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Disk Sources for Conformastationary Metrics

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Abstract

Conformastationary metrics – those of the form

$$ds^2 = f\left(dt - \mathcal{A}_k dx^k\right)^2 - f^{-1}\left(dx^2 + dy^2 + dz^2\right)$$

have been derived by Perjes and by Israel & Wilson as source-free solutions of the Einstein-Maxwell equations. By analogy with the conformastatic metrics which have charged dust sources it was assumed that conformastationary metrics would be the external metrics of charged dust in steady motion. However for axially symmetric conformastationary metrics we show that, as well as moving dust, hoop tensions are always necessary to balance the centrifugal forces induced by the motion. Exact examples of conformastationary metrics with disk sources are worked out in full. Generalisations to non-axially symmetric conformastationary metrics are indicated.

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1 Introduction

Using considerable mathematical ingenuity Perjes (1971) and later Israel & Wilson (1972) showed that metrics of the form

$$ds^2 = (VV^*)^{-1}\left(dt - \mathcal{A}_k dx^k\right)^2 - VV^*\left(dx^2 + dy^2 + dz^2\right), \qquad (1.1)$$

where V is a complex solution of Laplace's equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)V=0\ , \tag{1.2}$$

satisfy the Einstein-Maxwell equations for the electromagnetic fields,

$$\mathbf{E}+i\mathbf{H}=\boldsymbol{\nabla}\left(\frac{1}{V}\right), \tag{1.3}$$

$$\mathbf{D}+i\mathbf{B}=|V|(\mathbf{E}+i\mathbf{H})+i\boldsymbol{\mathcal{A}}\times(\mathbf{E}+i\mathbf{H})\ ,$$

whenever $\boldsymbol{\mathcal{A}}$ is a solution of

$$\boldsymbol{\nabla}\times\boldsymbol{\mathcal{A}}=i\left(V\boldsymbol{\nabla}V^*-V^*\boldsymbol{\nabla}V\right)\ . \tag{1.4}$$

In the above $\boldsymbol{\nabla}$ stands for

$$\left(\frac{\partial}{\partial x}\ ,\ \frac{\partial}{\partial y}\ ,\ \frac{\partial}{\partial z}\right)$$

and to obtain metrics that become flat at infinity we need $V\to 1$ there.

Hereafter we write

$$f=(VV^*)^{-1}\ . \tag{1.5}$$

However, except for extreme Kerr-Newman metrics (see below), such solutions are not known to be the external solutions of any real matter distributions. When $\boldsymbol{\mathcal{A}}\equiv 0$ the metrics are conformastatic and then we know that they are external metrics of static charged dust with the electric and gravitational forces in exact balance [see Synge 1960] and for the continuous case, e.g., Lynden-Bell et al. (1999). Most workers guessed that the conformastationary metrics would be the metrics of moving charged dust with those forces balancing because cylindrical conformastatic systems would make conformastationary balanced systems when set in uniform motion along the cylinder. However we show here that axially symmetrical conformastationary solutions with disk sources always need hoop tension to balance the centrifugal forces so that moving charged dust sources are insufficient to maintain equilibrium when the dust is accelerated with respect to the static frame. In such cases the electromagnetic and gravitational forces balance but the centrifugal forces would be unbalanced for charged dust without the introduction of these hoop tensions.

The theory of disk sources has a long history. Morgan & Morgan (1969) introduced pressureless counter-rotating disks while Lynden-Bell & Pineault (1978) gave a self-similar disk in real rotation. Lemos (1989) discussed the peculiarly interesting case of the counter-rotating photon disk which has accelerating Minkowski space on each side and generalised the self-similar disks to include surface pressures. More general solutions have been obtained by Chamorro et al (1987) and quite a number of exact solutions with and without any radial pressure are now known (Bičák et al 1993 a+b, Bičák and Ledvinka 1993, Pichon & Lynden-Bell 1996, Gonzáles & Letelier 1999). Discs can now be constructed with the full freedom in the pressure distribution and surface density but truly rotating disks can only be constructed for those special external metrics which are already known solutions of Einstein's equations. Essentially all others need to be computed although Neugebauer and Meinel (1995) made their calculations of the finite uniformly rotating dust disk as analytic as possible. A similar procedure to that of constructing the physical disk sources of the vacuum Kerr metrics has recently been used to find disks with rotating matter and electric currents which are sources of Kerr-Newman fields (Ledvinka et al. 1999). The extreme Kerr-Newman solutions are special cases of conformastationary fields.

The wonderful simplicity of the conformastatic metrics in which electrostatic forces balance gravity, see e.g., (Lynden-Bell et al. 1999), encouraged us to discover whether the conformastationary metrics had sources of similar simplicity. We were somewhat dismayed to find that hoop tensions were a necessity although we had gained the insight that they are only required to balance the centrifugal accelerations. With that exception the electromagnetic and gravitational forces balance with the latter including gravomagnetic forces.

The new aspect of electromagnetic disks is that while components of $\mathbf{E}$ within the surface must be continuous across the disk the surface components of $\mathbf{H}$ have a discontinuity equal to the surface currents in it. The normal component of $\mathbf{B}$ has to be continuous to preserve the magnetic flux through the surface. These conditions are readily satisfied if one chooses a complex solution of Laplace's equation $V(x,y,z)$ defined only in the region $z>0$ and then continues it into $z<0$ by the definition $V(x,y,-z)=V^{*}(x,y,z)$. With this definition the real part of V is continuous across $z=0$ although the imaginary part is not, and the imaginary part of $\frac{\partial V}{\partial z}$ is continuous although the real part is not. Furthermore, VV^{*} is continuous. It follows that V obeys Laplace's equation (1.2) everywhere excepting $z=0$ where the charges

and currents lie, and from (1.3) the continuity conditions on **E** and **B** are automatically accounted for by the above continuation of V to negative z values. The discontinuities in external curvatures give both the surface mass density and the matter currents as considered by Bičák & Ledvinka (1993) and by Pichon & Lynden-Bell (1996); see also Ledvinka et al. (1999).

We detail these equations below. The symmetry of the solutions above and below the disk already ensures that the intrinsic curvatures in the disk metric calculated from the metrics above and below exactly fit.

2 Conformastationary metrics with axial symmetry

We consider the metric (1.1) for axially symmetric spacetimes in

$$x^\mu = \left(x^0 = t, \quad x^1 = R, \quad x^2 = z, \quad x^3 = \phi\right) \text{coordinates.}$$

Following (1.1), $\mathcal{A}_k$ has then only one non-vanishing component in these coordinates

$$\mathcal{A}_k dx^k \equiv \mathcal{A} R d\phi \ . \tag{2.1}$$

The metric can be written

$$\begin{aligned} ds^2 &= f\left(dt - \mathcal{A} R d\phi\right)^2 - f^{-1}\gamma_{k\ell} dx^k dx^\ell \\ &= f\left(dt - \mathcal{A} R d\phi\right)^2 - f^{-1}\left(dR^2 + dz^2 + R^2 d\phi^2\right) \ , \end{aligned} \tag{2.2}$$

where f and $\mathcal{A}$ are functions of $(R,\, z)$. The components of the metric thus are:

$$g_{00} = f \ , \ g_{11} = g_{22} = -f^{-1} \ , \ g_{03} = -\mathcal{A} R f \ , \ g_{33} = -R^2 f^{-1}\left(1 - f^2\mathcal{A}^2\right)^2 \ , \tag{2.3}$$

with their inverse being

$$g^{00} = f^{-1}\left(1 - f^2\mathcal{A}^2\right) \ , \ g^{11} = g^{22} = -f \ , \ g^{03} = -\frac{1}{R} f \mathcal{A} \ , \ g^{33} = -\frac{1}{R^2} f \ , \tag{2.4}$$

and

$$\sqrt{-g} = \sqrt{-\det g_{\mu\nu}} = f^{-1} R \ . \tag{2.5}$$

The local tetrad $h_\mu^{(\alpha)}$ used below is the one appearing in (2.2), i.e.,

$$h_\mu^{(0)} = f^{1/2}\,(1,0,0,-\mathcal{A}R)\,,\ h_\mu^{(1)} = f^{-1/2}\,(0,1,0,0)\,,\ h_\mu^{(2)} = f^{-1/2}\,(0,0,1,0)\,,$$

$$h_\mu^{(3)} = f^{-1/2}\,(0,0,0,R)\ . \tag{2.6}$$

The dual tetrad reads (the tetrad indices being shifted by Minkowski metric)

$$h^\mu_{(0)} = f^{-1/2}(1,0,0,0),\ h^\mu_{(1)} = f^{1/2}(0,1,0,0),$$

$$h^\mu_{(2)} = f^{1/2}(0,0,1,0),\ h^\mu_{(3)} = f^{1/2}\left(\mathcal{A},0,0,R^{-1}\right). \tag{2.7}$$

This is the orthonormal frame used by static observers who are at rest with respect to infinity. The zero-angular-momentum observers whose worldlines are orthogonal to t = const. hypersurfaces will use "locally non-rotating frames" (e.g., Misner et al 1973) given by

$$e_\mu^{(0)} = f^{1/2}\left[(1-f^2\mathcal{A}^2)^{-1/2},\ 0,\ 0,\ 0\right]\ ,$$

$$e_\mu^{(1)} = h_\mu^{(1)}\ ,\quad e_\mu^{(2)} = h_\mu^{(2)}\ ,$$

$$e_\mu^{(3)} = f^{-1/2}\left[\mathcal{A}f^2\left(1-f^2\mathcal{A}^2\right)^{-1/2},\ 0,\ 0,\ R\left(1-f^2\mathcal{A}^2\right)^{1/2}\right]\ , \tag{2.8}$$

and

$$e^\mu_{(0)} = f^{-1/2}\left[(1-f^2\mathcal{A}^2)^{1/2},\ 0,\ 0,\ -\mathcal{A}f^2R^{-1}(1-f^2\mathcal{A}^2)^{-1/2}\right]\ ,$$

$$e^\mu_{(1)} = h^\mu_{(1)},\ e^\mu_{(2)} = h^\mu_{(2)},$$

$$e^\mu_{(3)} = f^{1/2}\left[0,\ 0,\ 0,\ R^{-1}\left(1-f^2\mathcal{A}^2\right)^{-1/2}\right]\ . \tag{2.9}$$

The coordinate angular velocity (i.e., velocity relative to infinity) of a zero-angular-momentum observer is given by

$$\omega = -\frac{g_{03}}{g_{33}} = -\frac{f^2\mathcal{A}}{1-f^2\mathcal{A}^2}\ . \tag{2.10}$$

3 Axially Symmetric Conformastatic Metrics with a disk at $z = 0$

The metric of the $z = 0$ hypersurface is

$$d\sigma^2 = f(dt - \mathcal{A}Rd\phi)^2 - f^{-1}\left(dR^2 + R^2 d\phi^2\right) = g_{ab}dx^a dx^b, \tag{3.1}$$

where indices $a, b = 0, 1, 3$ and $f = f(R\,,\,0)$, $\mathcal{A} = \mathcal{A}(R\,,\,0)$. The components g_{ab} and g^{ab} are given in (2.3) and (2.4) while

$$\sqrt{-\det g_{ab}} = f^{-1/2}R\ . \tag{3.2}$$

The unit normal vector to the hypersurface $z = 0$ is

$$n^\mu = \epsilon_n f^{1/2}(0, 0, 1, 0) \quad z = 0\ , \tag{3.3}$$

$\epsilon_n = 1 \quad (\epsilon_n = -1)$ if n^μ points in the positive (negative) z direction.

We shall now consider two spacetimes $\mathcal{M}_1$ and $\mathcal{M}_2$ with the metric (2.2) in which (see 1.5) $f = (VV^*)^{-1}$ and the complex V is a solution of e.g., (1.2) or, in (R, z, ϕ) coordinates,

$$\Delta V = \frac{1}{R}\partial_R\left(R\partial_R V\right) + \partial_z^2 V = 0\ . \tag{3.4}$$

In $\mathcal{M}_1$ sources of V are on the $z < 0$ side, in $\mathcal{M}_2$ V is exchanged with V^* and the "conjugate" sources are on the positive side $z > 0$ – see Figure 1. So $\mathcal{M}_2$ is the complexified mirror symmetric image of $\mathcal{M}_1$ in $z = 0$. The spacetime we are interested in is the third spacetime $\mathcal{M}$ which is composed of $\mathcal{M}_1$ for $z > 0$ and $\mathcal{M}_2$ for $z < 0$.
$\mathcal{M}$ is thus an empty space except on $z = 0$, where there is a thin disk of matter with metric (3.1) . The energy-momentum tensor of the disk $T^\mu_\nu \propto \delta(z)$. The discontinuity in the z direction of T^a_b defines the energy-momentum tensor of the disk τ^a_b, more precisely,

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} T^a_b f^{-1/2}dz = \tau^a_b\ , \tag{3.5}$$

other components are zero: $T^2_a = T^a_2 = T^2_2 = 0$. Thus, by integrating Einstein's equations, $G^\nu_\mu = \kappa T^\nu_\mu$ with $\kappa = 8\pi G/c^4$, across the disk we can calculate τ^a_b in terms of the metric components.

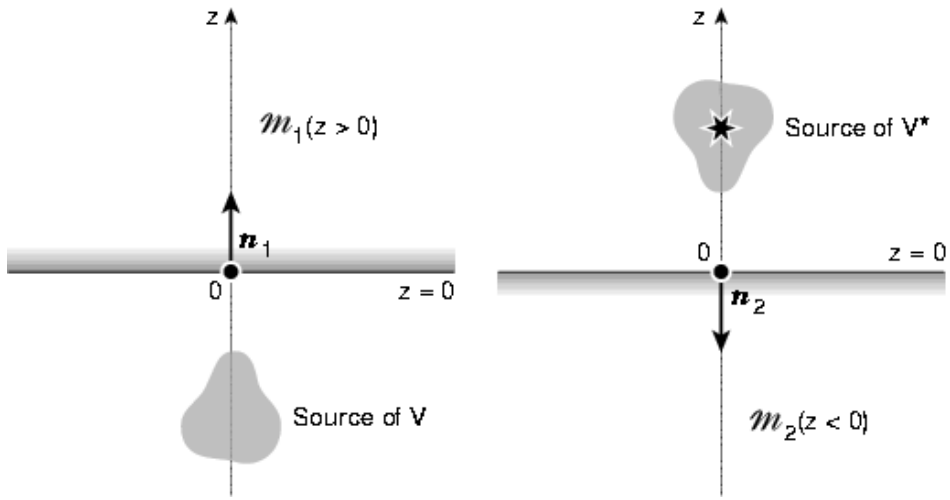

Figure 1:

The non-zero components of $\tau^{ab} = \tau^a_c g^{cb}$ are particularly simple if written in terms of two new quantities

$$\zeta = -\left(\partial_z f^{-1}\right)_{z=0} \ , \quad \chi = \left(\partial_z \mathcal{A}\right)_{z=0} . \tag{3.6}$$

We find that,

$$\kappa\tau^{00} = 2f^{1/2}\left(\zeta + f\mathcal{A}\chi\right) \ , \quad \kappa\tau^{03} = \frac{1}{R} f^{3/2}\chi \ , \quad \kappa\tau^{11} = \kappa\tau^{33} = 0 \ . \tag{3.7}$$

The projected components in the local tetrad (2.6) are respectively

$$\kappa\tau^{(0)(0)} = \kappa\tau^{(0)}_{(0)} = 2f^{3/2}\zeta \ , \ \kappa\tau^{(0)(3)} = \kappa\tau^{(3)}_{(0)} = f^{3/2}\chi \ , \ \kappa\tau^{(2)(2)} = \kappa\tau^{(3)(3)} = 0 \ . \tag{3.8}$$

The vanishing of τ^{33} and, hence, also of the stress $\tau^{(3)(3)}$ as measured by static observers, is the specific feature of conformastationary metrics (2.2). With more general metrics of the form

$$ds^2 = e^{2\nu}(dt - \mathcal{A}Rd\phi)^2 - e^{-2\nu}\left[e^{2\lambda}\left(dR^2 + dz^2\right) + R^2 d\phi^2\right] \ , \tag{3.9}$$

in which $\lambda\left(z, R\right) \neq 0$, as is the case e.g., with the Kerr-Newman metrics, one obtains (see Ledvinka et al. 1999)

$$\kappa\tau_{ab} = e^{\lambda}\left(e^{-2\lambda} g_{ab}\right)_{z=0} \ . \tag{3.10}$$

In our case this formula implies (3.7), (3.8) (even though the covariant component $\tau_{33} \neq 0$). However in general τ^{33} is non-vanishing. The frame components $\tau^{(3)(3)}$ are non-vanishing if expressed in some other than static frame. In particular the frames (2.8) of zero-angular momentum observers give non-vanishing $\tau^{(3)(3)}$ (see 4.11 below).

4 The material properties of the disk and its motion

If the disk is made of matter with proper rest-mass-energy surface density σ, surface "hoop pressure" Π and angular coordinate velocity Ω , the 3 velocity components are

$$U^a = U^0(1,\ 0,\ \Omega), \quad \Omega = \frac{d\phi}{dt}, \tag{4.1}$$

and

$$g_{ab}U^aU^b = 1\ . \tag{4.2}$$

The energy-momentum tensor is

$$\tau^{1b} = 0\ , \quad \tau^{ab} = (\sigma + \Pi)U^aU^b - \Pi g^{ab} \quad a, b = 0,\ 3 \text{ only.} \tag{4.3}$$

If we compare (3.7) with (4.3), we find that

$$\kappa\sigma = f^{3/2}\left[\zeta + \sqrt{\zeta^2 - \chi^2}\right]\ , \quad \kappa\Pi = -f^{3/2}\left[\zeta - \sqrt{\zeta^2 - \chi^2}\right]\ , \tag{4.4}$$

and

$$\Omega R = \frac{f\chi}{\zeta + \sqrt{\zeta^2 - \chi^2} + f\mathcal{A}\chi}\ . \tag{4.5}$$

Equations (4.4), (4.5) are the relationships between the matter parameters and the geometry. The co-moving mass energy density σ is positive of

$$\zeta = -\left(\partial_z f^{-1}\right)_{z=b} > 0\ . \tag{4.6}$$

It then follows that the pressure is actually a tension ($\Pi < 0$) which satisfies the dominant energy condition $-\Pi \leq \sigma$ here. The normalisation condition (4.2) implies

$$\frac{1}{(U^0)^2} = f^{-1}\left[f^2\left(1 - R\Omega\mathcal{A}\right)^2 - R^2\Omega^2\right]\ , \tag{4.7}$$

which, after substituting from (4.5), gives

$$U^0 = \frac{\zeta + \left(\zeta^2 - \chi^2\right)^{1/2} + f\mathcal{A}\chi}{\left\{2f\left[\zeta^2 - \chi^2 + \zeta\left(\zeta^2 - \chi^2\right)^{1/2}\right]\right\}^{1/2}} \quad . \tag{4.8}$$

One easily finds the physical velocity measured by local static observers to be

$$v_{\text{LOC}} = R\Omega/f\left(1 - R\Omega\mathcal{A}\right) \ , \tag{4.9}$$

whereas that measured by the zero-angular-momentum observers reads

$$v_{\text{ZAMO}} = f^{-1}R\left(\Omega - \omega\right)\left(1 - f^2\mathcal{A}^2\right) \ , \tag{4.10}$$

where ω is given by (2.10) .
In contrast to the vanishing stress in the ϕ direction in the static frames (see 3.8) the stress does not vanish in the zero-angular-momentum observers' frame:

$$\tau^{(3)(3)}{}_{\text{ZAMO}} = \left(\sigma + \Pi\right)\left(U^0\right)^2\mathcal{A}^2 f^3\left(1 - f^2\mathcal{A}^2\right)\left[R(\omega - \Omega)\right]^2 + \Pi \ , \tag{4.11}$$

where $(U^0)^2$ is given by (4.8) .

The disk can only exist if

$$|\chi| \leq \zeta \ , \tag{4.12}$$

otherwise σ , Π , and Ω are not real. The limit $\chi = 0$ represents a static disk ($\Omega = 0$) with $\kappa\sigma = 2f^{3/2}\zeta$, producing a conformastatic spacetime on both sides.

5 The electromagnetic currents in the disk

On both sides of the disk, spacetime is free of charges and currents. Maxwell's equations thus are

$$\partial_\nu \hat{F}^{\mu\nu} = 0 \ , \tag{5.1}$$

a "$\hat{}$" means multiplication by $\sqrt{-g} = f^{-1}R$ (see 2.5), and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. According to Perjes (1971) and Israel and Wilson (1972), and following (1.1) and (5.1), $\hat{F}^{k\ell}$ is of the form

$$\hat{F}^{k\ell} = -\eta^{k\ell m}\partial_m \Phi \ , \tag{5.2}$$

with

$$\eta^{k\ell m} = \frac{1}{\sqrt{\gamma}}\epsilon^{k\ell m} = \frac{1}{R}\epsilon^{k\ell m} \ . \tag{5.3}$$

$\epsilon^{k\ell m}$ is the alternating symbol, and Φ is some function of R, z . Both A_0 and Φ, which, together with $g_{\mu\nu}$, define $\hat{F}^{\mu\nu}$ completely, are given by

$$A_0 - i\Phi = fV^* - 1 \ . \tag{5.4}$$

The spatial components of the vector potential, A_k , are here not used. To find the charge and electric current densities in the disk we must integrate Maxwell's equations

$$\partial_\nu \hat{F}^{\mu\nu} = -4\pi \hat{j}^\mu_Q \tag{5.5}$$

accross the disk where the current density $\hat{j}^\mu_Q \sim \delta(z)$. The discontinuities in the z direction defines the components $\hat{i}^\mu_Q$ of the surface current density

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} \hat{j}^\mu_Q f^{-1/2} dz = \hat{i}^\mu_Q \ , \tag{5.6}$$

and the symmetry $z \to -z$ implies that the discontinuity in the $\hat{F}^{2\nu}$ components are just equal to twice $\hat{F}^{2\nu}$ on the positive side of the disk ($z = 0_+$). Thus, taking account of (2.5),

$$\left(F^{\mu 2}\right)_{z=0+} = -2\pi f^{1/2} i^\mu_Q \ . \tag{5.7}$$

In terms of $\hat{F}^{k\ell}$, given by (5.2) with (5.3), and with

$$F_{0k} = -\partial_k A_0 \ , \tag{5.8}$$

where A_0 is also defined by (5.4), we find that

$$\left(\partial_z A_0\right)_{z=0_+} = -2\pi f^{1/2}\left(i^0_Q + \omega R i^3_Q\right), \tag{5.9}$$

$$\frac{1}{R}\left(\partial_R \Phi\right)_{z=0_+} = -2\pi f^{-1/2} i^3_Q \ , \tag{5.10}$$

while $i^2_Q = 0$. Notice that if V is real, $\kappa\sigma = i^0_Q$ and the static disk is composed of charged dust in which gravitational attraction is in equilibrium with electrostatic repulsion.

6 Summarising the results obtained so far

Given a metric of the form (2.2), with two functions $f(R,z)$ and $\mathcal{A}(R,z)$, a solution exists, associated with any *complex* function $V(R,z)$ satisfying the Laplace equation (3.4). The corresponding gravitational potentials f and $\mathcal{A}$ are defined by (1.5) and (1.4) respectively, the electromagnetic "potentials" A_0 and Φ by (5.4). A disk at $z=0$ with $V(r,z>0)$ and $V^*(R,z<0)$ contains matter rotating with angular velocity Ω given by (4.5), proper mass-energy surface density σ and hoop tensions $-\Pi$ given by (4.4). The surface current i_Q^3 is given by (5.10), and the charge per unit surface, i_Q^0, is defined by (5.9). A simple example is given later.

7 Forces

The equations of motion of the disk are given, [see for instance equation (10) of Goldwirth and Katz (1995) which uses the same notations] in terms of the normal unit vectors to $z=0$ given in (3.3):

$$\nabla_b \tau_a^b = -\left[T_\mu^\nu \frac{\partial x^\mu}{\partial x^a} n_\nu\right]_-^+ = -2T_a^\nu \; n_\nu|_{0+} = +2f^{-1/2}T_a^2 \; . \tag{7.1}$$

T_μ^ν is the electromagnetic field energy-momentum tensor. The only non-trivial equation follows for $a=1$ for which

$$\nabla_b \tau_1^b - 2f^{-1/2}T_1^2 = -\frac{1}{2}\tau^{ab}\partial_1 g_{ab} - 2f^{-1/2}T_1^2 = 0. \tag{7.2}$$

This equation expresses the equilibrium of gravitational and electromagnetic forces. To see this explicitly, consider first the T_1^2 term in (7.2).

From

$$T_\nu^\mu = \frac{1}{4\pi}\left(F^{\mu\rho}F_{\rho\nu} + \frac{1}{4}\delta_\nu^\mu F^{\rho\sigma}F_{\rho\sigma}\right) \tag{7.3}$$

we get

$$-2f^{-1/2}T_1^2 = -f^{-1/2}\frac{1}{2\pi}\left(F^{20}F_{01} + F^{23}F_{31}\right), \tag{7.4}$$

or, in terms of (5.7),

$$F_1 = -2f^{-1/2}T_1^2 = i_Q^0 F_{01} + i_Q^3 F_{31}. \tag{7.5}$$

This is the radial component of the electromagnetic force,

$$F_k = i_Q^0 E_k + \eta_{k\ell m} i_Q^\ell B^m, \tag{7.6}$$

in which the components of the electric and magnetic fields are respectively defined by

$$E_k = F_{0k} = -\partial_k A_0, \quad B^k = -\frac{1}{2}\eta^{k\ell m} F_{\ell m}. \tag{7.7}$$

F_k is considered here as a 3-vector in $\gamma_{k\ell}$ space - see (2.2). In vector notations, (7.6) has a familiar look

$$\mathbf{F} = i_Q^0 \mathbf{E} + \mathbf{i}_Q \times \mathbf{B}. \tag{7.8}$$

Consider now the $\tau^{ab}-$ term in (7.2):

$$-\frac{1}{2}\tau^{ab}\partial_1 g_{ab} = -\frac{1}{2}\tau^{00}\partial_1 g_{00} - \tau^{03}\partial_1 g_{03} \ . \tag{7.9}$$

With $\tau_{(0)}^{(0)}$ and $\tau_{(0)}^{(3)}$ given by (3.8) and g_{ab} in (2.3), (7.9) may be rewritten as

$$-\frac{1}{2}\tau^{ab}\partial_1 g_{ab} = \tau_{(0)}^{(0)}\left(-\partial_1 \ln f^{1/2}\right) + \tau_{(0)}^{(3)} f \partial_1 \mathcal{A}_3. \tag{7.10}$$

This is the radial component of the gravitational force,

$$\mathcal{F}_k = \kappa\left(i_M^0 \mathcal{E}_k + \eta_{klm} i_M^l \mathcal{B}^m\right), \tag{7.11}$$

expressed in terms of the gravoelectric and gravomagnetic field components defined by

$$\mathcal{E}_k = -\partial_k \ln f^{1/2} \ , \quad \mathcal{B}^m = -\frac{1}{2}\eta^{mkl} f \partial_k \mathcal{A}_l \ , \tag{7.12}$$

and the matter current in the disk,

$$i_M^a = \tau_{(0)}^{(a)}. \tag{7.13}$$

With (7.8) and (7.11) - (7.13), (7.2) appears clearly as the balance of electromagnetic and gravitational forces; in vector notation,

$$\kappa\left(i_M^0 \boldsymbol{\mathcal{E}} + \mathbf{i}_M \times \boldsymbol{\mathcal{B}}\right) + i_Q^0 \mathbf{E} + \mathbf{i}_Q \times \mathbf{B} = 0 \ . \tag{7.14}$$

It is worth noting that electric and magnetic forces are not separately in equilibrium, i.e. $i_M^0 \boldsymbol{\mathcal{E}} + i_Q^0 \mathbf{E} \neq 0$ unless $\mathcal{A} = 0$.

Another notable point of equation (7.13) is the absence of centrifugal and pressure forces. These are in equilibrium by themselves; it is best seen by considering explicitly $\tau^{(3)(3)}$ which vanishes. With (4.3) we find that

$$\tau^{(3)(3)} = f^{-1} R^2 \tau^{33} = f^{-1} \left(\sigma + \Pi\right) \left(U^0\right)^2 \left(\Omega R\right)^2 + \Pi = 0 \tag{7.15}$$

In a weak gravitational field ($f \sim 1, \;\; \Pi \ll \sigma$) and with a slowly rotating disk ($U^0 \simeq 1$), equation (7.15) becomes the classical condition

$$\sigma \left(\Omega R\right)^2 + \Pi \simeq 0 \;, \tag{7.16}$$

for the equilibrium in a rotating narrow circular band between the centrifugal force and the tensions that keep it from flying apart. Of course, under this weak-field assumption, $\tau^{(3)(3)}{}_{\text{ZAMO}} \approx 0$ (see 4.11) implies (7.16) as well.

8 A simple example

A simple example is the following complex solution of (3.4):

$$V = 1 + \frac{q}{r} + i \frac{\mu(z+b)}{r^3} \;,\; r^2 = R^2 + (z+b)^2 \;,\; z > 0 \;, \tag{8.1}$$

$q > 0$, $b > 0$, and μ are constants. This solution represents a charge and a magnetic dipole at the same position $(0, \; -b)$. Following (1.5), the gravitational potential

$$f = (VV^*)^{-1} = \left[\left(1 + \frac{q}{r}\right)^2 + \frac{\mu^2 (z+b)^2}{r^6} \right]^{-1} \;, \tag{8.2}$$

while, following (8.3),

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{A}} = i \left(V^* \boldsymbol{\nabla} V - V \boldsymbol{\nabla} V^*\right) \;, \tag{8.3}$$

the solution of which is readily found to be, in $(R, \; z, \; \phi)$ coordinates, of the form $\boldsymbol{\mathcal{A}} = (0, 0, \mathcal{A}R)$ where

$$\mathcal{A} = \frac{\mu R}{r^3} \left(2 + \frac{q}{r}\right) . \tag{8.4}$$

Electromagnetic potentials A_0 and Φ follow from (5.4). Thus,

$$A_0 = 1 - f\left(1 + \frac{q}{r}\right) \ , \quad \Phi = -\frac{f\mu(z+b)}{r^3} \ , \tag{8.5}$$

and $F_{\mu\nu}$ or $\mathbf{E}$, $\mathbf{B}$ are readily deduced from (8.5). The disk is associated with two functions of R defined in (3.6);

$$\zeta = -\left(\partial_z f^{-1}\right)_{z=0} = \frac{2b}{r^3}\left(q + \frac{q^2}{r} - \frac{\mu^2}{r^3} + \frac{3\mu^2 b^2}{r^5}\right) \ , \tag{8.6}$$

and

$$\chi = +\left(\partial_z \mathcal{A}\right)_{z=0} = -\frac{\mu b R}{r^5}\left(6 + 4\frac{q}{r}\right) \ . \tag{8.7}$$

Notice that

$$r(z=0) = \sqrt{R^2 + b^2} \ . \tag{8.8}$$

Since $b > 0$ and $q > 0$, the energy condition $\zeta > 0$ is satisfied for every $b < r < \infty$.

Electromagnetic currents i_Q^0 , i_Q^3 are, according to (5.9), (5.10), defined in terms of

$$\left(\partial_z A_0\right)_{0+} = f^2\left[\frac{qb}{r^3} f^{-1} - \left(1 + \frac{q}{r}\right)\zeta\right] = -2\pi f^{1/2}\left(i_Q^0 - \mathcal{A} R i_Q^3\right) \ , \tag{8.9}$$

and

$$\frac{1}{R}\left(\partial_R \Phi\right)_{0+} = -f^2 \frac{\mu b}{r^5}\left[\frac{3\mu^2 b^2}{r^6} - \left(1 + \frac{q}{r}\right)\left(3 + \frac{q}{r}\right)\right] = -2\pi f^{-1/2} i_Q^3 \ . \tag{8.10}$$

Spacetime becomes flat at infinity in all directions. In particular, for $R \to \infty$ in the disk ($z = 0$)

$$f \simeq 1 - \frac{2q}{R} + \frac{3q^2}{R^2} \ , \quad \mathcal{A} \simeq +\frac{2\mu}{R^2} \ , \quad (R \to \infty, z = 0) \tag{8.11}$$

$$\zeta = \frac{2bq}{R^3} + \frac{2bq^2}{R^4} \ , \ \chi \simeq -\frac{6b\mu}{R^4} \ ; \ R \to \infty \ . \tag{8.12}$$

From this follows that $\kappa\sigma$ and $\kappa\Pi$ defined in (4.4) become

$$\kappa\sigma = \frac{4bq}{R^3} \ , \quad \kappa\Pi = -\frac{9b\mu^2}{qR^5} \ , \quad R \to \infty \ , \tag{8.13}$$

and

$$\Omega R \simeq -\frac{3\mu}{2qR} \ , \quad R \to \infty \ . \tag{8.14}$$

Electric charges and currents behave as follows:

$$2\pi i_Q^0 = \frac{bq}{R^3} \ , \quad 2\pi i_Q^3 = -\frac{3b\mu}{R^5} \ , \quad R \to \infty \ , \tag{8.15}$$

while

$$G i_M^0 \simeq G\sigma \simeq \frac{1}{2\pi}\frac{bq}{R^3} \ , \quad G i_M^3 = G\sigma\Omega \simeq -\frac{3}{4\pi}\frac{b\mu}{R^5} \ . \tag{8.16}$$

Thus,

$$G i_M^0 \simeq i_Q^0, \tag{8.17}$$

i.e., "mass and charge density become equal" at great distances while

$$G i_Q^3 = \frac{1}{2} i_Q^3 \tag{8.18}$$

Let us finally evaluate the "constants of motions" with all the G's and c's. According to (2.3) and following (8.2), $g_{00} = f = 1 - 2q/r + O(r^{-2})$. This implies, as is well known, that the total mass-energy of this spacetime is

$$M = \frac{qc^2}{G}. \tag{8.19}$$

On the other hand in x, y, z coordinates, the components of the gravomagnetic potential $\mathcal{A}_k$ defined by (2.1) can be calculated with $\mathcal{A}$ given in (8.4). Thus, in x, y, z coordinates we find that $g_{01} = \frac{2\mu}{r^3} y + O(r^{-3}), g_{02} = -\frac{2\mu}{r^3} x + O(r^{-3})$ and $g_{03} = O(r^{-3})$. So, following for instance Carmeli (1982),[17] equation (12), p.212, we see that the total angular momentum J_M is in the z direction and is given by

$$J_M = \frac{\mu c^3}{G} \tag{8.20}$$

The electric potential given in (8.5) can be written $\mathcal{A}_0 = \frac{q}{r} + O(r^{-2})$. Thus the total charge is

$$Q = \frac{qc^2}{\sqrt{G}} \tag{8.21}$$

On the other hand the components of the magnetic field which are defined in (7.7) can be calculated from (5.3) in which Φ is given in (8.5). At great distance $-B^k = B_k \simeq -\partial_k \Phi$ from which follows that $B_k = \frac{\mu}{r^3}(\delta^{k3} - 3n^k n^3) +$

$O(r^{-4})$ with $n^k = x^k/r$. This is formula (44.4) of Landau and Lifshitz (1971)[18] when the total magnetic momentum

$$J_Q = \frac{\mu c^3}{\sqrt{G}} \tag{8.22}$$

We have thus the following relations among the first integrals of this spacetime :

$$Q = \sqrt{G}M \quad , \quad J_Q = \sqrt{G}J_M \tag{8.23}$$

We see that the gyromagnetic ratio $J_Q/\sqrt{G}J_M = 1$.

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