi}) &= \delta(z)\frac{2M}{(L^2-\rho^2)^{3/2}}(\rho\gamma_0 + L\hat{\phi}) \\
\mathcal{G}(e_\rho) &= \delta(z)\frac{2M}{L(L^2-\rho^2)^{1/2}}e_\rho \\
\mathcal{G}(\gamma_3) &= 0.
\end{aligned} \tag{4.53}$$

These confirm that $\mathcal{G}(a)$ is symmetric, so there is no hidden torsion. We see immediately that e_ρ is an eigenvector of $\mathcal{G}(a)$, with eigenvalue $2M/(L(L^2-\rho^2)^{1/2})$, and also that there is no momentum flow in the γ_3 direction, which is physically obvious. The structure of the remaining terms is most easily seen by introducing the boost factor λ via

$$\tanh\lambda = \frac{\cos v}{\cosh u} \tag{4.54}$$

and defining the timelike velocity

$$v = e^{\lambda\boldsymbol{\sigma}_\phi}\gamma_0 = \cosh\lambda\gamma_0 + \sinh\lambda\hat{\phi}. \tag{4.55}$$

(This second use of the symbol v should not be confused with the vector $v = l{\cdot}\nabla l$ defined earlier.) With these definitions we see that, over the disk,

$$\mathcal{G}(v) = 0, \tag{4.56}$$

and

$$\mathcal{G}(\boldsymbol{\sigma}_\phi v) = \delta(z)\frac{2M}{L(L^2-\rho^2)^{1/2}}\boldsymbol{\sigma}_\phi v. \tag{4.57}$$

There is therefore zero energy density in the v direction, with isotropic tension of $M/(4\pi L(L^2-\rho^2)^{1/2})$ in the plane of the disk. This conclusion is gauge invariant, since it is based solely on the eigenvalue structure of the Einstein tensor. It is truly remarkable that such a simple picture emerges from the complicated set of calculations in Appendix A.

The velocity vector v defines the natural rest frame in the region of the disk. A second velocity is defined by the timelike Killing vector

$$g_t = \mathsf{h}^{-1}(\gamma_0) \tag{4.58}$$

(see [2] for details of how Killing vectors are handled within gauge-theory gravity.) The boost required to move between these velocities therefore provides an intrinsic definition of the field velocity in the disk region. In this region the function $\bar{\mathsf{h}}(a)$ reduces to the identity, so g_t is simply γ_0. It follows that the velocity is given by

$$\tanh\lambda = \cos v = \rho/L, \tag{4.59}$$

and the angular velocity is therefore $1/L$. This is precisely as expected for a rigid rotation, which fits in with the observation of [26] that the Kerr solution can be viewed as the limiting case of a rigidly-rotating matter distribution.

4.5 An alternative gauge and the matter ring

The form of the eigenvectors of $\mathcal{G}(a)$ suggest that the more appropriate gauge for the study of the Kerr solution is provided by the boost

$$R = e^{-\lambda\boldsymbol{\sigma}_\phi/2} \tag{4.60}$$

so that the new solution is generated by

$$\bar{\mathsf{h}}'(a) = R(a + M\alpha a\cdot n\, n)\tilde{R}. \tag{4.61}$$

In this gauge the tension lies entirely in the $I\boldsymbol{\sigma}_3$ eigenplane, and the characteristic bivector of the Riemann tensor becomes

$$R\boldsymbol{\sigma}_\gamma\tilde{R} = \frac{\boldsymbol{e}_u}{|\boldsymbol{e}_u|} \tag{4.62}$$

which is now a relative spatial vector. In this gauge the γ_0 frame is the rest frame defined by the Weyl and matter tensors, whereas the Killing vectors are now swept round.

The boost (4.60) is well-defined everywhere except for the ring singularity where the matter is located. This is unproblematic, since the fields are already singular there. We know that the integral of the Ricci scalar over the disk gives

$$8\pi M\int_0^L d\rho\,\frac{\rho}{L(L^2-\rho^2)^{1/2}} = 8\pi M \tag{4.63}$$

which accounts for the entire contribution to $\mathcal{R}$ found for integrals outside the disk. It follows that the contribution to the stress-energy tensor from the ring

singularity must have a vanishing trace, so that no contribution is made to the Ricci scalar. It is also clear that the entire contribution to the angular momentum must come from the ring, since the tension is isotropic over the disk. From these considerations it is clear that the contribution to $\mathcal{G}(a)$ from the ring singularity, in the gauge defined by (4.61), must be of the form

$$\mathcal{G}_{\rm ring}(a) = \frac{4M}{L}\delta(z)\,\delta(\rho - L)\,a\cdot(\gamma_0 + \hat{\phi})\,(\gamma_0 + \hat{\phi}). \tag{4.64}$$

This confirms that the ring current follows a lightlike trajectory — the natural endpoint for collapsing matter with angular momentum. The radius of the orbit, L, agrees with the minimum size allowed by special relativity (see exercise 5.6 of [27], for example).

4.6 The physics of the disk

A natural question is whether the tension field has a simple non-gravitational explanation. This is indeed the case. The equations for a special relativistic fluid are

$$(\varepsilon + P)(v\cdot\nabla v)\wedge v = \nabla P\wedge v \tag{4.65}$$

$$\nabla\cdot(\varepsilon v) = -P\nabla\cdot v, \tag{4.66}$$

where ε is the energy density, P is the pressure and v is the fluid velocity ($v^2 = 1$). For the case of a ring of particles surrounding a rigidly-rotating massless membrane under tension we see that (ignoring the factors of $\delta(z)$), $P = P(\rho)$ and

$$v = \cosh\lambda\,\gamma_0 + \sinh\lambda\,\hat{\phi}, \qquad \tanh\lambda = \rho/L. \tag{4.67}$$

If follows that

$$v\cdot\nabla v = \sinh^2\lambda\,\hat{\phi}\cdot\nabla\,\hat{\phi} = -\frac{\rho}{L^2 - \rho^2}\,e_\rho, \tag{4.68}$$

so the equation for P is

$$\frac{\partial P}{\partial\rho} - \frac{\rho}{L^2 - \rho^2}P = 0. \tag{4.69}$$

This has the solution

$$P = \frac{P_0}{(L^2 - \rho^2)^{1/2}}, \tag{4.70}$$

which has precisely the functional form of the tension distribution found above. The constant P_0 is found by requiring that the trace of the stress-energy tensor returns M when integrated over the disk. This fixes the tension to $M/(4\pi L(L^2 - \rho^2)^{1/2})$, precisely as is built into the gravitational fields. Of course, the required 'light' membrane cannot be made from any known matter. Indeed, the fact that the membrane generates a tension while having zero energy density means that it violates the weak energy condition. Nevertheless, it is quite remarkable that such a simple physical picture holds in a region of such extreme fields.

In writing the tension as $M/(4\pi L(L^2 - \rho^2)^{1/2})$ we are expressing it in terms of the radial coordinate ρ. This coordinate is given physical significance by the fact that the $\bar{\mathsf{h}}$ function is the identity over the disk region, so ρ is the

proper distance from the centre of the disk. This gives a simple gauge-invariant definition of L as the physical radius of the disk. Furthermore, the form of the Riemann tensor (4.4) shows that its complex eigenvalues are driven by M/ω^3, which is gauge-invariant, and hence a physically-measurable quantity. This affords gauge-invariant significance to M and ω, and hence to the coordinates ρ and z. It follows that both L and M have simple gauge-invariant definitions, without needing to appeal to the asymptotic properties of the solution.

5 Conclusions

Many of the significant solutions to the Einstein equations can be represented in Kerr–Schild form and gauge-theoretic approach of [2] is well suited to their analysis. For all solutions of Kerr–Schild type where the null vector l satisfies $l \cdot \nabla l = \phi l$ the Einstein tensor is a total divergence in flat spacetime. The structure of the sources generating the fields can therefore be elucidated by employing Gauss theorem to transform volume integrals to surface integrals. This approach is fully justified within the gauge-theory formulation, since one only ever deals with fields defined over a flat spacetime.

For the case of the Schwarzschild, Reissner–Nordstrom and Vaidya solutions the gravitational fields are seen to result from a δ-function point source of mass at the origin. For the Reissner–Nordstrom solution the δ-function point source is surrounded by a Coulomb field. An unexpected bonus of this approach is that the infinite self-energy of the Coulomb field is removed by the gravitational field. Similar techniques can be applied to Kinnersley's and Bonnor's work on accelerating and radiating masses [28, 29], as will be discussed elsewhere.

Applied to more general stationary, vacuum solutions we find that the complex structure at the heart of vacuum Kerr–Schild fields is the same as the natural complex structure inherent in the Weyl tensor through its self-duality symmetry. Further algebraic insights are obtained through the use of null vectors as idempotent elements, simplifying many of the derivations of the vacuum equations. Both of these insights highlight the algebraic advantages of the spacetime algebra approach. A further example of this is seen clearly in equation (3.40), which gives a remarkably simple and compact expression for the Riemann tensor.

The application of Gauss' theorem to the Kerr solution reveals some surprising features of the singularity. The ring of matter follows a lightlike trajectory and surrounds a disk of tension. The tension distribution over the disk is precisely that predicted by special relativity. The correct tension distribution was computed by Hamity [14], though he did not comment on its origin in terms of classical relativistic physics. We find no evidence of either the negative surface energy density or the superluminal speeds claimed by Isreal [5]. Both Hamity and Isreal asserted that they used the same results for surface layers in general relativity, but neither gave detailed calculations, so the reason for Isreal's disagreement with our result is hard to pin down.

Almost all trajectories in the Kerr geometry finish up on the disk, rather than the ring. Quite what happens when a particle encounters the δ-function tension over the disk is unclear and can only really be understood using a quantum framework to study the effect of the disk on a wavepacket. (A start on such an analysis in made in [30].) Assuming that all geodesics do terminate

on the disk then any non-causal features of the Kerr solution are removed [5], which is a physically attractive feature of the picture presented here. The fact that the tension membrane violates the weak energy condition raises a further interesting question — how can it be formed from collapsing baryonic matter? Furthermore, if baryonic matter cannot form the membrane, then what is the endpoint of the collapse process? Answers to these questions will only emerge when realistic collapse scenarios are formulated, though these are notoriously difficult computations to perform.

The discussion in this paper implicitly rules out considering any extensions to the manifold, such as obtained by converting the Schwarzschild solution to Kruskal coordinates. We therefore do not consider distinct universes connected by Schwarzschild 'throat', with separate future and past singularities [27, 31], or the maximum analytic extension of the Reissner–Nordstrom geometry with infinite ladder of possible 'universes' connected by wormholes [31, 32]. In such scenarios the applications of Gauss' theorem employed in this paper would not be valid. While infinite ladders of connected universes remain popular with science fiction writers, there is no reason to believe they could ever form physically in any collapse process. The descriptions presented here for both the Reissner–Nordstrom and Kerr solutions have a much more plausible physical feel to them, even if the final description of the singular region must ultimately involve quantum gravity. One final speculation concerns the nature of the membrane supporting the Kerr ring singularity. This bears a remarkable similarity to some of the structures encountered in string theory, and it would be of great interest to see if string theory can provide a quantum description of such a source.

A Surface integrals of the Einstein tensor

From the form of (4.52) there are 16 scalar functions to find using extensions of the technique described in Section 4.3. These are evaluated below.

A.1 $\mathcal{G}(\gamma_0)$

For this term we have

$$\mathcal{G}(\gamma_0) = \delta(z)(\alpha_1\gamma_0 + \beta_1\hat{\phi} + \delta_1 e_\rho + \epsilon_1\gamma_3), \tag{A.1}$$

and α_1 and ϵ_1 are computed directly from

$$\int_{\rho'\leq\rho} d^3x\,\mathcal{G}(\gamma_0) = 2\pi\int_0^\rho d\rho'\,\rho'\big(\alpha_1(\rho')\gamma_0 + \epsilon_1(\rho')\gamma_3\big). \tag{A.2}$$

On converting to a surface integral we obtain

$$\begin{aligned}\int_{\rho'\leq\rho} d^3x\,\mathcal{G}(\gamma_0) &= \rho\int_0^{2\pi} d\phi\int_{-\infty}^{\infty} dz\, e^\rho\cdot\big(\Omega(\gamma_0) - \gamma_0\wedge(\partial_a\cdot\Omega(a))\big)\\ &= M\rho\int_0^{2\pi} d\phi\int_{-\infty}^{\infty} dz\,\left(-\partial_\rho\alpha + (\alpha^2+\beta^2)\boldsymbol{e}_\rho\cdot\boldsymbol{n})\gamma_0 + \frac{1}{\rho}\partial_\phi\beta\,\gamma_3\right)\\ &= 2\pi M\rho^2\int_{-\infty}^{\infty} dz\,\left(\mathbb{R}(\gamma^3) + \frac{L\sinh u\gamma\gamma^*}{L^2\cosh^2 u}\right)\gamma_0,\end{aligned} \tag{A.3}$$

which shows that $\epsilon_1 = 0$. (Here γ^3 refers to the complex field $\gamma = 1/\omega$, and not to a reciprocal frame vector. When a γ_μ-vector is intended the γ will always appear with a subscript.) The final term on the right-hand side of (A.3) is the integral already performed for the Ricci scalar. For the remaining term we need

$$\begin{aligned}\rho^2 \int_{-\infty}^{\infty} dz\, \mathbb{R}(\gamma^3) &= \mathbb{R} \int_{-\infty}^{\infty} \frac{\rho^2\, dz}{(\rho^2 + (z + IL)^2)^{3/2}} \\ &= 2 - \frac{2L}{(L^2 - \rho^2)^{1/2}},\end{aligned} \tag{A.4}$$

where it is again crucial that the correct branch cuts are employed in the evaluation of the contour integral. Substituting this result into (A.3) we find that

$$2\pi \int_0^\rho d\rho'\, \rho' \alpha_1(\rho') = 4\pi M \left(2 - \frac{L}{(L^2 - \rho^2)^{1/2}} - \frac{(L^2 - \rho^2)^{1/2}}{L} \right), \tag{A.5}$$

and differentiating recovers

$$\alpha_1 = -\frac{2M\rho^2}{L(L^2 - \rho^2)^{3/2}}. \tag{A.6}$$

For the remaining terms we first observe that

$$\begin{aligned}\int_{\rho' \leq \rho} d^3x\, \rho e^\rho \cdot \mathcal{G}(\gamma_0) &= 2\pi \int_0^\rho d\rho'\, \rho'^2 \delta_1(\rho') \\ &= \int_{\rho' \leq \rho} d^3x\, (\rho e^\rho \wedge \overleftrightarrow{\nabla}) \cdot \big(\Omega(\gamma_0) - \gamma_0 \wedge (\partial_a \cdot \Omega(a))\big) \\ &= 0,\end{aligned} \tag{A.7}$$

so we must have $\delta_1 = 0$. To find β_1 we need to apply the divergence theorem twice:

$$\begin{aligned}2\pi \int_0^\rho d\rho'\, \rho'^2 \beta_1(\rho') &= \int_{\rho' \leq \rho} d^3x\, (-\rho' \hat{\phi}) \cdot \mathcal{G}(\gamma_0) \\ &= M\rho^2 \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, (I\boldsymbol{\sigma}_3) \cdot (-I\boldsymbol{\nabla}\beta) \\ &\quad - 2 \int_{\rho' \leq \rho} d^3x\, (I\boldsymbol{\sigma}_3) \cdot \big(\Omega(\gamma_0) - \gamma_0 \wedge (\partial_a \cdot \Omega(a))\big) \\ &= 2\pi M\rho^2 (\beta(0_-) - \beta(0_+)) + 2M \int_{\rho' \leq \rho} d^3x\, (I\boldsymbol{\sigma}_3) \cdot (\overrightarrow{\boldsymbol{\nabla}} \cdot (\alpha n) n) \\ &= -\frac{4\pi M\rho^2}{(L^2 - \rho^2)^{1/2}} + 4\pi M\rho \int_{-\infty}^{\infty} dz \frac{\alpha \cos v}{\cosh u} \\ &= -\frac{4\pi M\rho^2}{(L^2 - \rho^2)^{1/2}} + 8\pi M L W(\rho),\end{aligned} \tag{A.8}$$

with $W(\rho)$ as given by equation (4.48). This time, differentiating yields

$$\beta_1 = -\frac{2M\rho}{(L^2 - \rho^2)^{3/2}}. \tag{A.9}$$

Reassuringly, this term vanishes on the axis, as it must do for a valid axially-symmetric solution.

A.2 $\mathcal{G}(\gamma_3)$

For $\mathcal{G}(\gamma_3)$ we can write

$$\mathcal{G}(\gamma_3) = \delta(z)(\alpha_4\gamma_3 + \beta_4\hat{\phi} + \delta_4 e_\rho + \epsilon_4\gamma_0). \tag{A.10}$$

This time we find that

$$\begin{aligned}
\int_{\rho'\leq\rho} d^3x\, \mathcal{G}(\gamma_3) &= 2\pi \int_0^\rho d\rho'\, \rho'(\alpha_4(\rho')\gamma_3 + \epsilon_4(\rho')\gamma_0) \\
&= -2\pi M\rho\gamma_0 \int_{-\infty}^{\infty} dz\, \partial_\rho(\alpha\boldsymbol{\sigma}_3\cdot\boldsymbol{n}) \\
&\quad + M\rho\gamma_3 \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, (-I\boldsymbol{\sigma}_\phi)\cdot(-I\boldsymbol{\nabla}(\beta\boldsymbol{\sigma}_3\cdot\boldsymbol{n})) \\
&= -2\pi M\rho\gamma_0\, \partial_\rho \int_{-\infty}^{\infty} dz\, \alpha \sin v \\
&= 0.
\end{aligned} \tag{A.11}$$

The final term vanishes because $\alpha \sin v$ is an odd function of z. It follows that $\alpha_4 = \epsilon_4 = 0$. The same argument as at equation (A.7) shows that $\delta_4 = 0$, and for β_4 we construct

$$\begin{aligned}
2\pi \int_0^\rho d\rho'\, \rho'^2 \beta_4(\rho') &= \int_{\rho'\leq\rho} d^3x\, (-\rho'\hat{\phi})\cdot\mathcal{G}(\gamma_3) \\
&= M\rho^2 \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, (I\boldsymbol{\sigma}_3)\cdot(-I\boldsymbol{\nabla}(\beta\boldsymbol{\sigma}_3\cdot\boldsymbol{n})) \\
&\quad - 2\int_{\rho'\leq\rho} d^3x\, (I\boldsymbol{\sigma}_3)\cdot\big(\Omega(\gamma_3) - \gamma_3\wedge(\partial_a\cdot\Omega(a))\big) \\
&= -2M\rho \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, (I\boldsymbol{\sigma}_3)\cdot(-\alpha(I\boldsymbol{\sigma}_\phi)\cdot n\, n\big) \\
&= 4\pi M\rho \int_{-\infty}^{\infty} dz\, \frac{\alpha \sin v \cos v}{\cosh u} \\
&= 0,
\end{aligned} \tag{A.12}$$

where we have again used the fact that $\sin v$ is an odd function of z. We therefore have $\mathcal{G}(\gamma_3) = 0$, which is physically reasonable.

A.3 $\mathcal{G}(e_\rho)$, $\mathcal{G}(\hat{\phi})$

For these terms we write

$$\mathcal{G}(\hat{\phi}) = \delta(z)(\alpha_2\hat{\phi} + \beta_2\gamma_0 + \delta_2 e_\rho + \epsilon_2\gamma_3) \tag{A.13}$$

$$\mathcal{G}(e_\rho) = \delta(z)(\alpha_3 e_\rho + \beta_3\gamma_0 + \delta_3\hat{\phi} + \epsilon_3\gamma_3). \tag{A.14}$$

The calculations are now complicated by the fact that the vector arguments are functions of position. To get round this problem we must find equivalent

integrals in terms of the fixed γ_1 and γ_2 vectors. We first form

$$
\begin{aligned}
2\pi \int_0^\rho d\rho'\, \rho'^2 \epsilon_2(\rho') &= \int_{\rho'\le\rho} d^3x\, (-\rho'\gamma_3)\cdot\mathcal{G}(\hat\phi) \\
&= M\rho^2 \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, (-I\boldsymbol{\sigma}_\phi)\cdot\big(\Omega(\hat\phi) - \hat\phi\wedge(\partial_a\cdot\Omega(a))\big) \\
&\quad + \int_{\rho'\le\rho} d^3x\, (I\boldsymbol{\sigma}_1)\cdot\big(\Omega(\gamma_1) - \gamma_1\wedge(\partial_a\cdot\Omega(a))\big) \\
&\quad + \int_{\rho'\le\rho} d^3x\, (I\boldsymbol{\sigma}_2)\cdot\big(\Omega(\gamma_2) - \gamma_2\wedge(\partial_a\cdot\Omega(a))\big) \\
&= -2\pi M\rho^2 \int_{-\infty}^{\infty} dz\, \frac{\beta\,\mathrm{sinh}u\,\mathrm{cos}v}{\mathrm{cosh}u} + 2\pi M\rho \int_{-\infty}^{\infty} dz\, \frac{\alpha\,\mathrm{sin}v\,\mathrm{cos}v}{\mathrm{cosh}u} \\
&= 0
\end{aligned}
\tag{A.15}
$$

which shows that $\epsilon_2 = 0$. A similar calculation confirms that $\epsilon_3 = 0$.

If $\mathcal{G}(a)$ is symmetric then we expect to find $\beta_2 = -\beta_1$. This is confirmed by

$$
\begin{aligned}
2\pi \int_0^\rho d\rho'\, \rho'^2 \beta_2(\rho') &= \int_{\rho'\le\rho} d^3x\, \rho'\gamma_0\cdot\mathcal{G}(\hat\phi) \\
&= \rho^2 \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, \boldsymbol{e}_\rho\cdot\big(\Omega(\hat\phi) - \hat\phi\wedge(\partial_a\cdot\Omega(a))\big) \\
&\quad - \int_{\rho'\le\rho} d^3x\, \boldsymbol{\sigma}_1\cdot\big(\Omega(\gamma_2) - \gamma_2\wedge(\partial_a\cdot\Omega(a))\big) \\
&\quad - \int_{\rho'\le\rho} d^3x\, \boldsymbol{\sigma}_2\cdot\big(\Omega(\gamma_1) - \gamma_1\wedge(\partial_a\cdot\Omega(a))\big) \\
&= 2\pi M\rho^2\partial_\rho \int_{-\infty}^{\infty} dz\, \frac{\alpha\,\mathrm{cos}v}{\mathrm{cosh}u} - 2\pi M\rho \int_{-\infty}^{\infty} dz\, \frac{\alpha\,\mathrm{cos}v}{\mathrm{cosh}u} \\
&= 4\pi ML\rho^2\partial_\rho(W(\rho)/\rho) - 4\pi MLW(\rho) \\
&= -8\pi MLW(\rho) + \frac{4\pi M\rho^2}{(L^2-\rho^2)^{1/2}}.
\end{aligned}
\tag{A.16}
$$

A similar, though slightly more involved calculation confirms that $\beta_3 = 0$.

For the remaining four functions we need to consider various combinations of integrals. For example,

$$
\begin{aligned}
\pi \int_0^\rho d\rho'\, \rho'(\alpha_2(\rho') + \alpha_3(\rho')) &= \int_{\rho'\le\rho} d^3x\, \gamma^1\cdot\mathcal{G}(\gamma_1) \\
&= M\rho \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, \mathrm{sin}\phi(-I\boldsymbol{\sigma}_3)\cdot(-I\boldsymbol{\nabla}(\beta\boldsymbol{\sigma}_1\cdot\boldsymbol{n})) \\
&= \frac{\pi\rho^2 M}{L}(\beta(0_+) - \beta(0_-)) \\
&= \frac{2\pi M\rho^2}{L(L^2-\rho^2)^{1/2}},
\end{aligned}
\tag{A.17}
$$

from which we obtain

$$
\alpha_2 + \alpha_3 = \frac{4M}{L(L^2-\rho^2)^{1/2}} + \frac{2M\rho^2}{L(L^2-\rho^2)^{3/2}}. \tag{A.18}
$$

Similarly,

$$\pi \int_0^\rho d\rho'\, \rho'(\delta_2(\rho') - \delta_3(\rho')) = \int_{\rho' \le \rho} d^3x\, \gamma^1 \cdot \mathcal{G}(\gamma_2)$$
$$= M\rho \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, \sin\phi\, (-I\boldsymbol{\sigma}_3)\cdot(-I\boldsymbol{\nabla}(\beta\boldsymbol{\sigma}_2\cdot\boldsymbol{n}))$$
$$= 0 \tag{A.19}$$

(because $\sinh u = 0$ over the disk). This result confirms that $\mathcal{G}(a)$ is symmetric over the disk and therefore that there are no hidden sources of torsion.

The functions δ_2 and δ_3 are obtained from

$$\frac{\pi}{4} \int_0^\rho d\rho'\, \rho'^3(\delta_2(\rho') + \delta_3(\rho')) = \int_{\rho' \le \rho} d^3x\, \rho'^2 \sin\phi \cos\phi\, \gamma_1 \cdot \mathcal{G}(\gamma_1) \tag{A.20}$$

which evaluates to

$$M\rho^3 \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, \sin^2\phi \cos\phi (I\boldsymbol{\sigma}_3)\cdot(-I\boldsymbol{\nabla}(\beta\boldsymbol{\sigma}_1\cdot\boldsymbol{n}))$$
$$- \int_{\rho' \le \rho} d^3x\, \rho' \cos\phi\, (I\boldsymbol{\sigma}_3)\cdot\big(\Omega(\gamma_1) - \gamma_1 \wedge (\partial_a \cdot \Omega(a))\big)$$
$$= -M\rho^2 \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, \sin\phi \cos\phi\, \alpha (I\,\boldsymbol{\sigma}_3 \wedge \boldsymbol{n})^2 = 0, \tag{A.21}$$

which shows that $\delta_2 = \delta_3 = 0$. It follows that e_ρ is an eigenvector of the stress-energy tensor. A similar trick is used to evaluate the final term:

$$\frac{\pi}{4} \int_0^\rho d\rho'\, \rho'^3(\alpha_2(\rho') - \alpha_3(\rho')) = \int_{\rho' \le \rho} d^3x\, \rho'^2 \sin\phi \cos\phi\, \gamma_1 \cdot \mathcal{G}(\gamma_2). \tag{A.22}$$

This evaluates to

$$= M\rho^3 \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, \sin^2\phi \cos\phi (I\boldsymbol{\sigma}_3)\cdot(-I\boldsymbol{\nabla}(\beta\boldsymbol{\sigma}_2\cdot\boldsymbol{n}))$$
$$- \int_{\rho' \le \rho} d^3x\, \rho' \cos\phi (I\boldsymbol{\sigma}_3)\cdot\big(\Omega(\gamma_2) - \gamma_2 \wedge (\partial_b\, \cdot \Omega(b))\big)$$
$$= \frac{\pi M \rho^4}{2L(L^2 - \rho^2)^{1/2}} - 2\pi M \rho^2 W(\rho) + 4\pi M \int_0^\rho d\rho'\, \rho' W(\rho'), \tag{A.23}$$

where $W(\rho)$ is as defined in equation (4.48). We do not need to evaluate the final integral since we are only interested in the derivative of the right-hand side. This yields

$$\alpha_2 - \alpha_3 = \frac{2M\rho^2}{L(L^2 - \rho^2)^{3/2}} \tag{A.24}$$

which now gives us all of the terms in $\mathcal{G}(a)$.

References

[1] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt. *Exact Solutions of Einstein's Field Equations*. Cambridge University Press, 1980.

[2] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. Gravity, gauge theories and geometric algebra. *Phil. Trans. R. Soc. Lond. A*, 356:487–582, 1998.

[3] C.J.L Doran and A.N. Lasenby. *Geometric Algebra for Physicists*. Cambridge University Press, 2003.

[4] C.J.L. Doran and A.N. Lasenby. New techniques for analysing axisymmetric gravitational systems I. Vacuum fields. *Class. Quant. Gravity,*, 20(6):1077–1101, 2003.

[5] W. Isreal. Source of the Kerr metric. *Phys. Rev. D*, 2(1):641–646, 1970.

[6] R.P. Feynman, R.B. Leighton, and M. Sands. *The Feynman Lectures on Physics, Volume II*. Addison-Wesley, Reading MA, 1964.

[7] M.M. Schiffer, R.J. Adler, J. Mark, and C. Sheffield. Kerr geometry as complexified Schwarzschild geometry. *J. Math. Phys.*, 14(1):52, 1973.

[8] E.T. Newman and A.I. Janis. Note on the Kerr spinning-particle metric. *J. Math. Phys.*, 6(4):915, 1965.

[9] A.N. Lasenby, C.J.L. Doran, Y. Dabrowski, and A.D. Challinor. Rotating astrophysical systems and a gauge theory approach to gravity. In N. Sánchez and A. Zichichi, editors, *Current Topics in Astrofundamental Physics, Erice 1996*, page 380. World Scientific, Singapore, 1997.

[10] A. Krasinski. Ellipsoidal space-times, sources for the Kerr metric. *Ann. Phys.*, 112:22–40, 1978.

[11] J. Bicak and T. Ledvinka. Relativistic disks as sources of the Kerr metric. *Phys. Rev. Lett.*, 71(11):1669–1672, 1993.

[12] G. Neugebauer and R. Meinel. The Einsteinian gravitational field of the rigidly rotating disk of dust. *ApJ*, 414:L97, 1993.

[13] C. Pichon and D. Lynden-Bell. New sources for the Kerr and other metrics: Rotating relativistic discs with pressure support. *Mon. Not. R. Astron. Soc.*, 280:1007, 1996.

[14] V.H. Hamity. An "interior" of the Kerr metric. *Phys. Lett. A*, 56(2):77–78, 1976.

[15] D. Hestenes. *Space–Time Algebra*. Gordon and Breach, New York, 1966.

[16] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus*. Reidel, Dordrecht, 1984.

[17] C.J.L Doran, A.N. Lasenby, S.F. Gull, S.S. Somaroo, and A.D. Challinor. Spacetime algebra and electron physics. *Adv. Imag. & Elect. Phys.*, 95:271, 1996.

[18] S. Chandrasekhar. *The Mathematical Theory of Black Holes*. Oxford University Press, 1983.

[19] K.S. Virbhadra. Energy associated with a Kerr–Newman black hole. *Phys. Rev. D*, 41(4):1086, 1990.

[20] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. Astrophysical and cosmological consequences of a gauge theory of gravity. In N. Sánchez and A. Zichichi, editors, *Advances in Astrofundamental Physics, Erice 1994*, page 359. World Scientific, Singapore, 1995.

[21] C.J.L. Doran. *Geometric Algebra and its Application to Mathematical Physics*. PhD thesis, Cambridge University, 1994.

[22] R.P. Feynman, F.B. Morningo, and W.G. Wagner. *Feynman Lectures on Gravitation*. Addison–Wesley, Reading MA, 1995.

[23] K.P. Tod. Some examples of Penrose's quasi-local mass construction. *Proc. R. Soc. Lond. A*, 388:457, 1983.

[24] R. Penrose. Quasi-local mass and angular momentum in general relativity. *Proc. R. Soc. Lond. A*, 381:53, 1982.

[25] L.D. Landau and E.M. Lifshitz. *The Classical Theory of Fields. (Fourth Edition)*. Pergamon Press, 1975.

[26] F.J. Chinea and L.M. Gonzalez-Romero. A differential form approach for rotating perfect fluids in general relativity. *Class. Quantum Grav.*, 9:1271, 1992.

[27] C.W. Misner, K.S. Thorne, and J.A. Wheeler. *Gravitation*. W.H. Freeman and Company, San Francisco, 1973.

[28] W. Kinnersley. Field of an arbitrarily accelerating point mass. *Phys. Rev.*, 186(5):1335, 1969.

[29] W. B. Bonner. The photon rocket. *Class. Quantum Grav.*, 11:2007, 1994.

[30] C.J.L. Doran. New form of the Kerr solution. *Phys. Rev. D*, 61(6):067503, 2000.

[31] S.W. Hawking and G.F.R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge University Press, 1973.

[32] W.J. Kaufmann. *The Cosmic Frontiers of General Relativity*. Penguin Books, 1979.