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# Conservation Laws for Large Perturbations on Curved Backgrounds

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**Abstract**

Backgrounds are pervasive in almost every application of general relativity. Here we consider the Lagrangian formulation of general relativity for large perturbations with respect to a curved background spacetime. We show that Nœther's theorem combined with Belinfante's "symmetrization" method applied to the group of displacements provide a conserved vector, a "superpotential" and a energy-momentum that are independent of any divergence added to the Hilbert Lagrangian of the perturbations. The energy-momentum is symmetrical and divergenceless only on backgrounds that are Einstein spaces in the sense of A.Z.Petrov.

Here we use well tested methods in classical field theory to construct a conserved vector density with respect to arbitrary backgrounds. Backgrounds are pervasive in almost every physical application of general relativity: from gravitational radiation [1] to testing the theory in the solar system at higher orders of approximation in the

PPN formalism [2]; from stability theory of de Sitter or anti-de Sitter spacetimes [3] to stability of black holes [4]. Relativistic cosmology is studied on a Friedmann-Robertson-Walker background [5]. Isolated sources are analyzed on asymptotically flat backgrounds. Thus, backgrounds play an important role in practically all applications of general relativity. It is therefore interesting to have satisfactory differential conservation laws on curved as well as on flat backgrounds.

There are essentially two methods to obtain conserved vectors in general relativity. One method consists in rewriting Einstein's equations for the perturbations keeping on the left hand side terms linear in second order derivatives of the perturbed gravitational field components ([6][7] on a flat background; [3] on a Petrov space [8] background). Einstein's equations have been obtained in that form directly from *a variational principle* [9][10][11][12]. The right hand side of the equations is a symmetric "energy-momentum tensor" say $*T^{\mu\nu}$ and for any Killing vector of the background $\bar{\xi}_\nu$ there exist a conserved vector density $\sqrt{-g}(*T^{\mu\nu})\overline{\xi}_\nu$.

There are problems with that method. First, as pointed out by Boulware and Deser [13] and by Popova and Petrov [14], the perturbations of the gravitational field can be represented with the metric, the inverse metric, the metric density and so on. For each representation the conserved vector density will be different. Second conserved vectors have always been obtained for Killing vectors only and only on A.Z. Petrov spaces as backgrounds.

A second way of finding conservation laws consists in applying Nœther's method to a Lagrangian of the gravitational field (see for instance [15]). This leads to a "canonical" Nœther conserved vector, the divergence of a superpotential and to a canonical energy-momentum tensor. The method gives conserved vectors on *any*

*background* with any vector field defining a one parameter displacement [16], not only for Killing vectors of the background. This great multiplicity of conservation laws in general relativity is related to the relabeling of spacetime points and is not without analogy with circulation conservation in fluid dynamics. The relabeling of points in barotropic flows is associated with Nœther conserved vectors which in comoving coordinates are known (not too well) as conservation of "potential vorticities" [17][18]. The conservation of potential vorticities is the local expression of the more familiar non local Kelvin conservation law of circulation.

But there are problems with this method also. First the Lagrangian density is not unique. A divergence can and must be added to the Hilbert Lagrangian because the Hilbert Lagrangian leads to Komar's [19] conservation law which gives the wrong mass to angular momentum ratio in the weak field limit (the "anomalous factor 2" [20]) and does not give [21] the Bondi [22] mass. Various divergences have been added for different reasons [23][24][25]. They lead to different conserved vectors. Second, the canonical energy momentum tensor is not symmetrical nor in general divergenceless. On a flat background the energy-momentum *is* divergenceless but it is not symmetrical and does not provide a conserved angular momentum and when it does, the angular momentum does not include the helicity of the field. Thus this second method is not satisfactory even in the weak field limit.

Here we use Nœther's method, the Hilbert Lagrangian density but also Belinfante's [26] modification. In classical field theory, Belinfante's correction gives a symmetric field energy-momentum tensor which ensures conservation of angular momentum, helicity included. As we shall see, the Belinfante correction has the great advantage to provide a conserved vector and a superpotential that

are *independent* of any divergence added the the Hilbert Lagrangian and does not depend on a particular representation of the gravitational perturbation. The result looks like a peculiar blend of the two methods with none of their defects. The answer is unique.

Let us start with the Lagrangian for gravitational perturbations, the matter Lagrangian playing no role in our considerations:[1]

$$\hat{\mathcal{L}}'_G = -\frac{1}{2\kappa}\left(\hat{R} - \overline{\hat{R}}\right) = \hat{\mathcal{L}}' - \overline{\hat{\mathcal{L}}'}. \tag{1}$$

$\hat{R} = \sqrt{-g}R$ is the scalar curvature density of a spacetime with a metric $g_{\mu\nu}$, $\overline{\hat{R}}$ is that of a background with a metric $\overline{g}_{\mu\nu}$, both metrics have signature $-2$, and $\kappa = 8\pi G/c^4$. A hat "$\hat{}$" always means multiplication by $\sqrt{-g}$ not by $\sqrt{-\overline{g}}$, thus $\overline{\hat{R}} = \sqrt{-g}\overline{R}$ is different from $\overline{\hat{R}} = \sqrt{-\overline{g}}\overline{R}$.

We now apply Nœther's theorem to $\hat{\mathcal{L}}'$, not $\hat{\mathcal{L}}'_G$. For this we first calculate the Lie derivative $\pounds_\xi \hat{\mathcal{L}}'$ of $\hat{\mathcal{L}}'$ for an arbitrary displacement field $\xi^\mu$; it is equal to $\partial_\mu\left(\hat{\mathcal{L}}'\xi^\mu\right)$ and is of the form

$$\pounds_\xi \hat{\mathcal{L}}' = \frac{1}{2\kappa}\hat{G}^{\mu\nu}\pounds_\xi g_{\mu\nu} - \frac{1}{2\kappa}\partial_\mu\left(\hat{g}^{\rho\sigma}\pounds_\xi \Gamma^\mu_{\rho\sigma} - \hat{g}^{\mu\rho}\pounds_\xi \Gamma^\sigma_{\rho\sigma}\right) = \partial_\mu\left(\hat{\mathcal{L}}'\xi^\mu\right) \tag{2}$$

where $G^{\mu\nu}$ is Einstein's tensor, the left hand side of his equations. We then use the contracted Bianchi identities $D_\nu G^{\mu\nu} \equiv 0$ and Einstein's equations $G^\mu_\nu = \kappa T^\mu_\nu$ and obtain a conserved vector density $\hat{\imath}\,'^\mu$ which looks as follows

$$\partial_\mu \hat{\imath}\,'^\mu = 0, \qquad \hat{\imath}\,'^\mu = \hat{T}^\mu_\nu \xi^\nu - \frac{1}{2\kappa}\left(\hat{g}^{\rho\sigma}\pounds_\xi \Gamma^\mu_{\rho\sigma} - \hat{g}^{\mu\rho}\pounds_\xi \Gamma^\sigma_{\rho\sigma}\right) - \hat{\mathcal{L}}'\xi^\mu. \tag{3}$$

We then redo the same calculations with the Lie derivative $\pounds_\xi \overline{\hat{\mathcal{L}}'}$ of $\overline{\hat{\mathcal{L}}'}$ which is equal to $\partial_\mu(\overline{\hat{\mathcal{L}}'}\xi^\mu)$ and we obtain a conserved vector

[1] Symbols may seem unnecessarily complicated for this short paper but they are the same as in the full paper where they have some justification. Comparison will thereby be simpler.

density $\overline{\hat{\imath}\,'^{\mu}}$ that satisfies exactly the same equality (3) with bars over every symbol except $\xi^{\mu}$.

The conserved Nœther vector density $\hat{I}'^{\mu}$ associated with $\hat{\mathcal{L}}'_G = \hat{\mathcal{L}}' - \overline{\hat{\mathcal{L}}'}$ is equal to the difference $\hat{\imath}\,'^{\mu} - \overline{\hat{\imath}\,'^{\mu}}$. This difference is a linear homogeneous expression in $\xi^{\mu}$, in $\overline{g}_{\mu\nu}$-covariant derivatives $\overline{D}_{\rho}\xi_{\sigma}$ (with $\xi_{\sigma} = \overline{g}_{\sigma\mu}\xi^{\mu}$) and in *derivatives* of $\overline{D}_{(\rho}\xi_{\sigma)}$ which we denote by a special symbol $\bar{z}_{\rho\sigma}$; thus $\hat{I}'^{\mu}$ can be written in the following form

$$\begin{aligned} &\hat{I}'^{\mu} = \hat{\theta}'^{\mu}_{\nu}\xi^{\nu} + \hat{\sigma}'^{\mu\rho\sigma}\overline{D}_{\rho}\xi_{\sigma} + \hat{\eta}^{\mu}, \\ &2\kappa\hat{\eta}^{\mu} = \hat{l}^{\mu\lambda}\partial_{\lambda}\bar{z} + \hat{l}^{\rho\sigma}\left(\overline{D}^{\mu}\bar{z}_{\rho\sigma} - 2\overline{D}_{\rho}\bar{z}^{\mu}_{\sigma}\right) \end{aligned} \tag{4}$$

where $\hat{l}^{\rho\sigma} = \hat{g}^{\rho\sigma} - \overline{\hat{g}^{\rho\sigma}}$; indices are displaced with $\bar{g}_{\mu\nu}$, never with $g_{\mu\nu}$. In Eq. (4) $\hat{\theta}'^{\mu}_{\nu}$ is the relative energy momentum tensor density

$$\hat{\theta}'^{\mu}_{\nu} = \hat{T}^{\mu}_{\nu} - \overline{\hat{T}^{\mu}_{\nu}} + a\ field\ component. \tag{5}$$

The *field component* of the tensor is complicated as may be guessed from Eqs. (3) and, say, $\overline{(3)}$; however, its explicit form will not be needed and there is no point in writing it here in detail. The $\hat{\sigma}'^{\mu\rho\sigma}$ term is more important. Its antisymmetric part $\hat{\sigma}'^{\mu[\rho\sigma]}$ plays the role of a relative helicity in linearized quantum gravity [27] and is similar to the helicity in electromagnetism [28],

$$2\kappa\hat{\sigma}'^{\mu\rho\sigma} = \hat{g}^{\mu\rho}\overline{g}^{\sigma\nu}\Delta^{\lambda}_{\nu\lambda} + \hat{g}^{\nu\lambda}\overline{g}^{\mu\sigma}\Delta^{\rho}_{\nu\lambda} - 2\hat{g}^{\nu\rho}\overline{g}^{\sigma\lambda}\Delta^{\mu}_{\nu\lambda} \tag{6}$$

where the tensor $\Delta^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \overline{\Gamma}^{\lambda}_{\mu\nu}$ is the difference of the Christoffel symbols.

It is well known [29] that the conserved vector $\hat{I}'^{\mu}$ is equal to a divergence of an anti-symmetric tensor density, a superpotential, $\hat{K}^{\mu\nu}$ which for good reasons we call the "relative" Komar [19] superpotential, relative to the background

$$\begin{aligned} &\hat{I}'^{\mu} = \hat{\theta}'^{\mu}_{\nu}\xi^{\nu} + \hat{\sigma}'^{\mu\rho\sigma}\overline{D}_{\rho}\xi_{\sigma} + \hat{\eta}^{\mu} = \partial_{\nu}\hat{K}^{\mu\nu}, \\ &\hat{K}^{\mu\nu} = -\hat{K}^{\nu\mu} = \tfrac{1}{\kappa}\left(\hat{g}^{\rho[\mu}D_{\rho}\xi^{\nu]} - \overline{\hat{g}^{\rho[\mu}D_{\rho}\xi^{\nu]}}\right). \end{aligned} \tag{7}$$

$\hat{K}^{\mu\nu}$ contains $g_{\mu\nu}$-covariant derivatives $D_\rho$ as well as $\bar{g}_{\mu\nu}$-covariant derivatives $\overline{D}_\rho$.

We now apply Belinfante's [26] procedure. The method has been used in general relativity by Papapetrou [30] on a flat background with Killing vectors of rotation in Minkowski coordinates. It is here applied to curved ones with arbitrary $\xi$'s and in arbitrary coordinates. It works as follows. Replace the conserved vector density $\hat{I}'^\mu$ by the following new (also divergenceless) one $\hat{\mathcal{I}}^\mu$

$$\begin{aligned} &\hat{\mathcal{I}}^\mu = \hat{I}'^\mu + \partial_\nu \left( \hat{S}'^{\mu\nu\sigma} \xi_\sigma \right), \\ &\hat{S}'^{\mu\nu\sigma} = -\hat{S}'^{\nu\nu\mu\sigma} = \hat{\sigma}'^{\sigma[\mu\nu]} - \hat{\sigma}'^{\mu[\nu\sigma]} + \hat{\sigma}'^{\nu[\mu\sigma]}. \end{aligned} \tag{8}$$

In the new vector, there are no antisymmetric derivatives of $\xi$ anymore, the $S'$-addition cancels precisely the helicity-term. The $S'$-addition also modifies the energy-momentum tensor density $\hat{\theta}'^\mu_\nu$, the $\hat{\eta}^\mu$-vector density and the superpotential $\hat{K}^{\mu\nu}$. $\hat{\mathcal{I}}^\mu$ has the following form

$$\hat{\mathcal{I}}^\mu = \hat{\mathcal{T}}^\mu_\nu \xi^\nu + \hat{\mathcal{Z}}^\mu = \partial_\nu \hat{\mathcal{I}}^{\mu\nu}, \qquad \hat{\mathcal{I}}^{\mu\nu} = -\hat{\mathcal{I}}^{\nu\mu}, \qquad \partial_\mu \hat{\mathcal{I}}^\mu = 0. \tag{9}$$

The new energy-momentum tensor density $\hat{\mathcal{T}}^\mu_\nu$ and the new $\hat{\mathcal{Z}}^\mu$ are related to $\hat{\theta}'^\mu_\nu$ and $\hat{\eta}^\mu$ as follows:

$$\hat{\mathcal{T}}^\mu_\nu = \hat{\theta}'^\mu_\nu + \overline{D}_\rho \hat{S}'^{\mu\rho}{}_\nu, \qquad \hat{\mathcal{Z}}^\mu = \hat{\eta}^\mu + (\hat{\sigma}'^{\mu\rho\sigma} + \hat{S}'^{\mu\rho\sigma}) \bar{z}_{\rho\sigma} \tag{10}$$

while the Komar relative superpotential is replaced by a new superpotential

$$\hat{\mathcal{I}}^{\mu\nu} = \hat{K}^{\mu\nu} + \hat{S}'^{\mu\nu}{}_\rho \xi^\rho. \tag{11}$$

Notice that $\hat{\mathcal{Z}}^\mu$ like $\hat{\eta}^\mu$ is zero if $\xi^\mu$ is a Killing vector $\overline{\xi}^\mu$ of the background.

One crucial point is now this: let's add a divergence to $\hat{R} - \overline{\hat{R}}$ in the Lagrangian density (1) say $\partial_\mu \hat{k}^\mu$. This has the effect to produce another Nœther conserved vector density $\hat{I}^\mu \neq \hat{I}'^\mu$. Indeed, the Lie

derivative of the divergence, $\pounds_\xi(\partial_\mu \hat{k}^\mu) = \partial_\mu(\pounds_\xi \hat{k}^\mu) = \partial_\nu(\xi^\nu \partial_\mu \hat{k}^\mu)$. Since $\pounds_\xi \hat{k}^\mu$ contains at most first order derivatives of $\xi^\mu$ the conserved $\hat{I}^\mu$ will have a modified $\xi^\mu$ factor ($\hat{\theta}^\mu_\nu \neq \hat{\theta}'^\mu_\nu$) and a modified $\overline{D}_\rho \xi_\sigma$ factor ($\hat{\sigma}^{\mu\rho\sigma} \neq \hat{\sigma}'^{\mu\rho\sigma}$). Of course the superpotential is also changed and it is easy to find that $\hat{K}^{\mu\nu}$ is replaced by

$$\hat{I}^{\mu\nu} = \hat{K}^{\mu\nu} + \frac{1}{\kappa}\xi^{[\mu}\hat{k}^{\nu]}. \tag{12}$$

With a different $\hat{\sigma}^{\mu\rho\sigma}$ there is also another $\hat{S}^{\mu\nu\rho}$ as can be seen from Eq. (8) and it is equally easy to find how that $S$ is related to $S'$:

$$\hat{S}^{\mu\nu\rho} = \hat{S}'^{\mu\nu\rho} - \frac{1}{\kappa}\xi^{[\mu}\hat{k}^{\nu]}. \tag{13}$$

Thus, look at Eq. (11), $\hat{\mathcal{I}}^{\mu\nu}$ *does not depend on adding a divergence to the Hilbert Lagrangian.*[2] This is also true for $\hat{\mathcal{T}}^\mu_\nu$ and $\hat{\mathcal{Z}}^\mu$ as implied by Eq. (9).

The explicit structure of $\hat{\mathcal{T}}^\mu_\nu$ is not important here; it can be derived from Eq. (15) given below . What is important however are the properties of $\hat{\mathcal{T}}^\mu_\nu$ which we most easily obtain by Rosenfeld's [32] method. The modified conservation law $\partial_\mu \hat{\mathcal{I}}^\mu = 0$ is linear in $\xi^\mu$ with derivatives up to order three that come from $\partial_\mu \hat{\mathcal{Z}}^\mu$ as can be seen from Eqs. (9), (10) and (4). $\partial_\mu \hat{\mathcal{I}}^\mu = 0$ may thus be written in the form

$$\partial_\mu \hat{\mathcal{I}}^\mu \equiv \hat{\beta}_\nu \xi^\nu + \hat{\beta}^\mu{}_\nu \overline{D}_\mu \xi^\nu + \hat{\beta}^{\rho\sigma}{}_\nu \overline{D}_{(\rho\sigma)} \xi^\nu + \hat{\beta}^{\mu\rho\sigma}{}_\nu \overline{D}_{(\mu\rho\sigma)} \xi^\nu = 0. \tag{14}$$

This equation holds for arbitrary $\xi^\nu$. Therefore all the $\beta$'s must be equal to zero. These are the "Rosenfeld identities". The most interesting identities for now are those involving $\hat{\mathcal{T}}^\mu_\nu$ which, we can

[2] Bak, Cangemi and Jackiw [31] made already the interesting remark that Belinfante's modification of the Nœther currents obtained from Hilbert's or Einstein's Lagrangians lead to the same symmetric and divergenceless energy-momentum tensor relative to a flat background in Minkowski coordinates.

see from Eq. (9), are $\hat{\beta}^{\mu}{}_{\nu} = 0$ or rather $\hat{\beta}^{\mu\nu} = \hat{\beta}^{\mu}{}_{\rho}\overline{g}^{\rho\nu} = 0$ and $\hat{\beta}_{\nu} = 0$:

$$\begin{aligned} &\hat{\beta}^{\mu\nu} = \hat{\mathcal{T}}^{\mu\nu} + \overline{D}_{\rho}\left(\hat{\sigma}'^{(\rho\mu)\nu} + \hat{\sigma}'^{(\rho\nu)\mu} - \hat{\sigma}'^{(\mu\nu)\rho}\right) - \tfrac{1}{\kappa}\hat{l}^{\mu\lambda}\overline{R}^{\nu}_{\lambda} = 0, \\ &\hat{\mathcal{T}}^{\mu\nu} = \hat{\mathcal{T}}^{\mu}_{\rho}\overline{g}^{\rho\nu} \end{aligned} \tag{15}$$

and

$$\hat{\beta}_{\nu} = \overline{D}_{\mu}\hat{\mathcal{T}}^{\mu}_{\nu} - \frac{1}{2\kappa}\hat{l}^{\rho\sigma}\overline{D}_{\nu}\overline{R}_{\rho\sigma} = 0. \tag{16}$$

The new energy momentum tensor density has the following form

$$\hat{\mathcal{T}}^{\mu\nu} = \left(\hat{T}^{(\mu}_{\rho}\overline{g}^{\nu)\rho} - \overline{\hat{T}^{\mu\nu}}\right) + \hat{\tau}^{\mu\nu} + \frac{1}{\kappa}\left(\tfrac{1}{2}\hat{l}^{\rho\sigma}\overline{R}_{\rho\sigma}\bar{g}^{\mu\nu} + \hat{l}^{\lambda[\mu}\overline{R}^{\nu]}_{\lambda}\right). \tag{17}$$

It contains three types of terms, a symmetric matter energy-momentum of the perturbations, a *symmetric* field energy-momentum tensor[3] $\hat{\tau}^{\mu\nu} = \hat{\tau}^{\nu\mu}$ and two non-derivative coupling terms of the metric density perturbation to the background Ricci tensor, the last one *only* being anti-symmetric in $\mu\nu$. Therefore:

(i) Eq. (15) shows that $\hat{\mathcal{T}}^{\mu\nu} = \hat{\mathcal{T}}^{\nu\mu}$ if and only if $\hat{l}^{\lambda[\mu}\overline{R}^{\nu]}_{\lambda} = 0$, i.e. if $\overline{\hat{R}}_{\mu\nu} = -\overline{\Lambda}\overline{g}_{\mu\nu}$ with $\overline{\Lambda}$ necessarily constant. Thus the energy-momentum tensor is symmetrical only if the background belongs to the class of Einstein spaces in the sense of A.Z. Petrov [8]; these are not the "Einstein spacetimes" of Friedmann-Robertson-Walker cosmologies. However de Sitter and anti-de Sitter spacetimes are Einstein spaces.

(ii) For those particular backgrounds, Eq. (16) shows that $\hat{\mathcal{T}}^{\mu\nu}$ is also divergenceless: $\overline{D}_{\nu}\hat{\mathcal{T}}^{\mu\nu} = 0$. There are thus no divergenceless and symmetric field energy-momentum tensors of perturbations except on Einstein spaces.

(iii) If the background is an Einstein space and $\bar{\xi}_{\nu}$ one of its Killing vectors, the conserved vector density has the simplicity of a classical expression $\hat{\mathcal{J}}^{\mu} = \hat{\mathcal{T}}^{\mu\nu}\bar{\xi}_{\nu} = \partial_{\nu}\hat{\mathcal{J}}^{\mu\nu}$.

[3]This field energy-momentum tensor is a horrifying quadratic homogeneous expression in $\hat{l}^{\rho\sigma}$, $\overline{D}_{\mu}\hat{l}^{\rho\sigma}$ and $\overline{D}_{(\mu\nu)}\hat{l}^{\rho\sigma}$.

(iv) Notice that if the background is not an Einstein space and $\xi^\mu$ is not a Killing vector of the background there are still plenty of conserved vectors as can be seen in Eq. (9).

Having considered the properties of $\hat{\mathcal{I}}^{\mu\nu}$ we now turn our attention to the superpotential which has a rather simple form on *all backgrounds and with every vector* $\xi^\mu$:

$$\begin{aligned} \hat{\mathcal{I}}^{\mu\nu} &= \tfrac{1}{\kappa}\hat{l}^{\sigma[\mu}\overline{D}_\sigma\xi^{\nu]} + \hat{\mathcal{P}}^{\mu\nu}{}_\rho\xi^\rho = -\hat{\mathcal{I}}^{\nu\mu}, \\ \hat{\mathcal{P}}^{\mu\nu\rho} &= -\hat{\mathcal{P}}^{\nu\mu\rho} = \tfrac{1}{\kappa}\overline{D}_\sigma\left(\hat{l}^{\rho[\mu}\overline{g}^{\nu]\sigma} - \hat{l}^{\sigma[\mu}\overline{g}^{\nu]\rho}\right). \end{aligned} \tag{18}$$

This superpotential generalizes a number of well known particular cases:

— On flat backgrounds with Killing vectors of rotations and in Minkowski coordinates, $\hat{\mathcal{I}}^{\mu\nu}$ is the superpotential found by Papapetrou [30]. $\hat{\mathcal{P}}^{\mu\nu\rho}$ has been occasionally refered too (see for instance [33]) as Papapetrou's superpotential. This is strictly true with Killing vectors of translations only (and on a flat background in Minkowski coordinates).

— The tensor density $\hat{\mathcal{P}}^{\mu\nu\rho}$ is the same expression as Weinberg's [6] $\hat{Q}^{\mu\nu\rho}$ and Misner, Thorne and Wheeler's [7] $\partial_\sigma\hat{H}^{\mu\sigma\nu\rho}$ though in those expressions $\hat{l}^{\mu\nu}$ is the *linear approximation* of the inverse of $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$.

— $\hat{\mathcal{P}}^{\mu\nu\rho}$ is identical with the *linear approximation* on a flat background of Freud's [34] superpotential, and of Landau and Lifshitz's [15] superpotential.

— The *complete* superpotential (18) with Killing vectors, $\hat{\mathcal{J}}^{\mu\nu}$, is similar to that of Abbott and Deser [3]. To obtain their superpotential replace $\hat{l}^{\mu\nu}$ by $-\sqrt{-\overline{g}}$ times $H^{\mu\nu} = H_{\rho\sigma}\overline{g}^{\rho\mu}\overline{g}^{\sigma\nu}$ with $H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{g}^{\rho\sigma}h_{\rho\sigma}$. In the linear approximation $\hat{l}^{\mu\nu} \cong -\sqrt{-\overline{g}}H^{\mu\nu}$; the two superpotentials are thus (also) equal to the lowest order in $h_{\mu\nu}$, but not to higher orders. Both superpotentials give the total energy and angular momentum for stationary spacetimes at

spatial infinity. But, the superpotential (18) alone gives the four momentum of Bondi [22] and Sachs [35]'s radiating spacetimes at null infinity.

Eq. (11) can be looked at as a "corrected" Komar superpotential. There are other corrected Komar superpotentials in the literature [36][37]. These modified Komar superpotentials have the anomalous factor of 2 mentioned above and have also other unsatisfactory features [38].

Differential conservation laws on a curved background are useful in relativisitic cosmology and have indeed already been used. For instance, Friedmann-Robertson-Walker spacetimes admit a time translation *conformal* Killing vector. The corresponding conserved current helps solving Einstein's equations with scalar perturbations and topological defects in the limit of long wavelength [39][40]. Another instance is the "integral constraint vectors" which were used by Traschen and Eardley [41] to analyze measurable effects of the cosmic background radiation due to spatially localized perturbations. Those Traschen [42] vectors can be shown to be linear combinations with cosmic-time dependent coefficients of conserved vectors associated with the conformal Killing vectors of "accelerations" described in Fulton, Rohrlich and Witten [43].

We believe that finite volume integrals of the conserved vector densities, which are equal to closed surface integrals of the superpotential may be useful in numerical calculations for the same reason that the relativistic virial theorem of Gourgoulhon and Bonazzola [44] is useful to check numerical integrations of relativistic neutron stars [45].

Detailed calculations not included in this letter will appear in a full paper presumably in *Class. Quantum Grav.*.

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