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Integral Constraints on Cosmological Perturbations and their Energy

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Abstract. We show the relation between Traschen's integral equations and the energy, and "position of the centre of mass", of the matter perturbations in a Robertson-Walker spacetime. When perturbations are "localised" we get a set of integral constraints that includes hers. We illustrate them on a simple example.

1. Introduction

One "puzzle" in the theory of cosmological perturbations [1] is Traschen's "integral constraints" [2] (see also [3]) : besides the six standard Robertson-Walker Killing vectors, she extracted from Einstein's linearised equations four other vectors, that she called "integral constraint vectors". Each of those vectors yields an equation for the matter perturbations, relating a volume to a surface integral. The equations become constraints when perturbations are "localised", for which the surface integrals are zero. Those constraints have been widely used [4]–[7].

A first question we may ask is, *are there more than four such vectors ?* We will see that the answer is "yes", but that her vectors are particularly useful, especially when perturbations are localised. Indeed the constraints she obtains involve the matter variables only. However, other, simple, constraints on the *geometry* exist as well, as we shall see in Section 2.

Second, several authors [2]–[4], [6] have interpreted Traschen's equations as a generalisation of conservation laws for energy and momentum in cosmology. Such quantities however are not straightforward to define in general relativity. When Killing vector fields or an asymptotic Killing vector fields exist, then of course we can write integral quantities for the energy, momentum, angular momentum etc... But Traschen's four "integral constraint vectors" are not Robertson-Walker Killing vectors. Thus, are we allowed to interpret the conservation laws they imply as defining "energy" and "momentum" ?

A proper definition of conserved quantities such as energy, momentum etc, involves the introduction of a background spacetime [8] and hence depends a priori on the choice for the background, as well as on the way points of the physical spacetime and of the background are identified, i.e. on the mapping (see e.g. [9] and references therein). Applying this formalism to perturbed Robertson-Walker spacetimes (Section 3) we will first see how the choice of de Sitter spacetime as background is almost compulsory. Using its ten Killing vectors, we will write ten Noether conservation laws, that is ten equations relating volume to surface integrals. They will define, besides the known momentum and angular momentum, an energy, δE, and a "position of the centre of mass", $\delta\vec{Z}$, of the perturbations of the physical, perturbed Robertson-Walker spacetime. All will depend on the constant $\bar{R}$ defining the de Sitter background and on

the mapping. We shall thus see that Traschen's integrals are *not* conserved quantities. However, when the perturbations are localised, Traschen's constraints are equivalent to $\delta E = 0$ and $\delta \vec{Z} = 0$, independently of any mapping.

The comparison between Traschen's integrals and the conserved quantities is instructive in that it suggests to raise to a special status a particular mapping in which Traschen's integral constraint vectors become proportional to de Sitter Killing vectors (see [10] for the mathematical origin of this property). This is done in Section 4 where energy etc are expressed in that mapping, in a way where all explicit reference to the background has disappeared.

Finally, in Section 5, we dwell on what is meant by "localised" perturbations by looking at the simple case of spherically symmetric perturbations. We shall see that imposing the constraints amounts to imposing that not only the matter perturbations, but also the metric perturbations, be localised in space. Hence spacetime outside the perturbed region is strictly Robertson-Walker and the constraints can, as already shown in [1] on a Swiss cheese model, be interpreted as "fitting conditions" of the perturbed spacetime to a Robertson-Walker universe. That also shows that the constraints hold only for perturbations which are produced at some instant t in a finite region of space and then propagate in a up to then perfectly isotropic and homogeneous universe. The origin of such perturbations cannot be described by Einstein's equations : they must arise from local processes like "explosive" events or phase transitions producing bubbles of true vacuum, cosmic strings or other topological defects. And, indeed, it is in those contexts that Traschen's constraints have been used [2], [5]–[7].

With this paper we hope to throw some light on the meaning, and range of application, of integral constraints in cosmology. We will also clarify the issue of defining energy, momentum etc in spacetimes which are not asymptotically flat, in particular as regards the role of background spacetimes and mappings in cosmology.

2. Traschen's Vectors and Integral Constraints on Cosmological Perturbations

Traschen [4] has shown the existence of some vectors V^μ in Robertson-Walker

universes, which enter integral equations for arbitrary perturbations. In this Section, we find them in a simple way and recall what they are useful for. Let perturbed Robertson-Walker universes be described in coordinates $x^\mu = (x^0 \equiv t, x^k)$, $(\mu, \nu ... = 0, 1, 2, 3$; $i, j, ... = 1, 2, 3)$, such that the metric reads

$$ds^2 = dt^2 - a^2(t)(f_{ij} + h_{ij})dx^i dx^j \tag{1}$$

f_{ij} is the metric of a 3-sphere, plan or hyperboloid depending on whether the index $k = (+1, 0, -1)$ [11]

$$f_{ij} = \delta_{ij} + k\frac{\delta_{im}\delta_{jn}x^m x^n}{1 - kr^2} \qquad \text{with} \qquad r^2 \equiv \delta_{ij}x^i x^j \tag{2}$$

the scale factor $a(t)$ is determined by Friedmann's equation and $h_{ij}(x^\mu)$ is a small perturbation of f_{ij}. We choose to work in a synchronous gauge ($h_{00} = h_{i0} = 0$) merely to simplify calculations (we shall present gauge invariant calculations elsewhere).

If δT^μ_ν is the perturbation of the stress-energy tensor, the linearised Einstein constraint equations read [12]

$$\begin{cases} \delta G^0_0 \equiv \frac{1}{2a^2}\left(\nabla_m \nabla_n \tilde{h}^{mn} + k\tilde{h}\right) - \frac{\dot{a}}{2a}\dot{\tilde{h}} = \kappa \delta T^0_0 \\ \delta G^0_k \equiv \frac{1}{2}\nabla_l \dot{\tilde{h}}^l_k = \kappa \delta T^0_k \end{cases} \tag{3}$$

κ is Einstein's constant, all indices are raised with the metric f^{ij}, ∇ denotes the covariant derivative with respect to f_{ij}, a dot denotes time derivative and we have introduced the notation

$$\tilde{h}_{ij} \equiv h_{ij} - f_{ij}h \quad \text{with} \quad h \equiv f^{ij}h_{ij} = -\frac{1}{2}\tilde{h} \tag{4}$$

Let us now write equations (3) under an integral form

$$\frac{1}{\kappa}\int_\Sigma \delta G^0_\mu \hat{\zeta}^\mu d^3x = \int_\Sigma \delta T^0_\mu \hat{\zeta}^\mu d^3x \qquad , \qquad \frac{1}{\kappa}\int_\Sigma \delta G^0_k \hat{\zeta}^k d^3x = \int_\Sigma \delta T^0_k \hat{\zeta}^k d^3x \tag{5}$$

ζ^ν being an arbitrary vector field ; a hat denotes multiplication by $\sqrt{-g} = a^3\sqrt{f} = a^3/\sqrt{1 - kr^2}$ (at zeroth order) ; Σ is a volume in the hypersurface $t = Const$, and $d^3x \equiv dx^1 dx^2 dx^3$. If we perform the appropriate integrations by part to extract surface terms, equations (3-5) read

$$\int_\Sigma \sqrt{-g}\left[\delta T^0_\mu \zeta^\mu + \left(\nabla^{(l}\zeta^{k)} + Hf^{lk}\zeta^0\right)\frac{\dot{\tilde{h}}_{lk}}{2\kappa} - \left(\nabla^{lk}\zeta^0 + kf^{lk}\zeta^0\right)\frac{\tilde{h}_{lk}}{2\kappa a^2}\right] d^3x = \int_{\partial\Sigma} \hat{B}^l(\zeta) dS_l \tag{6}$$

and

$$\int_\Sigma \sqrt{-g}\left(\delta T^0_k \zeta^k + \frac{1}{2\kappa}\dot{\tilde{h}}^l_k \nabla_l \zeta^k\right) d^3x = \frac{1}{\kappa}\int_{\partial\Sigma}\sqrt{-g}\dot{\tilde{h}}^l_k \zeta^k dS_l \tag{7}$$

H is the Hubble parameter $H \equiv \dot{a}/a$, $\partial\Sigma$ is the boundary of the volume Σ, $dS_k \equiv \epsilon_{klm} dx^{[l} dx^{m]}$, parentheses mean symmetrisation, brackets antisymmetrisation, and

$$B^l(\zeta) \equiv \frac{1}{2\kappa}\left[\frac{1}{a^2}\left(\zeta^0 \nabla_k \tilde{h}^{kl} - \tilde{h}^{ml}\nabla_m \zeta^0\right) + \zeta^k \dot{\tilde{h}}^l_k\right] \tag{8}$$

If h_{ij} and $\partial_\rho h_{ij}$ vanish on the boundary $\partial\Sigma$, the surface terms in (6-7) disappear, in which case equations (6-7) become constraints (one for each vector ζ^μ) on the matter perturbations δT^0_μ.

Equations (6-7) are identically satisfied for all vector ζ^μ, if we take for the perturbations a solution of the Einstein equations. They simply relate a solution and its boundary conditions. Now, if one is looking for solutions satisfying some particular boundary conditions (like localised perturbations), then they *constrain* the set of solutions and can give some of their properties. Since ζ^μ is a priori arbitrary, there exists as many integral equations and constraints as independent vector fields, that is an infinite number.

However there are not so many vector fields which can be considered as useful. Indeed, for an arbitrary ζ^μ, one needs the full metric h_{kl} and hence one must solve the full Einstein equations to compute the integrals. Trashen's vectors $\zeta^\mu = V^\mu$ [4] are such that

$$\nabla^{(l}V^{k)} + Hf^{lk}V^0 = 0 \qquad , \qquad \nabla^{lk}V^0 + kf^{lk}V^0 = 0 \tag{9}$$

so that the coefficients of $\tilde{h}_{kl}$ and $\dot{\tilde{h}}_{lk}$ in (6) separately vanish. Traschen's vectors therefore enable to decouple the perturbations of the matter andthose of the geometry and give informations on the matter perturbations (density, pressure...) alone, without having to solve the full Einstein equations. Indeed equation (6) then becomes

$$\delta P_{Tr}(V) \equiv \int_\Sigma \delta T^0_\mu \hat{V}^\mu \, d^3x = \int_{\partial\Sigma} \hat{B}^l(V)\, dS_l \tag{10}$$

which are the ten Traschen's integral equations [4]. As for Equation (7), it becomes

$$\int_\Sigma \sqrt{-g}\left(\delta T^0_k V^k - \frac{H}{2\kappa}\dot{\tilde{h}} V^0\right) d^3x = \frac{1}{2\kappa}\int_{\partial\Sigma}\sqrt{-g}\dot{\tilde{h}}^l_k V^k dS_l \tag{11}$$

Traschen's vectors V^μ are ten linearly independent, particular, solutions of equations (9) and any solution of (9) is a linear combination of the V^μ with time dependent coefficients. The explicit expressions of the V^μ are given in Appendix 1. They split into two families. The first one contains the six Robertson-Walker Killing vectors of spatial translations, $V^\mu = P^\mu$, and spatial rotations, $V^\mu = R^\mu$ (see equations (A1-A2) for their explicit expression). For those six vectors Traschen's equations (10) as well as equations (11) are equivalent to

$$\begin{cases} \int_\Sigma \delta T_i^0 d^3x = \frac{1}{2\kappa}\int_{\partial\Sigma} \dot{\tilde{h}}_i^l dS_l & \text{for } k=0 \\ \int_\Sigma \delta T_i^0 \frac{(\delta^{ir}x^s - \delta^{is}x^r)}{\sqrt{1-kr^2}} d^3x = \frac{1}{2\kappa}\int_{\partial\Sigma} \dot{\tilde{h}}_i^l \frac{(\delta^{ir}x^s - \delta^{is}x^r)}{\sqrt{1-kr^2}} dS_l & \text{for } k\neq 0 \end{cases} \tag{12}$$

The second family of vectors contains four vectors, one T^μ and three K^μ, that Traschen called "integral constraint vectors" or ICVs : see equations (A6) and (A7) in Appendix 1 for their explicit expressions. Note that they are *not* Robertson-Walker conformal Killing vectors. For those four vectors Traschen's equations (10) read

$$\delta P_{Tr}(T) \equiv a^3 \int_\Sigma \left(\delta\rho - H\delta T_l^0 x^l\right) d^3x = \int_{\partial\Sigma} \hat{B}^l(T) dS_l \tag{13}$$

$$\begin{cases} \delta P_{Tr}^i(K) \equiv a^3 \int_\Sigma \left[x^i\delta\rho + H\delta T_l^0\left(k\delta^{li} - x^l x^i\right)\right] \frac{d^3x}{\sqrt{1-kr^2}} = \int_{\partial\Sigma} \hat{B}^{li}(K) dS_l & \text{for } k\neq 0 \\ \delta P_{Tr}^i(K) \equiv a^3 \int_\Sigma \left[x^i\delta\rho + H\delta T_l^0\left(\frac{1}{2}\delta^{li}r^2 - x^l x^i\right)\right] d^3x = \int_{\partial\Sigma} \hat{B}^{li}(K) dS_l & \text{for } k=0 \end{cases} \tag{14}$$

As for Equations (11) they become

$$\int_\Sigma \left(\delta T_l^0 x^l + \frac{1}{2\kappa}\dot{\tilde{h}}\right) d^3x = \frac{1}{2\kappa}\int_{\partial\Sigma} \dot{\tilde{h}}_k^l x^k dS_l \tag{15}$$

$$\begin{cases} \int_\Sigma \left[\delta T_l^0\left(k\delta^{li} - x^l x^i\right) - \frac{1}{2\kappa}\dot{\tilde{h}} x^i\right] \frac{d^3x}{\sqrt{1-kr^2}} = \frac{1}{2\kappa H}\int_{\partial\Sigma} \dot{\tilde{h}}_k^l K^{ki} \frac{dS_l}{\sqrt{1-kr^2}} & \text{for } k\neq 0 \\ \int_\Sigma \left[\delta T_l^0\left(\frac{1}{2}\delta^{li}r^2 - x^l x^i\right) - \frac{1}{2\kappa}\dot{\tilde{h}} x^i\right] d^3x = \frac{1}{2\kappa H}\int_{\partial\Sigma} \dot{\tilde{h}}_k^l K^{ki} dS_l & \text{for } k=0 \end{cases} \tag{16}$$

Since T^μ and K^μ are not Robertson-Walker Killing vectors the interpretation of $\delta P_{Tr}(T)$ and $\delta P_{Tr}^i(K)$ is not straightforward.

Traschen considered perturbations that are "localised", for which the surface integrals vanish. Equations (12-14) then become constraints which read

$$\int_\Sigma \delta T_i^0 d^3x = 0 \quad , \quad \int_\Sigma \delta T_i^0 \frac{(\delta^{ir}x^s - \delta^{is}x^r)}{\sqrt{1-kr^2}} d^3x = 0 \tag{17}$$

$$\int_\Sigma \left(\delta\rho - H\delta T_l^0 x^l\right) d^3x = 0 \tag{18}$$

$$\begin{cases} \int_\Sigma \left[x^i\delta\rho + H\delta T^0_l \left(k\delta^{li} - x^l x^i\right)\right] \frac{d^3x}{\sqrt{1-kr^2}} = 0 & \text{for } k \neq 0 \\ \int_\Sigma \left[x^i\delta\rho + H\delta T^0_l \left(\frac{1}{2}\delta^{li} r^2 - x^l x^i\right)\right] d^3x = 0 & \text{for } k = 0 \end{cases} \tag{19}$$

which are useful when studying localised (or "causal") density perturbations, especially when they are scalar that is such that $\delta T^0_k = 0$, which is the case in most practical applications [2]–[4], [5]–[7].

The constraints (17-19) are the only ones which involve only the matter perturbations. However when perturbations are localised Equations (15-16) also become constraints

$$\int_\Sigma \left(\delta T^0_l x^l + \frac{1}{2\kappa}\dot{\tilde{h}}\right) d^3x = 0 \tag{20}$$

$$\begin{cases} \int_\Sigma \left[\delta T^0_l \left(k\delta^{li} - x^l x^i\right) - \frac{1}{2\kappa}\dot{\tilde{h}} x^i\right] \frac{d^3x}{\sqrt{1-kr^2}} = 0 & \text{for } k \neq 0 \\ \int_\Sigma \left[\delta T^0_l \left(\frac{1}{2}\delta^{li} r^2 - x^l x^i\right) - \frac{1}{2\kappa}\dot{\tilde{h}} x^i\right] d^3x = 0 & \text{for } k = 0 \end{cases} \tag{21}$$

This simple new constraints which involve only the geometry when the perturbations are scalar could be useful in numerical calculations. We shall use them in a simple case in Section 5.

3. Defining energy and motion of the centre of mass in perturbed Robertson-Walker universes

Several authors have interpreted equations (13-14) as defining the energy and momentum of the perturbations of a Robertson-Walker universe. However, to define properly energy, momentum, angular momentum etc we shall introduce a background, as in Katz [8] and Katz Bičak and Lynden-Bell [9].

Consider a spacetime $(\mathcal{M}, g_{\mu\nu}(x^\lambda))$, a background $(\bar{\mathcal{M}}, \bar{g}_{\mu\nu}(x^\lambda))$ and a mapping between these two spacetimes, i.e. a way to identify points of $\mathcal{M}$ and $\bar{\mathcal{M}}$.

We take as lagrangian density for gravity

$$\hat{\mathcal{L}}_G = \frac{1}{2\kappa}[\hat{g}^{\mu\nu}(\Delta^\rho_{\mu\nu}\Delta^\sigma_{\rho\sigma} - \Delta^\rho_{\mu\sigma}\Delta^\sigma_{\rho\nu}) - (\hat{g}^{\mu\nu} - \bar{\hat{g}}^{\mu\nu})\bar{R}_{\mu\nu}] \tag{22}$$

where we have introduced the difference $\Delta^\lambda_{\mu\nu}$ between Christoffel symbols in $\mathcal{M}$ and $\bar{\mathcal{M}}$ and where $\bar{R}_{\mu\nu}$ is the Ricci tensor of the background. We recall that a hat denotes multiplication by $\sqrt{-g}$. $\hat{\mathcal{L}}_G$ vanishes when $g_{\mu\nu} = \bar{g}_{\mu\nu}$, and is quadratic in the first

order derivatives of $g_{\mu\nu}$. It reduces to the familiar "$\Gamma\Gamma - \Gamma\Gamma$" form when the Riemann tensor of the background is zero and when the coordinates are cartesian (such that $\bar{\Gamma}^{\lambda}_{\mu\nu} = 0$). Since the "$\Delta$" are tensors, $\hat{\mathcal{L}}_G$ is a true scalar density.

If we now perform a small displacement $\Delta x^\mu = \zeta^\mu \Delta\lambda$, where ζ^μ is an arbitrary vector field and $\Delta\lambda$ an infinitesimal parameter, and use the fact that $\hat{\mathcal{L}}_G$ is a scalar density, we have that, with L_ζ denoting the Lie derivative,

$$\mathrm{L}_\zeta \hat{\mathcal{L}}_G - \partial_\mu(\hat{\mathcal{L}}_G \zeta^\mu) = 0 \tag{23}$$

Computing explicitely $\mathrm{L}_\zeta \hat{\mathcal{L}}_G$ from (22), it can be shown (cf [9]) that there exists an identically conserved vector $\hat{I}^\mu$ (that is such that $\partial_\mu \hat{I}^\mu \equiv 0$), and hence an antisymmetric tensor $\hat{J}^{[\mu\nu]}$ such that

$$\hat{I}^\mu = \partial_\nu \hat{J}^{[\mu\nu]} \tag{24}$$

The explicit expression for $\hat{I}^\mu$ is

$$\hat{I}^\mu = \left[(\hat{T}^\mu_\nu - \bar{\hat{T}}^\mu_\nu) + \frac{1}{2\kappa} \hat{l}^{\rho\sigma} \bar{R}_{\rho\sigma} \delta^\mu_\nu + \hat{t}^\mu_\nu \right] \zeta^\nu + \hat{\sigma}^{\mu[\rho\sigma]} \partial_{[\rho} \zeta_{\sigma]} + \hat{Z}^\mu(\zeta^\nu) \tag{25}$$

with $\hat{l}^{\mu\nu} \equiv \hat{g}^{\mu\nu} - \bar{\hat{g}}^{\mu\nu}$. (In equation (25) indices are moved with the background metric $\bar{g}_{\mu\nu}$.) We can interpret the first term $(\hat{T}^\mu_\nu - \bar{\hat{T}}^\mu_\nu)$ as the energy-momentum tensor density of matter with respect to the background. The second term can be seen as a coupling between the spacetime and the background. The third one reduces to the Einstein pseudo-tensor density when the background is flat and the coordinates are cartesian. The next term, quadratic in the metric, is the helicity tensor density of the gravitational field with respect to the background. The last term is a function of the vectors ζ^μ which vanishes when those vectors are Killing vectors of the background. The explicit expressions for the various quantities introduced, as well as that for $J^{[\mu\nu]}$ can be found in [9] (see also Appendix 2).

Let us stress that the equality $\hat{I}^\mu(\{g^{\mu\nu}, \bar{g}^{\mu\nu}, \zeta^\nu\}) = \partial_\nu \hat{J}^{[\mu\nu]}(\{g^{\mu\nu}, \bar{g}^{\mu\nu}, \zeta^\nu\})$ and the integral equation that can be deduced from it, are valid for all $\{g^{\mu\nu}, \bar{g}^{\mu\nu}, \zeta^\nu\}$. We have written identities which involve an arbitrary vector ζ^μ (just as in Eq (5-7)), two metrics and their derivatives. Such identities are, in the terminology of Bergman [13], *strong conservation laws*. They reduce to the Noether conservation laws when the vectors ζ^μ become Killing vectors $\bar{\xi}^\mu$ of the background.

This means that, in order to obtain the maximum number of Noether conservation laws, one must consider a background with maximal symmetry, in which case ten integral equations (one for each Killing vector $\zeta^\mu = \bar{\xi}^\mu$) can be written. They are

$$P(\bar{\xi}) \equiv \int_\Sigma \hat{I}^\mu d\Sigma_\mu \equiv \int_\Sigma \left\{ \left[(\hat{T}^\mu_\nu - \bar{\hat{T}}^\mu_\nu) \ + \frac{1}{2\kappa} \hat{l}^{\rho\sigma} \bar{R}_{\rho\sigma} \delta^\mu_\nu + \hat{t}^\mu_\nu \right] \bar{\xi}^\nu \right.$$
$$\left. + \hat{\sigma}^{\mu[\rho\sigma]} \partial_{[\rho} \bar{\xi}_{\sigma]} \right\} d\Sigma_\mu \equiv \int_{\partial\Sigma} \hat{J}^{\mu\nu} d\Sigma_{\mu\nu} \quad (26)$$

where $d\Sigma_\mu$ is the volume element of a spacelike hypersurface Σ, $d\Sigma_{\mu\nu}$ the surface element of its boundary $\partial\Sigma$.

We know two maximally symmetric spacetimes, Minkowski and de Sitter spacetimes (we shall not consider here the perhaps interesting anti-de Sitter possibility). If $\bar{\xi}^\mu = \bar{T}^\mu$ refers to the time translations in Minkowski spacetime or the quasi-time translations of de Sitter spacetime, then the quantity $P(\bar{T})$ will be called energy. When one uses the three Killing vectors associated with the Lorentz rotations of Minkowski or the quasi-Lorentz rotations of de Sitter spacetimes, $\bar{\xi}^\mu = \bar{K}^\mu$, $P(\bar{K})$ will be the "position of the centre of mass" [12]. The introduction of a maximally symmetric background thus allows to define an energy etc, even if the physical spacetime does not possess symmetries, globally or asymptotically. The justification for defining energy etc by (26) can be found in e.g. [9]. Minkowski spacetime has been extensively used as background to study spacetimes which are asymptotically flat (even if the role of the background is not apparent, as is the case with pseudo-tensors when cartesian coordinates are used from the start). We want to define energy etc in cosmology, and that will, as we shall see shortly, make us choose de Sitter rather than Minkowski spacetime as background.

We now apply the formalism summarized above to a perturbed Robertson-Walker spacetime with metric (1). The maximally symmetric background will be chosen with the same spatial topology as the physical perturbed Robertson-Walker spacetime and the metric for the background will be written as

$$d\bar{s}^2 = \Psi(t)^2 dt^2 - \bar{a}(t)^2 f_{ij} dx^i dx^j \quad (27)$$

Equation (27) contains a definition of the mapping for each point of the $t = Const.$ hypersurface, up to an isometry. The function $\Psi(t)$ defines the mapping of the cosmic times (and theexplicit expression for the scale factor $\bar{a}(t)$). Those restrictions

on the mapping render the choice of de Sitter rather than Minkowski spacetime almost compulsory. Indeed when the Robertson-Walker sections are closed, Minkowski spacetime as background is excluded, since there is no coordinate system in which it has closed spatial sections. When the Robertson-Walker sections are flat, in order to have a one-to-one correspondance between Ψ and $\bar{a}$, Minkowski spacetime, which has $\bar{a} = 1\ \forall\Psi$ when $k = 0$, must again be excluded as background. Hence, only when $k = -1$ is Minkowski spacetime (in Milne coordinates) possible as background. One may also note that in the flat and open cases the physical spacetime is mapped on only a patch of the de Sitter hyperboloid. This is not a problem as we need not fix the patch : the de Sitter Killing vectors corresponding to the quasi-time translations and quasi-Lorentz rotations do not apply the patch onto itself but displace it on the de Sitter hyperboloid.

The explicit expressions of the ten de Sitter Killing vectors when the metric is written under the form (27) are given in Appendix 1 (equations (A1-5)). They satisfy equations very similar to the equations (9) satisfied by Traschen's ICVs, to wit :

$$\nabla^{(l}\bar{\xi}^{k)} + \frac{\dot{\bar{a}}}{\bar{a}} f^{lk}\bar{\xi}^0 = 0 \qquad , \qquad \nabla^{lk}\bar{\xi}^0 + k f^{lk}\bar{\xi}^0 = 0 \tag{28}$$

The zeroth order conservation quantities $P_{RW}(\bar{\xi})$ have been defined and studied by Katz Bičak and Lynden-Bell [9]. Here we focus on their perturbations at first order. A fairly long but straightforward calculation brings equation (26) to the form : $P(\bar{\xi}) = P_{RW}(\bar{\xi}) + \delta P(\bar{\xi})$, where $\delta P(\bar{\xi})$ is the sum of equation (6), with ζ^μ a de Sitter Killing vector satisfying equation (28), and a surface term

$$\delta P(\bar{\xi}) \equiv \int_\Sigma \sqrt{-g}\left(\delta T^0_\mu \bar{\xi}^\mu + \frac{1}{2}\beta\dot{\bar{h}}\bar{\xi}^0\right) d^3x + \int_{\partial\Sigma} \hat{M}^l(\bar{\xi}) dS_l = \int_{\partial\Sigma} (\hat{B}^l + \hat{M}^l)(\bar{\xi}) dS_l \tag{29}$$

where

$$M^l(\bar{\xi}) \equiv \frac{h}{2\kappa}\left\{\left[-2H + \frac{\bar{H}}{2\Psi}\left(\frac{\bar{a}^2}{a^2} + 3\Psi^2\right)\right]\bar{\xi}^l + \frac{1}{2}\left(\Psi^2 - \frac{\bar{a}^2}{a^2}\right)\frac{f^{kl}}{\bar{a}^2}\nabla_k\bar{\xi}^0\right\} \tag{30}$$

B^l is given by equation (8) and we have introduced the notation

$$\kappa\beta \equiv \frac{\dot{a}}{a} - \frac{\dot{\bar{a}}}{\bar{a}} \tag{31}$$

as well as the Hubble parameter of the background $\bar{H} \equiv \dot{\bar{a}}/\Psi\bar{a}$.

Using the explicit expressions of the De Sitter/Robertson-Walker Killing vectors corresponding to spatial translations, $\bar{\xi}^\mu = P^\mu$ (see equation (A1)), the total linear momentum of the perturbations is thus defined as

$$\delta P_i(P) \equiv a^3 \int_\Sigma d^3x \delta T_i^0 + \int_{\partial\Sigma} \hat{M}_i^l(P) dS_l = \int_{\partial\Sigma} (\hat{B}_i^l + \hat{M}_i^l)(P) dS_l \tag{32}$$

with

$$\hat{M}_i^l(P) = \frac{a^3 h}{2\kappa}\left[-2H + \frac{\bar{H}}{2\Psi}\left(\frac{\bar{a}^2}{a^2} + 3\Psi^2\right)\right]\delta_i^l \quad , \quad \hat{B}_i^l(P) = \frac{a^3}{2\kappa}\dot{\tilde{h}}_i^l \tag{33}$$

and a similar expression for their total angular momentum corresponding to $\bar{\xi}^\mu = R^\mu$ as given by equation (A2). One sees that the total linear (and angular) momentum is the sum of a background and mapping independent volume integral plus a surface term which does depend on the background and the mapping.

When perturbations are localised equation (32) becomes a constraint which is Traschen's constraint (17). When it comes now to the de Sitter Killing vectors corresponding to quasi-time translations ($\bar{\xi}^\mu = \bar{T}^\mu$) and quasi-Lorentz rotations ($\bar{\xi}^\mu = \bar{K}^\mu$), not only the definitions, as written in (29-30), of the corresponding energy and motion of the centre of mass of the perturbations, but also the constraints which follow when the perturbations are localised, seem to depend on the background and the mapping. Now, that the *definition* of conserved quantities be dependent on conventions for the choice of background or mapping is not a problem. On the other hand *constraints*, which contain measurable information (for example they imply a drastic reduction of the cosmic microwave background anisotropies [2], [5]) cannot be mapping dependent. To show explicitely that indeed thedefinition of energy and motion of the centre of mass depends on the mapping and the background (just as the total linear and angular momentum), but that the constraints do not, we rearrange equations (29-30) using the explicit expressions (A3-A5) for $\bar{T}^\mu$ and $\bar{K}^\mu$ as well as the relations (15-16) to eliminate $\dot{\tilde{h}}$ in the volume integral of (28). We obtain

$$\delta E = \frac{1}{\Psi}\delta P_{Tr}(T) + \int_{\partial\Sigma}\left[\hat{M}^l(\bar{T}) + \frac{\sqrt{-g}}{2\kappa\Psi}\left(1 - \frac{\dot{a}\bar{a}}{a\dot{\bar{a}}}\right)\dot{\tilde{h}}_k^l \bar{T}^k\right] dS_l \tag{34}$$

$$\begin{cases} \delta Z^i = \frac{1}{\Psi}\delta P^i_{Tr}(K) + \int_{\partial\Sigma}\left[\hat{M}^{li}(\bar{K}) + \frac{\sqrt{-g}}{2\kappa\Psi}\left(1 - \frac{\dot{a}\bar{a}}{a\dot{\bar{a}}}\right)\dot{\tilde{h}}_k^l \bar{K}^{ik}\right] dS_l & \text{for } k \neq 0 \\ \delta Z^i = \frac{1}{\Psi}\delta P^i_{Tr}(K) - \frac{1}{2\bar{H}\bar{a}^2}\delta P^i(P) + \int_{\partial\Sigma}\left[\hat{M}^{il}(\bar{K}) + \frac{\sqrt{-g}}{2\kappa\Psi}\left(1 - \frac{\dot{a}\bar{a}}{a\dot{\bar{a}}}\right)\dot{\tilde{h}}_k^l \bar{K}^{ik}\right] dS_l & \text{for } k = 0 \end{cases} \tag{35}$$

We have introduced the short-hand notation $\delta E \equiv \delta P(\bar{T})$ and $\delta Z^i \equiv \delta P^i(\bar{K})$, and the background and mapping independent δP_{Tr} are given by equations (13-14). Hence, the energy and motion of the centre of mass of the perturbations are the sum of volume integrals which are, up to the overall function of time Ψ, background and mapping independent, plus surface terms which do depend on the background and the mapping. We thus see the announced relationship between the energy and motion of the centre of mass of the perturbations and Traschen's integrals (10) (13-14). Turning to localised perturbations for which all surface integrals vanish, we finally see on the form (34-35) for the conserved quantities that the resulting constraints are background and mapping independent, and are Traschen's constraints (18-19).

4. Mapping the cosmic times

The conserved quantities defined in the previous section are background and mapping dependent. We show in this section that there is a mapping of the cosmic times of particular significance. To see that, we shall use the relationship found in [9] between Traschen's ICVs V^μ and de Sitter Killing vectors $\bar{\xi}^\mu$. The four ICVs which are not de Sitter Killing vectors are given by equations (A6-A7); as for the four de Sitter Killing vectors corresponding to quasi-time translations and quasi-Lorentz rotations they are given by equations (A3-A5), so that we have

$$T^0 = \Psi \bar{T}^0 \qquad , \qquad T^k = \frac{H}{\bar{H}} \bar{T}^k \tag{36}$$

$$\begin{cases} K^0 = \Psi \bar{K}^0 \qquad , \qquad K^k = \frac{H}{\bar{H}} \bar{K}^k & \text{for } k \neq 0 \\ K^0 = \Psi \bar{K}^0 \qquad , \qquad K^k = \frac{H}{\bar{H}} \left(\bar{K}^k + \frac{1}{2H\bar{a}^2} P^k \right) & \text{for } k = 0 \end{cases} \tag{37}$$

(As for the remaining six Traschen and de Sitter Killing vectors, they are identical and correspond to the six Robertson-Walker Killing vectors P^μ and R^μ. We also note that Traschen's ICVs become combinations of full-fledged Killing vectors when the Robertson-Walker spacetime becomes a de Sitter spacetime [5], [10].)

Now, as emphasised in [9], in the particular mapping

$$a = \bar{a} \qquad \Longrightarrow \qquad \Psi = H/\bar{H} \quad \text{and} \quad \beta = 0 \tag{38}$$

Traschen's ICVs become (for $k = \pm 1$) strictly proportional to the de Sitter Killing vectors : $V^\mu = \Psi \bar{\xi}^\mu$, where the function Ψ is completely determined once the Robertson-Walker scale factor $a(t)$ is known. For example, in the case of flat spatial sections, $\Psi = 2H\sqrt{3/\bar{R}}$, where $\bar{R}$ is the scalar curvature of the de Sitter background.

This property suggests to raise the mapping (38) to a special status. Moreover the surface terms then acquire a particularly simple form. If, finally, one normalises $\bar{R}$ to 12, for which $\bar{H} = 1$ when $k = 0$, then all explicit reference to the de Sitter background disappears from the definitions (34-35). For example, the energy of the perturbations of a flat Roberston-Walker spacetime becomes

$$\delta E \equiv \frac{a^3}{H} \int_\Sigma (\delta\rho - H\delta T^0_l x^l) d^3x + \frac{a^3}{H} \frac{H^2 - 1}{4\kappa} \int_{\partial\Sigma} h x^l dS_l \tag{39}$$

5. Integral constraints and "localised perturbations"

Ellis and Jaklitsch [1] have given an interpretation of Traschen's Integral Constraints in terms of "fitting conditions", using as an example the "Swiss cheese" model. We shall do the same for another simple case, that of spherical perturbations. This will clarify further what is meant by "localised" perturbations and examplify the use of our constraints (20-21).

Consider a spherical symmetric perturbation of a spatially flat dust universe. Spherical perturbations are scalar. The integral equations (13) (15) for $\delta\rho$ and $\dot{h}$ reduce to

$$\int_0^R \dot{h} r^2 dr = G(R) \quad , \quad \int_0^R \delta\rho r^2 dr = F(R) \tag{40}$$

where R is the radius of the sphere on which the integration is performed and where G and F are some fonctions of the metric perturbations and their derivatives. Imposing that perturbations be localised has meant, in the context of this paper, that the surface terms in (40) be zero for all surfaces outside a sphere of radius one, say. The integral equations then become constraints,

$$\int_0^R \dot{h} r^2 dr = \int_0^R \delta\rho\, r^2 dr = 0 \quad \forall \quad R > 1 \tag{41}$$

which imply

$$\delta\rho = \dot{h} = 0 \quad \forall \quad r > 1 \qquad \Rightarrow \qquad h = h(r) \quad \text{for} \quad r > 1 \tag{42}$$

However, imposing (41) means more than just (42). To see that let's go back to Einstein's equations. Their solution is known. It is the linearisation of a Tolman-Bondi solution [14]. It depends on two arbitrary functions $t_0(r)$, the delayed Big-Bang, and $\epsilon(r)$, the local curvature. A flat Robertson-Walker universe corresponds to $t_0 = \epsilon = 0$. In the case where $t_0(r) = 0$ and $\epsilon(r) << 1$, the metric reads, with $a(t) \equiv (\frac{3}{2}t)^{2/3}$

$$ds^2 = dt^2 - a^2(t)\left\{\left[1 + r^2\epsilon(r) - \frac{2}{3}a(t)(\epsilon + r\epsilon')\right]dr^2 + \left[1 - \frac{2}{3}a(t)\epsilon\right]r^2 d\Omega\right\} \tag{43}$$

and we have

$$\begin{cases} \delta\rho(r,t) & = & \frac{1}{\kappa a(t)^2}\Xi(r) \\ h(r,t) & = & \frac{2}{3}a(t)\Xi(r) - r^2\epsilon(r) \\ \dot{h}(r,t) & = & \frac{2}{3}\dot{a}(t)\Xi(r) \end{cases} \tag{44}$$

where we have introduced the function $\Xi(r) = \frac{1}{r^2}(r^3\epsilon)'$.

Therefore, the conditions (42) only amount to imposing that $\Xi(r) = 0$ for $r > 1$ or, equivalently, that the perturbation $\epsilon(r)$ be of the form

$$\epsilon(r) = \epsilon(1)/r^3 \quad \forall \quad r > 1 \qquad \Rightarrow \qquad h = -\epsilon(1)/r \quad \forall \quad r > 1 \tag{45}$$

where $\epsilon(1)$ is a constant, whereas the stronger constraints (41) add the extra condition

$$\epsilon(1) = 0 \qquad \Rightarrow \qquad h = 0 \quad \forall \quad r > 1 \tag{46}$$

We therefore see on this simple example that "localised" perturbations, that is perturbations such that the surface terms vanish outside a certain region, are not simply perturbations for which $\delta\rho = 0$, but perturbations for which $\delta\rho = 0$ and $h_{ij} = 0$ outside a certain region. Outside that region, spacetime is strictly Robertson-Walker. Hence, the constraints hold only for perturbations that arise from local processes like"explosive" events, or phase transitions producing bubbles of true vacuum, cosmic strings or other topological defects [2], [5–7].We can thus interpret the constraints in the following way : if spacetime is strictly Robertson-Walker outside a certain region, then the metric can be chosen so that $h_{ij} = 0$ outside that region, and Einstein's equations then tell us that the conserved quantities of the perturbations inside that region are all zero. Moreover, since the "background" scale factor, $a(t)$, is the same as that of the outside Robertson-Walker universe, the constraints can also be interpreted as "fitting" conditions [1].

Appendix 1. The ten de Sitter Killing vectors and Traschen's ICVs

We write the de Sitter metric as :

$$d\bar{s}^2 = \Psi^2(t)dt^2 - \bar{a}^2(t)f_{ij}dx^i dx^j$$

where Ψ is an arbitrary function of time and where x^i are Weinberg's [11] coordinates, so that

$$f_{ij} = \delta_{ij} + k\frac{\delta_{im}\delta_{jn}x^m x^n}{1-kr^2} \quad \text{with} \quad r^2 = \delta_{ij}x^i x^j$$

where $k = +1, 0, -1$ depending on whether the spatial sections are closed, flat, or hyperbolic. We have that $\sqrt{f} = 1/\sqrt{1-kr^2}$. Let us also introduce the quantities

$$\bar{H} \equiv \frac{1}{\Psi}\frac{\dot{\bar{a}}}{\bar{a}} \qquad , \qquad \tau \equiv \frac{\Psi}{\dot{\bar{a}}}$$

Ten independent Killing vectors describe three spatial translations, three spatial rotations, one quasi-time translation and three quasi-Lorentz rotations. Their explicit expression is [9]

(a) spatial translations : $\bar{\xi}^\mu = P^\mu$

$$P^0 = 0 \qquad , \qquad P^k = \delta^k_r\sqrt{1-kr^2} \tag{A1}$$

(b) spatial rotations : $\bar{\xi}^\mu = R^\mu$

$$R^0 = 0 \qquad , \qquad R^k = \delta^{kr}x^s - \delta^{ks}x^r \tag{A2}$$

(c) quasi-time translations : $\bar{\xi}^\mu = \bar{T}^\mu$

$$\bar{T}^0 = \frac{1}{\Psi}\sqrt{1-kr^2} \qquad , \qquad \bar{T}^k = -\bar{H}x^k\sqrt{1-kr^2} \tag{A3}$$

(d) quasi-Lorentz rotations : $\bar{\xi}^\mu = \bar{K}^\mu$

$$\bar{K}^0 = \frac{1}{\Psi}x^r \quad , \quad \bar{K}^k = \bar{H}(k\delta^{kr} - x^k x^r) \qquad \text{if} \quad k = \pm 1 \tag{A4}$$

$$\bar{K}^0 = \frac{1}{\Psi}x^r \quad , \quad \bar{K}^k = \bar{H}\left[\frac{1}{2}\delta^{kr}(r^2-\tau^2) - x^k x^r\right] \qquad \text{if} \quad k = 0 \tag{A5}$$

In analogy with special relativistic definitions [12], the conserved quantity corresponding to spatial translations is momentum, angular momentum corresponds

to spatial rotations, energy to quasi-time translations and position of the centre of mass to quasi-Lorentz rotations.

We write Roberston-Walker metrics as

$$ds^2 = dt^2 - a^2(t) f_{ij} dx^i dx^j$$

We also introduce

$$H \equiv \frac{\dot{a}}{a}$$

With these coordinates we also have that $\sqrt{-g} = a^3\sqrt{f} = a^3/\sqrt{1-kr^2}$.

In that coordinate system, six of the ten Traschen vectors V^μ are nothing but the previous Robertson-Walker/de Sitter Killing vectors corresponding to spatial translations and rotations. The extra four, the "integral constraint vectors" T^μ and K^μ, read

$$T^0 = \sqrt{1-kr^2} \quad , \qquad T^k = -Hx^k\sqrt{1-kr^2} \tag{A6}$$

$$K^0 = x^r \quad , \qquad K^k = H(k\delta^{kr} - x^k x^r) \qquad \text{for} \quad k = \pm 1 \tag{A7}$$

$$K^0 = x^r \quad , \qquad K^k = H\left(\frac{1}{2}\delta^{kr} r^2 - x^k x^r\right) \qquad \text{for} \qquad k = 0 \tag{A8}$$

They are related to the de Sitter Killing vectors corresponding to quasi-time translations, quasi-Lorentz rotations, and, in the flat case, spatial translations, by [9]

$$T^0 = \Psi \bar{T}^0 \qquad , \qquad T^k = \frac{H}{\bar{H}} \bar{T}^k \qquad \forall \quad k \tag{A9}$$

$$K^0 = \Psi \bar{K}^0 \qquad , \qquad K^k = \frac{H}{\bar{H}} \bar{K}^k \qquad , \qquad \text{for} \quad k = \pm 1 \tag{A10}$$

$$K^0 = \Psi \bar{K}^0 \qquad , \qquad K^k = \frac{H}{\bar{H}}\left(\bar{K}^k + \frac{1}{2\bar{H}\bar{a}^2} P^k\right) \qquad \text{for} \qquad k = 0 \tag{A11}$$

All vectors $\tilde{V}^\mu = F(t) V^\mu$, with $F(t)$ an arbitrary function of time, are solutions of Traschen's equations (9). If one chooses $F(t) \equiv \Psi^{-1}$, and the function Ψ such that $H = \Psi \bar{H}$, then the vectors $\tilde{V}^\mu$ become a combination of the de Sitter Killing vectors.

Appendix 2.

One can extract frome Einstein's equations a conserved current $\hat{I}^\mu$ and an anti-symmetric tensor $\hat{J}^{\mu\nu}$ such that

$$\hat{I}^\mu = \partial_\nu \hat{J}^{[\mu\nu]} \qquad \Leftrightarrow \qquad \partial_\mu \hat{I}^\mu = 0 \tag{B1}$$

We give here the expressions of these two quantities [see ref.[9]].

First introduce

$$\hat{l}^{\mu\nu} \equiv \hat{g}^{\mu\nu} - \bar{\hat{g}}^{\mu\nu} \qquad \text{and} \qquad \Delta^\lambda_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu} - \bar{\Gamma}^\lambda_{\mu\nu} \tag{B2}$$

where Γ and $\bar{\Gamma}$ are the Christoffel symbols of the spacetime and of the background, $\bar{D}_\mu$ and D_μ the two covariant derivatives and for a given quantity A, $\hat{A}$ denotes $\sqrt{-g}A$ and $\bar{A}$ the value of A on the background. Notice that $\bar{\hat{A}} = \sqrt{-\bar{g}}\bar{A} \neq \hat{\bar{A}} = \sqrt{-g}\bar{A}$.

expression for $\hat{J}^{\mu\nu}$

For any vector ζ^μ we have,

$$\kappa \hat{J}^{\mu\nu} = \hat{l}^{[\mu\rho}\bar{D}_\rho \zeta^{\nu]} + \hat{g}^{[\mu\rho}\Delta^{\nu]}_{\rho\lambda}\zeta^\lambda + \zeta^{[\mu}\hat{g}^{\nu]\rho}\Delta^\sigma_{\rho\sigma} - \zeta^{[\mu}\Delta^{\nu]}_{\rho\sigma}\hat{g}^{\rho\sigma} \tag{B3}$$

expression for $\hat{I}^\mu$

$$\hat{I}^\mu = \left[\left(\hat{T}^\mu_\nu - \bar{\hat{T}}^\mu_\nu\right) + \frac{1}{2\kappa}\hat{l}^{\rho\sigma}\bar{R}_{\rho\sigma}\delta^\mu_\nu + \hat{t}^\mu_\nu\right]\zeta^\nu + \hat{\sigma}^{\mu[\rho\sigma]}\partial_{[\rho}\zeta_{\sigma]} + \hat{Z}^\mu(\zeta^\nu) \tag{B4}$$

$\hat{T}^\mu_\nu$ and $\bar{\hat{T}}^\mu_\nu$ are the two energy-momentum tensors.

$$\begin{aligned} 2\kappa\hat{t}^\mu_\nu = \hat{g}^{\rho\sigma}\left(\Delta^\lambda_{\rho\lambda}\Delta^\mu_{\sigma\nu} + \Delta^\mu_{\rho\sigma}\Delta^\lambda_{\lambda\nu} - 2\Delta^\mu_{\rho\lambda}\Delta^\lambda_{\sigma\nu}\right) + \hat{g}^{\mu\rho}\left(\Delta^\sigma_{\lambda\sigma}\Delta^\lambda_{\rho\nu} - \Delta^\sigma_{\sigma\rho}\Delta^\lambda_{\lambda\nu}\right) \\ - \hat{g}^{\rho\sigma}\left(\Delta^\eta_{\rho\sigma}\Delta^\lambda_{\lambda\eta} - \Delta^\eta_{\rho\lambda}\Delta^\lambda_{\eta\sigma}\right)\delta^\mu_\nu \end{aligned} \tag{B5}$$

This term reduces to the Einstein pseudo-tensor density when the background is Minkowski spacetime in cartesian coordinates.

$$2\kappa\hat{\sigma}^{\mu[\rho\sigma]} = \left(\hat{l}^{\mu[\rho}\bar{g}^{\sigma]\lambda} - \bar{g}^{\mu[\rho}\hat{l}^{\sigma]\lambda}\right)\Delta^\nu_{\lambda\nu} - 2\hat{l}^{\lambda[\rho}\bar{g}^{\sigma]\nu}\Delta^\mu_{\lambda\nu} \tag{B6}$$

$$\begin{aligned} 4\kappa\hat{Z}^\mu(\zeta^\nu) = \left(Z^\mu_\rho\hat{g}^{\rho\sigma} + \hat{g}^{\mu\rho}Z^\sigma_\rho - \hat{g}^{\mu\sigma}Z\right)\Delta^\lambda_{\sigma\lambda} + \left(\hat{g}^{\rho\sigma}Z - 2\hat{g}^{\rho\lambda}Z^\sigma_\lambda\right)\Delta^\mu_{\rho\sigma} + \hat{g}^{\mu\lambda}\partial_\lambda Z \\ + \hat{g}^{\rho\sigma}\left(\bar{D}^\mu Z_{\rho\sigma} - 2\bar{D}_\rho Z^\mu_\sigma\right) \end{aligned} \tag{B7}$$

with

$$Z_{\rho\sigma} = \mathrm{L}_\zeta \bar{g}_{\rho\sigma} = \bar{D}_{(\rho}\zeta_{\sigma)} \ \text{ and } \ Z = Z_{\rho\sigma}\bar{g}^{\rho\sigma} \tag{B8}$$

When ζ^μ is a Killing vector of the background, $Z_{\rho\sigma} = 0$ and thus $Z^\mu = 0$.

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