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Fermion absorption cross section of a Schwarzschild black hole

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Abstract

We study the absorption of massive spin-half particles by a small Schwarzschild black hole by numerically solving the single-particle Dirac equation in Painlevé–Gullstrand coordinates. We calculate the absorption cross section $\sigma(E)$ for a range of gravitational couplings $Mm/{m_P}^2$ and incident particle energies E. At high couplings, where the Schwarzschild radius R_S is much greater than the wavelength λ, we find that $\sigma(E)$ approaches the classical result for a point particle. At intermediate couplings, $R_S \sim \lambda$, we find oscillations around the classical limit whose precise form depends on the particle mass. These oscillations give quantum violations of the equivalence principle. At high energies the cross section converges on the geometric-optics value of $27\pi R_S^2/4$, and at low energies we find agreement with an approximation derived by Unruh. When the hole is much smaller than the particle wavelength we confirm that the minimum possible cross section approaches $\pi R_S^2/2$.

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1 Introduction

It is widely accepted that general relativity and quantum mechanics are incompatible in their current form, yet a theory reconciling the two remains elusive. Despite this problem, it is only at the smallest length scales ($l < l_P$), or highest energies, that we expect substantial modification to existing theory. At low energies, away from spacetime singularities, the propagation of quantum fields on gravitational backgrounds is well understood (see the books by Birrell & Davies [1] or Chandrasekhar [2]).

Interest in the absorption of quantum waves by black holes was reignited in the 1970s, following Hawking's discovery that black holes can emit, as well as scatter and absorb, radiation [3]. Hawking showed that the evaporation rate is proportional to the total absorption cross section. More recently, absorption cross sections (or "grey body factors") have been of interest in the context of higher-dimensional string theories.

In a series of papers [4, 5, 6] Sanchez considered the scattering and absorption of massless scalar particles by an uncharged, spherically-symmetric (Schwarzschild) black hole. Using numerical methods she showed that the total absorption cross section (as a function of incident frequency) exhibits oscillations around the geometric-optics limit characteristic of diffraction patterns.

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Unruh [7] studied the absorption of massive spin-half particles by piecing together analytic solutions to the Dirac equation across three regions. He showed that, in the low-energy limit, the scattering cross section for the fermion is exactly 1/8 of that for the scalar particle. He also derived an approximation to the total cross section valid at low energies, which we revisit in section 5.

In this paper we return to the massive fermion absorption problem studied by Unruh. We employ a different coordinate system, but retain equivalent ingoing boundary conditions at the horizon. Instead of using analytical approximations we numerically integrate the Dirac equation to calculate the absorption cross section over a range of energies and gravitational couplings. We compare our results with the classical cross section for a massive particle, and with Unruh's low-energy approximation.

The natural dimensionless parameter to describe the strength of the gravitational coupling between a black hole (of mass M) and a quantum particle (of mass m) is given by

$$\alpha = \frac{GMm}{\hbar c} = \frac{Mm}{m_P^2} = \frac{\pi R_S}{\lambda_C} \tag{1}$$

where R_S is the Schwarzschild radius of the hole, λ_C is the Compton wavelength of the quantum particle, and m_p is the Planck mass. We use the symbol α because it has an analogous role in gravitation to the fine-structure constant in electromagnetism. We expect quantum effects to be important when $\alpha \sim 1$, whereas in the high-α limit classical effects should dominate.

In first-quantised theory the capture of light and matter by a black hole is a one-way process. The direction of time implied by this process is not revealed in Schwarzschild coordinates, however, as these are manifestly time-reverse symmetric and are invalid at the horizon. Time-asymmetric coordinates, such as Eddington–Finkelstein coordinates, allow the continuation of the metric across the horizon and allow us to correctly study the properties of wavefunctions [8, 9, 10]. We then find that ingoing states correspond precisely to those that are regular at the horizon.

Here, rather than using Eddington–Finkelstein coordinates, we prefer to work with the coordinates first introduced by Painlevé [11] and Gullstrand [12]. In these coordinates the metric becomes

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \sqrt{\frac{8M}{r}} dt\, dr - dr^2 - r^2 d\Omega^2 \tag{2}$$

The utility of this form of the Schwarzschild solution has recently been highlighted by Martel & Poisson [13] and others [9]. For black holes (as opposed to white holes) the negative sign for the crossterm $dt\, dr$ is the correct choice, as this guarantees that all particles fall in across the horizon in a finite proper time. This sign is also uniquely picked out by models in which the black hole is formed by a collapse process [8]. One advantage of this system is that the time coordinate has a natural interpretation as the proper time measured by an observer in freefall starting from rest at infinity.

According to general relativity the classical absorption cross section of a Schwarzschild black hole is given by

$$\sigma_{\text{abs}} = \frac{\pi M^2}{2u^4} \left(8u^4 + 20u^2 - 1 + (1 + 8u^2)^{3/2}\right) \tag{3}$$

where u is the velocity of the particle [7]. In accordance with the equivalence principle, the cross section is independent of the particle mass m. We expect that the quantum cross section will approach this value in the limit $\alpha \gg 1$ (that is, $R_S \gg \lambda$).

We start with the radial separation of the Dirac equation in Painlevé–Gullstrand coordinates. We then study the properties of solutions around the horizon, identifying the physical branch of regular solutiuons. For unbound states $E > mc^2$ we find that the physical solutions are composed of ingoing and outgoing waves at infinity. By numerically finding the ratios of these waves in any given angular mode we are able to compute the absorption spectrum. We use natural coordinates $G = \hbar = c = 1$, except in cases where inclusion of the factors adds clarity.

2 The Dirac equation

We let $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ denote the gamma matrices in the Dirac–Pauli representation, and introduce spherical polar coordinates (r, θ, ϕ). From these we define the unit polar matrices

$$\begin{aligned}
\gamma_r &= \sin\theta(\cos\phi\,\gamma_1 + \sin\phi\,\gamma_2) + \cos\theta\,\gamma_3 \\
\gamma_\theta &= \cos\theta(\cos\phi\,\gamma_1 + \sin\phi\,\gamma_2) - \sin\theta\,\gamma_3 \\
\gamma_\phi &= -\sin\phi\,\gamma_1 + \cos\phi\,\gamma_2.
\end{aligned} \tag{4}$$

In terms of these we define four position-dependent matrices $\{g_t, g_r, g_\theta, g_\phi\}$ by

$$\begin{aligned}
g_t &= \gamma_0 + \sqrt{\frac{2M}{r}}\gamma_r & g_\theta &= r\gamma_\theta \\
g_r &= \gamma_r & g_\phi &= r\sin\theta\gamma_\phi.
\end{aligned} \tag{5}$$

These satisfy the anti-commutation relations

$$\{g_\mu, g_\nu\} = 2g_{\mu\nu}I$$

where $g_{\mu\nu}$ is the Painlevé–Gullstrand metric of equation (2). The reciprocal matrices $\{g^t, g^r, g^\theta, g^\phi\}$ are defined by the equation

$$\{g^\mu, g_\nu\} = 2\delta^\mu_\nu I, \tag{6}$$

and both sets are well-defined everywhere except at the origin.

The Dirac equation for a spin-half particle of mass m is

$$ig^\mu\nabla_\mu\psi = m\psi, \tag{7}$$

where

$$\nabla_\mu\psi = (\partial_\mu + \frac{i}{2}\Gamma^{\alpha\beta}_\mu\Sigma_{\alpha\beta})\psi, \qquad \Sigma_{\alpha\beta} = \frac{i}{4}[\gamma_\alpha, \gamma_\beta]. \tag{8}$$

The components of the spin connection are found in the standard way [14] and are particularly simple in the Painlevé–Gullstrand gauge [10],

$$g^\mu\frac{i}{2}\Gamma^{\alpha\beta}_\mu\Sigma_{\alpha\beta} = -\frac{3}{4r}\sqrt{\frac{2M}{r}}\gamma_0. \tag{9}$$

An advantage of our choice of metric is that the Dirac equation can now be written in a manifestly Hamiltonian form

$$i\partial\!\!\!/\psi - i\gamma^0 \left(\frac{2M}{r}\right)^{1/2} \left(\frac{\partial}{\partial r} + \frac{3}{4r}\right) \psi = m\psi, \tag{10}$$

where $\partial\!\!\!/$ is the Dirac operator in flat Minkowski spacetime. The interaction term is non-Hermitian, as the singularity acts as a sink for probability density, making absorption possible.

The Dirac equation is clearly seperable in time, so has solutions that go as $\exp(-iEt)$. The energy E conjugate to time-translation is independent of the chosen coordinate system, and has a physical definition in terms of the Killing time [10]. We can further exploit the spherical symmetry to seperate the spinor into

$$\psi = \frac{e^{-iEt}}{r} \begin{pmatrix} u_1(r)\chi^\mu_\kappa(\theta,\phi) \\ u_2(r)\sigma_r\chi^\mu_\kappa(\theta,\phi) \end{pmatrix} \tag{11}$$

where

$$\sigma_r = \sin\theta(\cos\phi\,\sigma_1 + \sin\phi\,\sigma_2) + \cos\theta\,\sigma_3. \tag{12}$$

The angular eigenmodes are labeled by κ, which is a positive or negative nonzero integer, and μ, which is the total angular momentum in the $\theta = 0$ direction. Our convention for these eigenmodes is that

$$(\boldsymbol{\sigma}\cdot\boldsymbol{L} + \hbar)\chi^\mu_\kappa = \kappa\hbar\chi^\mu_\kappa, \quad \kappa = \pm(j+1/2) = \ldots, -2, -1, 1, 2, \ldots. \tag{13}$$

The positive and negative κ modes are related by

$$\sigma_r\chi^\mu_\kappa = \chi^\mu_{-\kappa} \tag{14}$$

and are normalised so that

$$\int d\phi \int d\theta \sin\theta\ \chi^\mu_\kappa(\theta,\phi)^\dagger\ \chi^{\mu'}_{\kappa'}(\theta,\phi) = \delta_{\kappa\kappa'}\delta_{\mu\mu'}. \tag{15}$$

The trial function (11) results in a pair of coupled first-order equations

$$(1-2M/r)\frac{d}{dr}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 2M/r \\ 2M/r & 1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} \kappa/r & i(E+m) - (2M/r)^{1/2}(4r)^{-1} \\ i(E-m) - (2M/r)^{1/2}(4r)^{-1} & -\kappa/r \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{16}$$

The equations have regular singular points at the origin and horizon, as well as an irregular singular point at $r = \infty$. As far as we are aware, the special function theory required to deal with such equations has not been developed. Instead we use series solutions around the singular points as initial data for a numerical integration scheme.

3 Series Solutions and Boundary Conditions

As is clear from (16), there is a regular singular point in the coupled equations at the horizon, $r = 2M$. We look for series solutions

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (r-2M)^s \sum_{k=0} \begin{pmatrix} a_k \\ b_k \end{pmatrix} (r-2M)^k \tag{17}$$

where s is the lowest power in the series, and a_k, b_k as coefficients to be determined. On substituting into (16) and setting $r = 2M$ we obtain an eigenvalue equation for s, which has solutions

$$s = 0 \qquad \text{and} \qquad s = -\tfrac{1}{2} + 4iME. \tag{18}$$

The regular root $s = 0$ ensures that we can construct solutions which are finite and continuous at the horizon. We will see later that regular solutions automatically have an ingoing current at the horizon. The singular branch gives rise to discontinuous, unnormalisable solutions with an outgoing current at the horizon [8]. We therefore restrict attention to the regular, physically-admissable solutions. The eigenvector for the regular solution has

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} \kappa - 2iM(E+m) + 1/4 \\ \kappa + 2iM(E-m) - 1/4 \end{pmatrix}. \tag{19}$$

In order to expand about infinity we need to take care of the irregular singularity present there. There are two sets of solutions, $U^{(\text{out})}$ and $U^{(\text{in})}$, which asymptotically resemble outgoing and ingoing radial waves with additional radially-dependent phase factors. To lowest order,

$$\begin{aligned} U^{(\text{out})} &= e^{ipr} e^{i(\phi_1+\phi_2)} \begin{pmatrix} 1 \\ p/(E+m) \end{pmatrix} \\ U^{(\text{in})} &= e^{-ipr} e^{i(\phi_1-\phi_2)} \begin{pmatrix} 1 \\ -p/(E+m) \end{pmatrix} \end{aligned} \tag{20}$$

where the phase factors $\phi_1(r), \phi_2(r)$ are given by

$$\phi_1(r) = E\sqrt{8Mr}, \qquad \phi_2(r) = \frac{M}{p}\left(m^2 + 2p^2\right)\ln(pr) \tag{21}$$

and the momentum p is defined in the usual way, $p^2 = E^2 - m^2$. The general (regular) solution as $r \to \infty$ is a superposition of the ingoing and outgoing waves,

$$U(r \to \infty) = \alpha_\kappa U^{(\text{in})} + \beta_\kappa U^{(\text{out})} \tag{22}$$

for each angular mode. The magnitudes of α_κ and β_κ determine the amount of scattered and absorbed radiation present.

4 Absorption

The spatial probability current is conserved for states with real energy, $E > m$. For each angular eigenmode we obtain a conserved Wronskian W_κ

$$W_\kappa = (u_1 u_2^\dagger + u_1^\dagger u_2) - \sqrt{2M/r}(u_1 u_1^\dagger + u_2 u_2^\dagger) \tag{23}$$

which measures the total outward flux over a surface of radius r. At the horizon

$$W_\kappa = -\left|u_1 - u_2\right|^2 \propto -\left|a_0 - b_0\right|^2 \tag{24}$$

so the flux is inwards for all regular solutions. On substituting the asymptotic forms of equations (20) and (22) into equation (23) we find an expression for the Wronskian in the large-r limit,

$$W_\kappa = -\frac{2p}{E+m}\left(\left|\alpha_\kappa\right|^2 - \left|\beta_\kappa\right|^2\right) \tag{25}$$

The coefficients α_κ and β_κ can be determined (up to an overall magnitude and phase) by matching the ingoing (regular) solution at the horizon to the asymptotic form in the large-r limit. We choose the normalisation of each angular mode so that $W_\kappa = -1$, and write the most general solution to the wave equation as

$$\psi = \sum_{k\neq 0} g_\kappa \psi_\kappa \tag{26}$$

where ψ_κ are spinors of the trial form with u_1, u_2 as given by (20), (22), and g_κ are complex coefficients. The total absorbed flux is then just

$$W_{tot} = \sum_{\kappa\neq 0} |g_\kappa|^2 . \tag{27}$$

We now employ a partial wave analysis to derive a simple expression for the absorbed cross section. We write the asymptotic behaviour of ψ as the sum of a plane wave (propagating in the $\theta = 0$ direction) and an outgoing scattered wave,

$$\psi = e^{ipr\cos\theta}\Psi_1 + \frac{f(\theta)}{r} e^{ipr}\Psi_2 \tag{28}$$

where Ψ_1, Ψ_2 are constant spinors. The plane wave can be decomposed into ingoing and outgoing radial waves in the large-r limit. We equate the ingoing part of the plane wave with the ingoing part of the asymptotic wave (22). Normalising the plane wave to $2E$ particles per unit volume we find

$$g_\kappa \alpha_\kappa e^{i(\phi_1-\phi_2)} = i(-1)^{\kappa+1} \frac{\sqrt{4\pi(E+m)}}{2p} \frac{\kappa}{\sqrt{|\kappa|}} \tag{29}$$

for each angular mode. The total absorption cross section $\sigma_{\rm abs}$ is the ratio of the ingoing flux (27) to the incident flux of the plane wave ($2p$),

$$\sigma_{\rm abs} = \frac{\pi}{2p(E-m)} \sum_{\kappa\neq 0} \frac{|\kappa|}{|\alpha_\kappa|^2}. \tag{30}$$

At low energies, the $|\kappa| = 1$ states dominate the absorption, but at higher energies we need to sum over a range of κ.

5 Results

We determine the coefficients α_κ required for (30) by matching the ingoing solution at the horizon to the asymptotic form (22) at infinity. In a similar calculation, Unruh [7] used analytic approximations to the radial functions to find the leading contributions to the cross section. Here, we use numerical integration of the wavefunction out from the horizon to match the solutions, and compare our results with the analytic approach.

Figure 1 compares the result of our matching calculation at $\alpha = 0.2$ with the classical cross section of a point particle (3) over a range of energies. We see that the quantum absorption cross section oscillates around the classical value, as found by Sanchez [4, 5] for the massless scalar wave. For a given α we find the period of these oscillations is approximately constant, but the amplitude

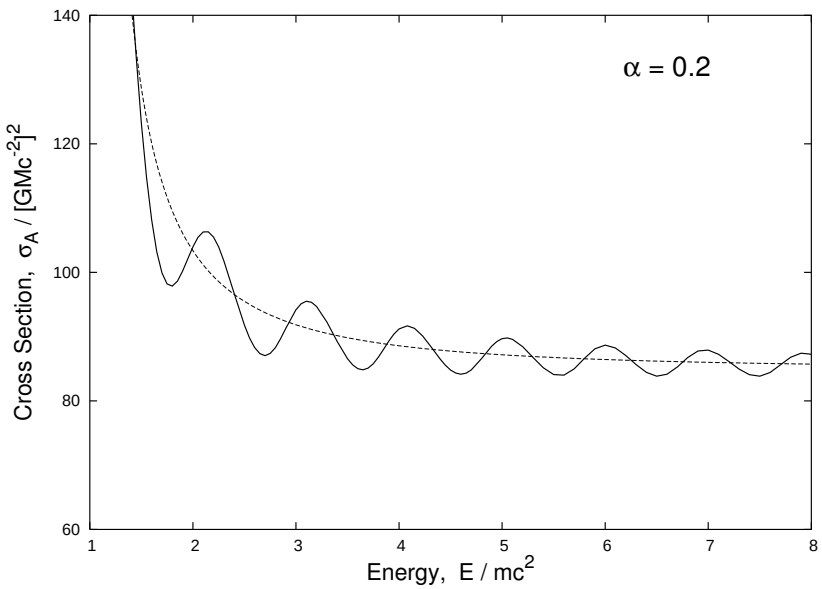

Figure 1: *Classical and quantum absorption cross section.* The plot compares the absorption cross section for the Dirac wave [solid] with the classical prediction for a point particle [dotted], for a gravitational coupling of $\alpha = Mm/{m_p}^2 = 0.2$. The cross section is plotted in units of $(GM/c^2)^2$ (proportional to the event horizon area), and the energy in units of the rest mass energy mc^2.

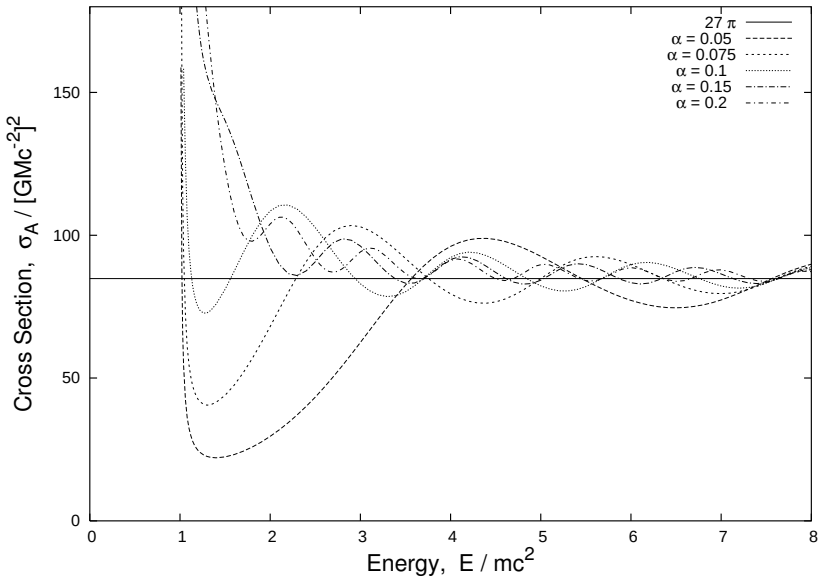

Figure 2: *Quantum absorption cross section for a range of couplings.* The plot shows the cross section at a range of gravitational couplings, $0.05 \leq Mm/{m_p}^2 \leq 0.2$. The horizontal line is the photon limit.

decays as $E \to \infty$. As α is increased we find that the magnitude and period of the oscillations decreases. In the $\alpha \gg 1$ limit we recover the classical cross section.

Figure 2 illustrates that the precise form of the oscillation depends on α, and therefore on the mass of the quantum particle. This represents a quantum-mechanical violation of the equivalence principle. At sufficiently high energies, we see that all cross sections tend to the photon limit of $\sigma_{\text{abs}} = 27\pi(GMc^{-2})^2$. As noted by Unruh, all particles travelling close enough to light speed, $u \approx 1$, see a black hole of roughly the same size, regardless of mass or spin.

Unruh also showed that in the low-energy limit, the Dirac cross section is 1/8 of the scalar cross section, and absorption is dominated by the lowest angular momentum modes, $|\kappa| = 1$. In this limit he showed

$$\frac{\sigma_{\text{abs}}}{(GMc^{-2})^2} \approx \frac{4\pi^2(1+u^2)\alpha}{u^2(1-u^2)^{1/2}\left\{1-\exp(-2\pi\alpha(1+u^2)/u(1-u^2)^{1/2})\right\}}. \tag{31}$$

This proves to be an excellent fit to the numerical cross sections in the low-energy regime, such as those shown in figure 3. A minimum-possible cross section can be found by considering the $\alpha \sim 0$ limit of equation (31), which reduces to

$$\frac{\sigma_{\text{abs}}}{(GMc^{-2})^2} \approx \frac{2\pi}{u}. \tag{32}$$

The minimum value of this occurs in the $u = 1$ limit, and figure 3 confirms that the minimum cross section approaches 2π at low couplings.

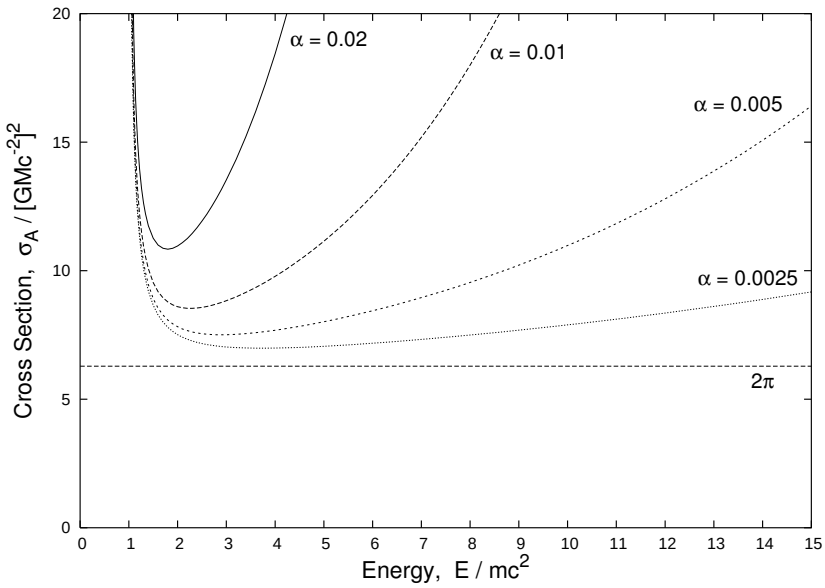

Figure 3: *Quantum absorption cross sections in the $\lambda_C \gg r_S$ limit.* The plot shows the absorption cross section as a function of energy, for small couplings, $\alpha = Mm/{m_p}^2 \ll 1$. In the low-energy region plotted here, the wavelength of the Dirac particle is large compared to the black hole event horizon. Absorption is dominated by the lowest $j = 1/2$ angular momentum states ($\kappa = 1, -1$ states), and the low-energy approximation of Unruh [7] is valid (see text). As $\alpha \to 0$, the minimum of the cross section approaches 2π (dotted line). In the high energy limit, all cross sections return to the photon limit, 27π.

6 Discussion

We have shown that the absorption cross section for a Dirac wave in a classical Schwarzschild background can be calculated by matching ingoing solutions at the horizon to appropriate asymptotic forms at infinity. The analysis proved particularly simple in the Painlevé–Gullstrand metric, though we stress that the cross sections calculated here do not depend on the particular choice of gauge. Furthermore, antiparticle solutions are be generated from particle solutions by the transformation

$$(u_1, u_2, E, \kappa) \mapsto (u_1^*, u_2^*, -E^*, -\kappa). \tag{33}$$

It follows that the absorption cross section is invariant under charge conjugation.

In the large-α limit ($R_S \gg \lambda$) we find that the cross section approaches the classical prediction of equation (3). When $\alpha \sim 1$ ($R_S \sim \lambda$) we observe energy-dependent oscillations about the classical value in the σ-vs-E plot (figure 1). Oscillations of this nature were previously found by Sanchez for the massless scalar wave. The form of the oscillations depends on m, so represents a quantum-mechanical violation of the equivalence principle (figure 2).

In the low-energy regime, the $j = 1/2$ cross section (31) given by Unruh is an excellent fit to our numerical results. In the $\alpha \to 0$ ($R_S \ll \lambda$) limit we see that the minimum-possible cross section approaches $2\pi(GMc^{-2})^2$. In the high-energy limit, all cross sections eventually converge on the photon limit of $27\pi(GMc^{-2})^2$.

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