arXiv:hep-th/0010215 v2 22 Jan 2001

Perturbations of brane worlds

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(October 6, 2002)

We consider cosmological models where the universe, governed by Einstein's equations, is a piece of a five dimensional double-sided anti-de Sitter spacetime (that is, a "Z_2-symmetric bulk") with matter confined to its four dimensional Robertson-Walker boundary or "brane". We study the perturbations of such models. We use conformally minkowskian coordinates to disentangle the contributions of the bulk gravitons and of the motion of the brane. We find the restrictions put on the bulk gravitons when matter on the brane is taken to be a scalar field and we solve in this case the brane perturbation equations.

98.80.Cq, 98.70.Vc

I INTRODUCTION

In a now classic paper [1] Randall and Sundrum indicated how one could recover the linearized Einstein equations on a four dimensional minkowskian "brane", a brane being in that context a boundary of a five dimensional double-sided anti-de Sitter spacetime (AdS_5), that is of a "Z_2-symmetric bulk". This discovery was soon followed by the building of cosmological models, where the brane, instead of flat, is taken to be a Robertson-Walker spacetime, and it was shown that such "brane worlds" can tend at late times to the standard big bang model and hence represent the observed universe (see e.g. [2] for early models and [3-5] for fully relativistic ones).

More recently various theoretical set ups to study the perturbations of such cosmological models have been proposed [6-13]. The purpose of these analyses is, in particular, to eventually calculate the Cosmic Microwave Background anisotropies predicted by brane worlds. However,

they all have up to now stalled on the problem of solving, in a general manner, the Lanczos-Darmois-Israel equations (that is the Einstein equations integrated across the brane, often called "junction conditions") which relate the matter perturbations on the brane and the perturbations in the bulk.

In order to be in a position to solve these equations, we present in this paper the perturbation theory of brane worlds from a geometrical point of view, in the line of [6], [7] and [10]. This approach, which uses conformally minkowskian coordinates that are well adapted to the geometry of the bulk, will first allow a clear and simple distinction between the perturbations in the brane due to perturbations in the bulk and the perturbations in the brane due to its motion. (As we shall comment upon, the latter, so-called "brane-bending" effect, is more difficult to describe when gaussian coordinates are used, as in e.g. [8], [9], [11] or [13].)

We shall then write the Lanczos-Darmois-Israel equations and see that only a sub-class of bulk gravitational waves is compatible with a given type of matter on the brane. As an example we shall consider the case when matter on the brane is imposed to be a scalar field and find explicitly in that case the allowed bulk gravitational waves. We shall then solve the Lanczos-Darmois-Israel equations and give in a closed form the perturbed metric and scalar field in the brane.

The paper is organized as follows. In section 2 we present the formalism and notations used and describe the background brane and bulk. Section 3 treats the geometry and matter perturbations in the brane induced by a bending of the brane in a strictly anti-de Sitter bulk. As for section 4, it considers the changes in the brane induced by perturbations of the bulk. There is nothing essentially new in these sections 3 and 4, but the presentation is, we hope, more straightforward and pedagogical than some. In section 5 we dwell on gauge issues, count the degrees of freedom of the perturbations in the brane and comment upon the use of gaussian normal coordinates in which, as we shall argue, the brane bending effect is described in a fairly awkward manner. We recall in section 6 standard results of the linearized Einstein equations in an anti-de Sitter spacetime in conformally minkowskian coordinates. In section 7 the Lanczos-Darmois-Israel equations are written and solved when matter on the brane is taken to be a scalar field.

II THE BACKGROUND BULK AND BRANE IN CONFORMALLY MINKOWSKIAN COORDINATES

The "bulk" is a piece of a five dimensional spacetime of which the four dimensional edge, or "brane", is supposed to represent our universe. At zeroth order in perturbation theory this background bulk will be chosen to be an anti-de Sitter spacetime (see below a reason why we do not consider a de Sitter bulk and e.g. [14-15] and references therein for more general backgrounds).

Many different coordinate systems can be used to describe anti-de Sitter spacetime: see e.g. [1] or [5] for normal gaussian coordinates in which the surface $y = 0$ represents the brane, e.g. [16] for Schwarzschild-like coordinates and [17] for their local equivalence. In this paper we

shall use conformally minkowskian coordinates X^A, in which the metric of a five dimensional anti-de Sitter spacetime reads

$$d\bar{s}^2|_5 = \bar{g}_{AB}\, dX^A dX^B \quad \text{with} \quad \bar{g}_{AB} = \frac{1}{(\mathcal{K}X^4)^2}\, \eta_{AB} \tag{1}$$

where the upper case indices A, B run from 0 to 4, $\eta_{AB} = \mathrm{diag}(-1,1,1,1,1)$ and $\mathcal{K}$ is a positive constant. Note that the coordinates X^A do not cover the whole AdS_5 spacetime; see appendix A for details.

The background brane is a four dimensional surface in AdS_5 with maximally symmetric spatial sections. We shall restrict our attention to those Robertson-Walker branes which have euclidean spatial sections. The equation for such a brane $\bar{\Sigma}$ is

$$X^A = \bar{X}^A(x^\mu) \qquad \text{with} \qquad \bar{X}^0 = T(\eta) \quad , \quad \bar{X}^i = x^i \quad , \quad \bar{X}^4 = A(\eta) \tag{2}$$

where the four coordinates $x^\mu\,(x^0 \equiv \eta, x^i)$, lower case latin indices running from 1 to 3, parametrize the brane, where $A(\eta)$ is an a priori arbitrary function of η and where $T(\eta)$ is defined up to an arbitrary constant by

$$T' = \sqrt{1 + A'^2} \tag{3}$$

a prime denoting a derivative with respect to η. This condition defines η as conformal time. Four independent tangent vectors to the brane are

$$\bar{V}^A_\mu \equiv \frac{\partial \bar{X}^A}{\partial x^\mu} \qquad \text{that is} \qquad \bar{V}^A_\eta = (T', 0, 0, 0, A') \quad , \quad \bar{V}^A_i = (0, \delta^A_i, 0). \tag{4}$$

The induced metric on the brane is also conformally minkowskian and reads

$$d\bar{s}^2|_4 = \bar{g}_{AB}|_{\bar{\Sigma}}\, \bar{V}^A_\mu \bar{V}^B_\nu\, dx^\mu dx^\nu = \frac{1}{(\mathcal{K}A)^2}\, \eta_{\mu\nu}\, dx^\mu dx^\nu. \tag{5}$$

It will be useful in the following to introduce the scale factor $a(\eta)$, the cosmic time t, and the Hubble parameter H defined by

$$a \equiv \frac{1}{\mathcal{K}A} \quad , \quad dt \equiv a d\eta \quad , \quad H \equiv \frac{\dot{a}}{a} \tag{6}$$

where a dot denotes a derivative with respect to t.

At this stage one can note in passing that if the bulk had been chosen to be a de Sitter rather than an anti-de Sitter spacetime, given by metric (1) with conformal factor $(\mathcal{K}X^0)^{-2}$ instead of $(\mathcal{K}X^4)^{-2}$, then the induced metric on a brane defined by $X^0 = A(\eta),\ X^i = x^i,\ X^4 = T(\eta)$ would have been (5) with the condition (3) replaced by $T' = \sqrt{A'^2 - 1} \Longrightarrow A'^2 \geq 1 \Longrightarrow H \geq \mathcal{K}$, a condition which is not fulfilled by standard cosmological scenarios. Hence the bulk cannot be chosen to be a de Sitter spacetime, at least when the branes are defined as above. A Minkowski bulk does not lead either to an acceptable cosmological scenario, see [3] and e.g. [18].

Let us recall now how the extrinsic curvature of a brane is calculated. One introduces the normal $\bar{n}^A(x^\mu)$ to the brane as

$$\bar{g}_{AB}|_{\bar{\Sigma}}\,\bar{n}^A\bar{V}^B_\mu = 0 \quad , \quad \bar{g}_{AB}|_{\bar{\Sigma}}\,\bar{n}^A\bar{n}^B = 1 \quad \text{that is} \quad \bar{n}^A = \mathcal{K}A\,(A',0,0,0,T') \tag{7}$$

where the sign has been chosen arbitrarily. The extrinsic curvature of the brane is then defined as (introducing $\bar{D}$, the covariant derivative associated with $\bar{g}_{AB}$) $\bar{K}_{\mu\nu} = -\bar{V}^A_\mu\bar{V}^B_\nu\bar{D}_A\bar{n}_B$, that is, using the symmetry $\bar{K}_{\mu\nu} = \bar{K}_{\nu\mu}$,

$$\bar{K}_{\mu\nu} = -\frac{1}{2}\left[\bar{g}_{AB}|_{\bar{\Sigma}}\,(\bar{V}^A_\mu\partial_\nu\bar{n}^B + \bar{V}^A_\nu\partial_\mu\bar{n}^B) + \bar{V}^A_\nu\bar{V}^B_\mu\bar{n}^C(\partial_C\bar{g}_{AB})|_{\bar{\Sigma}}\right] \tag{8}$$

which gives

$$\begin{aligned}\bar{K}_{\eta\eta} &= \frac{1}{\mathcal{K}A^2T'}(AA'' - 1 - A'^2) = -\frac{a^2}{\sqrt{\mathcal{K}^2+H^2}}\left(\mathcal{K}^2 + \frac{\ddot{a}}{a}\right)\\ \bar{K}_{ij} &= \frac{T'}{\mathcal{K}A^2}\delta_{ij} = a^2\sqrt{\mathcal{K}^2+H^2}\,\delta_{ij}\,.\end{aligned} \tag{9}$$

Physics, gravity and matter, are introduced in this up to now purely geometrical picture by means of Einstein's equations, $G_{AB} = \kappa T_{AB}$, where G_{AB} is Einstein's tensor, κ a five dimensional gravitational coupling constant and T_{AB} the stress energy tensor of matter. The AdS_5 bulk is then interpreted as a solution of Einstein's equations with matter a cosmological constant: $\kappa T_{AB}|_{\text{bulk}} = 6\mathcal{K}^2\bar{g}_{AB}$. Furthermore, matter is introduced on the brane by means of the so-called Z_2 symmetry which amounts to

(1) cut AdS_5 along the brane (see e.g. [10] or [17] for conformal diagrams and appendix A for embedding descriptions of this cut),

(2) keep the side between the brane and $X^4 \to +\infty$ [10],

(3) make a copy of this "half" AdS_5 spacetime and join it to the original along the brane (hence the description of the bulk as a doubled-sided piece of AdS_5),

(4) integrate Einstein's equations across this singular surface and obtain the Lanczos-Darmois-Israel equations often called "junctions conditions" [19] and thus get the stress energy tensor $\bar{\mathcal{T}}_{\mu\nu}$ of the matter on the brane in terms of its extrinsic curvature as

$$\kappa\left(\bar{\mathcal{T}}^\nu_\mu - \frac{1}{3}\delta^\nu_\mu\,\bar{\mathcal{T}}\right) = 2\bar{K}^\nu_\mu \tag{10}$$

where the indices are raised by means of the inverse metric $a^{-2}\eta^{\mu\nu}$ and $\bar{\mathcal{T}} \equiv \bar{\mathcal{T}}^\mu_\mu$. Condition (2) above together with the choice of sign in equation (7) ensures that the energy density on the brane is definite positive [20]. The spatial components of (10) give, using (8) and noting $\bar{\mathcal{T}}^0_0 = -\rho$, $\bar{\mathcal{T}}^i_j = p\delta^i_j$,

$$\kappa\rho = 6\mathcal{K}\sqrt{1+A'^2} \quad \Longleftrightarrow \quad \kappa\rho = 6\sqrt{\mathcal{K}^2+H^2}\,. \tag{11}$$

As for the $(0,\mu)$ components of (10) they are equivalent to the conservation of $\bar{\mathcal{T}}^\nu_\mu$, because matter in the bulk reduces to a cosmological constant (see e.g. [20]). They read

$$\dot{\rho} + 3H(\rho+p) = 0 \quad \Longleftrightarrow \quad \kappa p = 2\frac{\mathcal{K}}{T'}(AA'' - 3T'^2)\,. \tag{12}$$

One thus sees in particular that Einstein's equations impose that minkowskian branes, such that $A(\eta) =$ constant, must contain matter under the form of a "tension" such that $\kappa\bar{\mathcal{T}}_\nu^\mu = -6\mathcal{K}\delta_\nu^\mu$ [1]. More generally, an equation of state $\rho = 6\mathcal{K}/\kappa + \rho_m$, $p = -6\mathcal{K}/\kappa + p_m$ with $p_m = p_m(\rho_m)$ being chosen, equations (11-12) together with (3) give $A(\eta)$ (or $a(t)$) as well as $\rho(\eta)$. Various cosmological scenarios can hence be built [2-5].

III BRANE-BENDING IN AN ANTI-DE SITTER BULK

In this section we shall consider an unperturbed anti-de Sitter bulk that we shall describe using the conformally minkowskian coordinates (1). We therefore do not allow here for perturbations of the coordinate system in the bulk; cf. section V for gauge related issues. On the other hand we allow for perturbations in the position of the brane - this is the "brane-bending" effect analyzed by e.g. [21] in the case of a minkowskian background brane. In other words we consider in AdS_5 with metric (1) a brane Σ defined by

$$X^A = \bar{X}^A(x^\mu) + \epsilon^A(x^\mu) \tag{13}$$

where $\bar{X}^A(x^\mu)$ are given by (2-3) and where the five "small" functions $\epsilon^A(x^\mu)$ can be conveniently decomposed along the four tangent vectors to the brane (4) and its normal (7) according to

$$\epsilon^A = \xi^\lambda\, \bar{V}_\lambda^A + \zeta\, \bar{n}^A \tag{14}$$

with $\xi^\lambda(x^\mu)$ and $\zeta(x^\mu)$ five arbitrary functions of the coordinates $x^\mu = (\eta, x^i)$ which parametrize the brane. Strictly speaking the tangent and normal vectors $\bar{V}_\lambda^A$ and $\bar{n}^A$ are defined on the unperturbed brane only. The vectors which appear in (14) are Lie transported to the perturbed brane. (See [12] for equivalent parallel transport.)

When $\zeta = 0$ there is no deformation of the brane and the perturbation ϵ^A amounts to a slight change in its parametrization, which can be absorbed into the infinitesimal change of coordinates: $x^\lambda = \tilde{x}^\lambda - \xi^\lambda$ (as it is easy to show explicitly; cf. appendix B). We shall therefore set $\xi^\lambda = 0$ and describe the deformation of the brane by the single function ζ. Hence the gauge is completely fixed, in the brane as well as in the bulk.

A short calculation then shows that the induced metric on the perturbed brane

$$ds^2|_4 = \bar{g}_{AB}|_\Sigma \left[\bar{V}_\mu^A + \partial_\mu(\zeta\bar{n}^A)\right]\left[\bar{V}_\nu^B + \partial_\nu(\zeta\bar{n}^B)\right] dx^\mu dx^\nu \tag{15}$$

can be expressed in terms of the background brane extrinsic curvature $\bar{K}_{\mu\nu}$ (8) as

$$ds^2|_4 = \frac{1}{(\mathcal{K}A)^2}\left(\eta_{\mu\nu} + \gamma_{\mu\nu}^{(p)}\right) dx^\mu dx^\nu \quad \text{with} \quad \gamma_{\mu\nu}^{(p)} = -2(\mathcal{K}A)^2\zeta\bar{K}_{\mu\nu} \tag{16}$$

where the index (p) stands for perturbation of the *position* of the brane. (This perturbation cannot be gauged away, unless the background brane extrinsic curvature tensor vanishes.) In terms of the scale factor, cosmic time and Hubble parameter it reads

$$ds^2|_4 = -\left[1 - \frac{2\zeta}{\sqrt{\mathcal{K}^2 + H^2}}\left(\mathcal{K}^2 + \frac{\ddot{a}}{a}\right)\right] dt^2 + a^2\left(1 - 2\zeta\sqrt{\mathcal{K}^2 + H^2}\right)\delta_{ij}dx^i dx^j. \tag{17}$$

Hence, for a scale factor behaving as t^p, the induced metric remains bounded if the function ζ behaves as t^q, $q \geq 1$, at early times.

The normal vector to the perturbed brane and its intrinsic curvature $K_{ij}^{(p)}$ are obtained from the definitions (7-8) with all bars dropped, apart from the one on $\bar{g}_{AB}$. One obtains, denoting the perturbation of the extrinsic curvature of the brane due to its bending as $\delta^{(p)}K_j^i \equiv K_j^{i\,(p)} - \bar{K}_j^i$,

$$\begin{aligned}\delta^{(p)}K_j^i &= (\mathcal{K}A)^2\left[\partial_j^i\zeta + \delta_j^i\left(\frac{A'\zeta'}{A} + \frac{A'^2\zeta}{A^2}\right)\right] \\ &= \frac{1}{a^2}\partial_j^i\zeta + H(H\zeta - \dot{\zeta})\,\delta_j^i. \end{aligned}\tag{18}$$

(The indices of $K_j^{i\,(p)}$ are raised by means of the metric (17) and $\partial_j^i\zeta \equiv \delta^{ik}\partial_{jk}\zeta$.)

IV PERTURBING THE GEOMETRY OF THE BULK

In this section we consider a perturbed anti-de Sitter bulk with metric

$$ds^2|_5 = g_{AB}dX^A dX^B = \frac{1}{(\mathcal{K}X^4)^2}(\eta_{AB} + h_{AB})dX^A dX^B \tag{19}$$

where, among the fifteen function $h_{AB}(X^C)$, five have been chosen to fix the gauge in the bulk and the remaining ten are imposed not to be reducible to zero by a change of coordinates. As for the brane $\bar{\Sigma}$ it is defined by the *same* equations as in the unperturbed case, that is by equations (2-3). Of course this brane, despite its notation, is geometrically different from the unperturbed Robertson-Walker brane of section II.

The induced metric on $\bar{\Sigma}$ is

$$ds^2|_4 = g_{AB}|_{\bar{\Sigma}}\,\bar{V}_\mu^A\bar{V}_\nu^B dx^\mu dx^\nu = \frac{1}{(\mathcal{K}A)^2}(\eta_{\mu\nu} + \gamma_{\mu\nu}^{(b)})dx^\mu dx^\nu\,, \tag{20}$$

with

$$\begin{aligned}\gamma_{\eta\eta}^{(b)} &= T'^2 h_{00}|_{\bar{\Sigma}} + 2T'A'h_{04}|_{\bar{\Sigma}} + A'^2 h_{44}|_{\bar{\Sigma}} \\ \gamma_{\eta i}^{(b)} &= T'h_{0i}|_{\bar{\Sigma}} + A'h_{4i}|_{\bar{\Sigma}} \\ \gamma_{ij}^{(b)} &= h_{ij}|_{\bar{\Sigma}} \end{aligned}\tag{21}$$

where the index (b) means that these perturbations are induced by the perturbations of geometry of the *bulk*.

As for the normal vector to the brane and its extrinsic curvature $K_{ij}^{(b)}$ they are again defined by (7-8) where, here, all bars are kept, apart from the one on g_{AB}. One obtains for the perturbation of the extrinsic curvature of the brane due to the perturbations of geometry of the bulk, $\delta^{(b)}K_{ij} \equiv K_{ij}^{(b)} - \bar{K}_{ij}$,

$$\delta K^{(b)}_{ij} = \frac{1}{2\mathcal{K}A}\left[A'(\partial_j h_{0i} + \partial_i h_{0j} - \partial_0 h_{ij})|_{\bar\Sigma} + T'(\partial_j h_{4i} + \partial_i h_{4j} - \partial_4 h_{ij})|_{\bar\Sigma}\right] + \\ + \frac{1}{\mathcal{K}A^2}\left\{\delta_{ij}\left[T'(A'^2 - \tfrac{1}{2}T'^2)h_{44}|_{\bar\Sigma} + A'^3 h_{04}|_{\bar\Sigma} + \tfrac{1}{2}A'^2 T' h_{00}|_{\bar\Sigma}\right] + T' h_{ij}\right\}. \tag{22}$$

This expression can be rewritten in a more compact and geometrical form as

$$\delta^{(b)} K^i_j = \pi^i_j - \tfrac{1}{2}\sigma^i_j - a\left(H\gamma^{(b)}_\eta + \frac{a}{2}\sqrt{\mathcal{K}^2 + H^2}\,\gamma^{(b)}\right)\delta^i_j \tag{23}$$

where $\delta^{(b)} K^i_j = K^{i\,(b)}_j - \bar K^i_j$ (indices being raised by means of the metric(20)) and where we have introduced

$$\sigma_{ij} = \bar n^A(\partial_A h_{ij})|_{\bar\Sigma} \qquad \text{and} \qquad \pi_{ij} = \frac{1}{2}[\bar n^A(\partial_j h_{Ai})|_{\bar\Sigma} + \bar n^A(\partial_i h_{Aj})|_{\bar\Sigma}]\,, \tag{24}$$

(with $\pi^i_j \equiv \delta^{ik}\pi_{jk}$, $\sigma^i_j \equiv \delta^{ik}\sigma_{jk}$) as well as

$$\gamma^{(b)}_\eta = h_{AB}|_{\bar\Sigma}\,\bar n^A \bar V^B_\eta \qquad \text{and} \qquad \gamma^{(b)} = h_{AB}|_{\bar\Sigma}\,\bar n^A \bar n^B. \tag{25}$$

Using (4) and (7) one finds explicitly

$$\gamma^{(b)}_\eta = \frac{1}{\mathcal{K}^2 a}\left[-H\sqrt{\mathcal{K}^2 + H^2}(h_{00}|_{\bar\Sigma} + h_{44}|_{\bar\Sigma}) + (\mathcal{K}^2 + 2H^2)h_{04}|_{\bar\Sigma}\right] \\ \gamma^{(b)} = \frac{1}{\mathcal{K}^2 a^2}\left[H^2 h_{00}|_{\bar\Sigma} - 2H\sqrt{\mathcal{K}^2 + H^2}h_{04}|_{\bar\Sigma} + (\mathcal{K}^2 + H^2)h_{44}|_{\bar\Sigma}\right]. \tag{26}$$

We postpone until section VII the interpretation of the perturbations of extrinsic curvature in terms of matter perturbations on the brane.

V GAUGE RELATED ISSUES

In section III we considered a strictly anti-de Sitter bulk in conformally minkowskian coordinates and we perturbed the position of the brane along its normal. In so doing, as we have seen, we fixed the gauge completely, in the bulk as well as in the brane, and introduced a single function $\zeta(x^\mu)$ the effect of which on the induced metric and extrinsic curvature of the brane cannot be gauged away and is given by equations (16-17) and (18).

In the previous section we geometrically perturbed the bulk in a given coordinate system and fixed the position of the brane in that system. We have thus introduced ten functions $h_{AB}(X^C)$ which characterize in a given coordinate system a geometrical perturbation of the bulk, i.e. we imposed that they cannot be gauged away. Their effect on the brane is given by equations (20-21) and (22) or (23).

The brane bending perturbation adds a degree of freedom to these ten perturbations of the bulk since it can be present in the geometrically unperturbed background AdS_5 bulk. We note in passing that this effect can be described alternatively as a brane perturbation induced by a general coordinate shift performed in the background AdS_5 bulk *without* changing accordingly the equation for the brane. See appendix B for details.

Hence, the eleven functions introduced describe completely the geometrical perturbations of the bulk and the position of the brane in that bulk. Now these eleven independent functions will be constrained in the next sections to satisfy Einstein's equations. Imposing in a first step Einstein's equations in the five dimensional bulk, where matter is chosen to be a cosmological constant, will reduce these eleven functions to six (according to the rule "the gauge hits twice"). These six functions will be interpreted as the five degrees of freedom of the AdS_5 gravitational waves plus a "radion" describing the motion of the brane. Imposing in a second step the Z_2 symmetry and Einstein's equations across the singular brane will define the matter perturbations on the brane in terms of these six arbitrary functions, which is just the right number to describe, in a given brane coordinate system, the most general four dimensional perturbed universe.

This counting can be generalized to any N-dimensional brane in a $D = N + 1$-dimensional bulk. The number of gauge independent metric perturbations (or, equivalently, the number of independent metric perturbations in a given gauge) in a D-dimensional bulk is $\frac{1}{2}D(D+1)-D = \frac{1}{2}D(D-1)$. The number of freely propagating degrees of freedom (gravitational waves) in a D-dimensional bulk is $\frac{1}{2}D(D-1) - D = \frac{1}{2}D(D-3)$. The deformation of a $D - 1 = N$-dimensional brane is described by the normal vector $\zeta\bar{n}^A$, that is by one function. Now we have $\frac{1}{2}D(D-3) + 1 = \frac{1}{2}N(N-1)$ which is the number of gauge independent metric perturbations in a N-dimensional brane.

We would like to argue at this point that normal gaussian coordinates (used by e.g. [7], [9], [12], [13]) in which the perturbed bulk metric is written as

$$ds^2|_5 = (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dy^2 \tag{27}$$

where $y = 0$ is the position of the brane and where the explicit expression of the background anti-de Sitter metric coefficients $g_{\mu\nu}$ can be found in [15], seem less appropriate to treat the problem at hand than the conformally minkowskian coordinates advocated here. Indeed one can certainly use the form (27) of the metric to study linearized gravity on an anti-de Sitter background. However

(1) the linearized Einstein equations are much simpler, and their boundary conditions much easier to implement, when written in conformally minkowskian coordinates (as recalled in the next section),

(2) imposing that the brane is at $y = 0$ means choosing among all coordinate systems such that $h_{yy} = h_{y\mu} = 0$, the particular sub-class which is adapted to the bending of the brane [7]. This implies that when solving the linearised Einstein equations in the bulk one can no longer simplify them by choosing the best adapted coordinate system within the class (1) (like, for example, an harmonic system). In practice this means that one must solve the constraint equations in full generality. This introduces an arbitrary function $\zeta(x^\mu)$ which encodes the brane bending effect as well as the transformation to the coordinate system in which the solution of the constraint equations is simple and the brane located at $y = \zeta(x^\mu)$. (A similar procedure must be applied in synchronous gauge descriptions of the surface defining reheating in inflationary scenarios, see e.g. [22] for further discussion of that point.)

When the background brane is Minkowski spacetime, as in the Randall-Sundrum scenario, conformally minkowskian and gaussian normal coordinates are almost identical, so that objection (1) falls in that case. As for objection (2) it falls as well since it is then as simple to solve the linearised Einstein equations in an harmonic gauge where the brane is located at $y = \zeta(x^\mu)$ than in a gaussian normal gauge (cf. e.g. [21] and [23]). When, on the other hand the background brane is a Robertson-Walker spacetime, the simplicity of the linearised Einstein equations in conformally minkowskian coordinates discussed in section VI will, as we hope to convince the reader, compensate for the slightly more complicated form of the junction conditions given in section VII.

VI EINSTEIN'S EQUATIONS IN THE BULK

The metric perturbations of the bulk we considered in section IV are now forced to obey Einstein's equations, matter being chosen to be a cosmological constant: $G_{AB} = 6\mathcal{K}^2 g_{AB}$. The metric being given by (19) their linearization gives the following equations for the perturbations h_{AB} everywhere outside the brane [23]

$$\begin{aligned} &\frac{1}{2}\Big[\partial_{AL}h^L_B + \partial_{BL}h^L_A - \partial_{AB}h - \Box_5 h_{AB} - \eta_{AB}(\partial_{LM}h^{LM} - \Box_5 h)\Big] \\ &-\frac{6}{(X^4)^2}\eta_{AB}h_{44} - \frac{3}{2X^4}\Big[\partial_A h_{4B} + \partial_B h_{4A} - \partial_4 h_{AB} + \eta_{AB}(\partial_4 h - 2\partial_L h^L_4)\Big] = 0 \end{aligned} \tag{28}$$

where all indices are raised with η^{AB}, $h \equiv h^L_L$ and $\Box_5 \equiv \partial_L\partial^L$.

These equations must be solved in a given gauge. If we impose the conditions

$$h_{4A} = 0 \tag{29}$$

the fifteen functions $h_{AB}(X^A)$ reduce to the ten functions $h_{\mu\nu}(X^A)$ which satisfy

$$\begin{aligned} \partial_{\rho\sigma}h^{\rho\sigma} - \Box_4 h + \frac{3}{X^4}\partial_4 h &= 0 \\ \partial_4(\partial_\rho h^\rho_\mu - \partial_\mu h) &= 0 \\ \partial_{44}h - \frac{1}{X^4}\partial_4 h &= 0 \\ \Box_4 h_{\mu\nu} + \partial_{44}h_{\mu\nu} - \frac{3}{X^4}\partial_4 h_{\mu\nu} &= \partial_{\mu\rho}h^\rho_\nu + \partial_{\nu\rho}h^\rho_\mu - \partial_{\mu\nu}h + \frac{\eta_{\mu\nu}}{X^4}\partial_4 h\,. \end{aligned} \tag{30}$$

The first three constraint equations are easily solved and one then chooses (a choice that one cannot make when using gaussian normal coordinates) the coordinate system satisfying condition (29) such that the solution reduces to

$$h \equiv \eta^{\rho\sigma}h_{\rho\sigma} = 0 \qquad , \qquad \partial_\rho h^\rho_\mu = 0\,. \tag{31}$$

Hence the choice of gauge together with the constraint equations reduce the ten functions $h_{\mu\nu}(X^A)$ to five, which represent the five degrees of freedom of AdS_5 gravitational waves.

As for the fourth evolution equation (30) it is solved by separation of variables. Inserting the ansatz

$$\hat{h}_{\mu\nu} = (mX^4)^2 Z_2(mX^4)\, e_{\mu\nu}(k^i, m)\, e^{\mathrm{i}\, k_\rho X^\rho} \tag{32}$$

where m and k^i are the four separation constants one obtains

$$\frac{d}{dX^4}\frac{dZ_2}{dX^4} + \frac{1}{X^4}\frac{dZ_2}{dX^4} + \left(m^2 - \frac{4}{(X^4)^2}\right) Z_2 = 0 \tag{33}$$

and

$$k^\rho k_\rho = -m^2 \qquad \Longrightarrow \qquad k_0 = -\sqrt{k_i k^i + m^2}\,. \tag{34}$$

For $m \neq 0$ the general solution of (33), which represents the "Kaluza-Klein excitations" introduced in this context by [1], is a combination of Bessel functions of order 2 [24]

$$Z_2(mX^4) = a_m J_2(mX^4) + b_m N_2(mX^4) \tag{35}$$

where a_m and b_m are a priori arbitrary constants.

A word of caution is in order here. Since the conformally flat coordinates are unsuited to describe the universal covering of AdS_5 (see appendix A and references quoted therein), one may have to impose a boundary condition on Z_2 at $X^4 \to +\infty$. There does not seem to be an agreement on that point in the literature. For example the authors of ref. [1], [21], [23] do not impose any condition at $X^4 \to +\infty$. As for [25] they impose $b_m = 0$ whereas [10], [26] or [27] choose $a_m = -\mathrm{i} b_m$ (cf. also [28]). We shall leave this question open here and rather make the following remark. The "zero-mode" perturbation is the $m = 0$ bounded and normalizable solution of (33). It behaves as $Z_2 \propto (X^4)^{-2}$ so that in that case $\hat{h}_{\mu\nu}$ does not depend on X^4 and can be considered as the limit when $mX^4 \to 0$ of the Bessel function $N_2(mX^4)$. One may therefore advocate the condition

$$a_m = 0 \tag{36}$$

so that the bounded zero mode and the $m \neq 0$ modes form a uniform family of states.

Finally the constraint equations (31) impose

$$k^\rho e_{\rho\mu} = 0 \qquad \text{and} \qquad \eta^{\rho\sigma} e_{\rho\sigma} = 0\,. \tag{37}$$

To summarize, the general solution of the linearized Einstein equations in an AdS_5 background is

$$ds^2|_5 = \frac{1}{(\mathcal{K}X^4)^2}(\eta_{AB} + h_{AB}) dX^A dX^B$$

with

$$h_{A4} = 0 \quad \text{and} \quad h_{\mu\nu} = \int dm\, d^3k\, \hat{h}_{\mu\nu}(X^A, k^i, m) \tag{38}$$

where the mode, or gravitational wave, $\hat{h}_{\mu\nu}$ is given by (32) (34) and (35), where $e_{\mu\nu}$ is transverse and traceless (eq. (37)) and where the additional condition (36) ensures that the massive modes tend to the bounded zero mode when $m \to 0$.

VII THE LANCZOS-DARMOIS-ISRAEL EQUATIONS

We now turn to the matter perturbations on the brane. They are obtained by imposing the Lanczos-Darmois-Israel equations (10) with all bars dropped and with $K^i_j = \bar{K}^i_j + \delta^{(p)}K^i_j + \delta^{(b)}K^i_j$, $\delta^{(p)}K^i_j$ and $\delta^{(b)}K^i_j$ being given respectively by equations (18) and (23), it being understood now that the perturbations in the bulk are given by (38). Hence, just using the fact that we chose the gauges $h_{4A} = 0$, we have that

$$\frac{\kappa}{2}\delta\left(\mathcal{T}^i_j - \frac{1}{3}\delta^i_j\mathcal{T}\right) = \frac{1}{a^2}\,\partial^i_j\zeta + H\,\delta^i_j\left(H\zeta - \dot{\zeta}\right) + \frac{H^2}{2\mathcal{K}^2}\,\delta^i_j\,\sqrt{\mathcal{K}^2 + H^2}\,h_{00}|_{\bar{\Sigma}} + \frac{1}{2\mathcal{K}a}\left[H(\partial_0 h^i_j)|_{\bar{\Sigma}} - \sqrt{\mathcal{K}^2 + H^2}\,(\partial_4 h^i_j)_{\bar{\Sigma}} - H(\partial_j h^i_0 + \partial^i h_{0j})|_{\bar{\Sigma}}\right] \tag{39}$$

where, on the right-hand side, spatial indices are raised with δ^{ij}.

As for the $(0,\mu)$ components of the junction conditions, they are still equivalent to the conservation of $\mathcal{T}_{\mu\nu}$ because matter in the bulk reduces to a cosmological constant [20]

$$\nabla_\mu \mathcal{T}^\mu_\nu = 0 \tag{40}$$

∇_μ being the covariant derivative associated to the induced metric on the brane $a^2(\eta_{\mu\nu} + \gamma_{\mu\nu})$ with $\gamma_{\mu\nu} = \gamma^{(p)}_{\mu\nu} + \gamma^{(b)}_{\mu\nu}$.

Equations (39-40) are the central result of this paper. They look more complicated than analogous expressions obtained in gaussian normal coordinates (cf. e.g. [7], [9], [11], [13]) but they include the brane bending effect explicitly and are expressed in terms of the bulk gravitational waves written in conformally minkowskian coordinates which are known and simple, as recalled in the previous section.

There are several ways to interpret these equations. If the gravitational waves in the bulk are given by some underlying physics (they may be for example the zero point fluctuations of quantum gravitons) and if the perturbation of the position of the brane is also governed by some theory then equations (39-40) just define a tensor which has no reason, a priori, to be the stress-energy tensor of any realistic matter (although one can, of course, interpret it in terms of "new physics"). Conversely, if matter on the brane is imposed to be of a certain type, e.g. a scalar field or a perfect fluid with or without topological defects etc., then equations (39-40) become "junction conditions" which restrict the gravitational waves in the bulk and the position of the brane to those which are compatible with the imposed brane stress-energy tensor. Now it may be that some compromise has to be made for the junction conditions to have a solution. In fact this is already the case when solving the background equations. As we saw in section II a Robertson-Walker brane can be the edge of a given anti-de Sitter bulk, but at some price: matter on the brane has to include a fine-tuned, fairly unphysical, tension in order for the scale factor of the brane to obey a reasonable quasi Friedmannian evolution equation.

To gain some insight on the restrictive aspect of equations (39-40) and in order to show how they can be solved explicitly, we shall for the sake of the example impose that matter in the brane reduces to a single scalar field $\phi(x^\mu)$ with potential $V(\phi)$ plus a tension σ:

$$\mathcal{T}^\mu_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\left(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi + V(\phi)\right) - \sigma\delta^\mu_\nu\,. \tag{41}$$

Setting $\phi(x^\mu) = \Phi(\eta) + \chi(x^\mu)$ one first obtains for the background brane equations (11-12)

$$\begin{aligned} \ddot{\Phi} + 3H\dot{\Phi} + \frac{dV}{d\Phi} &= 0 \\ \frac{\kappa}{6}\left(\frac{\dot{\Phi}^2}{2} + V + \sigma\right) &= \sqrt{\mathcal{K}^2 + H^2} \end{aligned} \tag{42}$$

in which, in order to recover standard cosmological scenarios, the tension must be fine tuned to $\kappa\sigma = 6\mathcal{K}$. A potential $V(\Phi)$ and initial conditions being chosen equations (42) give $\Phi(t)$ and $a(t)$ (cf. e.g. [29] where these equations are studied in detail).

At linear order in χ and the brane metric perturbations $\gamma_{\mu\nu}$ the left hand-side of equation (39) reads

$$\frac{\kappa}{2}\delta\left(\mathcal{T}^i_j - \frac{1}{3}\delta^i_j\mathcal{T}\right) = \frac{\kappa}{6}\delta^i_j\left[\dot{\Phi}\dot{\chi} + \chi\frac{dV}{d\Phi} + \frac{\dot{\Phi}^2}{2}\gamma_{\eta\eta}\right]\,. \tag{43}$$

Introducing the spatial tensor

$$F^i_j \equiv \frac{1}{a^2}\,\partial^i_j\zeta + \frac{1}{2\mathcal{K}a}\left[H(\partial_0 h^i_j)|_{\bar{\Sigma}} - \sqrt{\mathcal{K}^2 + H^2}\,(\partial_4 h^i_j)_{\bar{\Sigma}} - H(\partial_j h^i_0 + \partial^i h_{0j})|_{\bar{\Sigma}}\right] \tag{44}$$

equation (39) splits into a traceless and trace part

$$\begin{aligned} F^i_j &= \frac{1}{3}\delta^i_j\,F \\ F &= \frac{\kappa}{2}\left[\dot{\Phi}\dot{\chi} + \chi\frac{dV}{d\Phi} + \frac{\dot{\Phi}^2}{2}\gamma_{\eta\eta}\right] - 3H(H\zeta - \dot{\zeta}) - \frac{3H^2}{2\mathcal{K}^2}\sqrt{\mathcal{K}^2 + H^2}\,h_{00}|_{\bar{\Sigma}} \end{aligned} \tag{45}$$

so that the junction conditions transform into (44-45) plus the conservation equations (40), which is the Klein-Gordon equation for ϕ:

$$\ddot{\chi} - \frac{1}{a^2}\Delta\chi + 3H\dot{\chi} + \frac{d^2V}{d\Phi^2}\chi + (\ddot{\Phi} + 3H\dot{\Phi})\gamma_{\eta\eta} - \frac{1}{a}\dot{\Phi}\partial_i\gamma^i_\eta + \frac{\dot{\Phi}}{2}(\dot{\gamma}_{\eta\eta} + \dot{\gamma}^i_i) = 0\,. \tag{46}$$

We now enter in equations (44-46) the explicit solution of the bulk Einstein equations. First, gathering (16) and (21), the brane metric perturbations are

$$\begin{aligned} \gamma_{\eta\eta} &= \frac{1}{\mathcal{K}^2}(\mathcal{K}^2 + H^2)h_{00}|_{\bar{\Sigma}} + \frac{2\zeta}{\sqrt{\mathcal{K}^2 + H^2}}\left(\mathcal{K}^2 + \frac{\ddot{a}}{a}\right) \\ \gamma^i_\eta &= \frac{1}{\mathcal{K}}\sqrt{\mathcal{K}^2 + H^2}h^i_0|_{\bar{\Sigma}} \\ \gamma^i_j &= h^i_j|_{\bar{\Sigma}} - 2\zeta\,\sqrt{\mathcal{K}^2 + H^2}\,\delta^i_j\,. \end{aligned} \tag{47}$$

Second, the perturbations $h_{\mu\nu}$ are given by (38). More explicitly, we have for each mode $\hat{h}_{\mu\nu}$

$$
\begin{aligned}
\hat{h}_{\mu\nu}|_{\bar{\Sigma}} &= \left(\frac{m}{\mathcal{K}a}\right)^2 Z_2\left(\frac{m}{\mathcal{K}a}\right) e_{\mu\nu} e^{\mathrm{i}\left(-\sqrt{k^2+m^2}\,T(t)+k_i x^i\right)} \\
(\partial_0 \hat{h}_{\mu\nu})|_{\bar{\Sigma}} &= -\mathrm{i}\sqrt{k^2+m^2}\,\hat{h}_{\mu\nu}|_{\bar{\Sigma}} \\
(\partial_i \hat{h}_{\mu\nu})|_{\bar{\Sigma}} &= \mathrm{i}k_i\,\hat{h}_{\mu\nu}|_{\bar{\Sigma}} \\
(\partial_4 \hat{h}_{\mu\nu})|_{\bar{\Sigma}} &= m\left(\frac{m}{\mathcal{K}a}\right)^2 Z_1\left(\frac{m}{\mathcal{K}a}\right) e_{\mu\nu} e^{\mathrm{i}\left(-\sqrt{k^2+m^2}\,T(t)+k_i x^i\right)}
\end{aligned}
\tag{48}
$$

where $T(t)$ is given by (3), that is $T(t) = \int dt\sqrt{\mathcal{K}^2+H^2}/\mathcal{K}a$, and where some standard properties of the Bessel functions have been used [24]. Finally, we can without loss of generality consider only the modes such that $k_1 = k_2 = 0$, $k_3 \equiv k$. The transverse and traceless properties of $e_{\mu\nu}$ then imply that the five possible polarisations are characterised by e_{11}, e_{12}, e_{13}, e_{23} and e_{33}, the other components being $e_{0i} = -ke_{i3}/\sqrt{k^2+m^2}$, $e_{00} = k^2 e_{33}/(k^2+m^2)$, and $e_{22} = -e_{11} - m^2 e_{33}/(k^2+m^2)$.

We are now in a position to try and solve explicitly the junction conditions for each mode. The traceless equation (45) first reduces for $m \neq 0$ the five a priori possible polarisations to only one, characterized by $e_{33} \equiv e(k,m)$. The others are

$$
\begin{aligned}
& e_{12} = e_{13} = e_{23} = e_{01} = e_{02} = 0 \\
& e_{11} = e_{22} = -\frac{1}{2}\frac{m^2}{k^2+m^2}\,e \\
& e_{03} = -\frac{k}{\sqrt{k^2+m^2}}\,e \\
& e_{00} = \frac{k^2}{k^2+m^2}\,e\,.
\end{aligned}
\tag{49}
$$

When $m = 0$ on the other hand not only does e_{33} survive but e_{13} and e_{23} as well. The latter two polarisations will correspond to gravitational waves freely propagating in the brane (the so-called tensorial modes).

The traceless equation (45) also forces the perturbation of the position of the brane to be a linear superposition of the following modes

$$
\zeta = \frac{e}{2\mathcal{K}k^2}\,a\left[\mathrm{i}\frac{H}{\sqrt{k^2+m^2}}\left(k^2-\frac{3}{2}m^2\right)G_2 - m\frac{\sqrt{\mathcal{K}^2+H^2}}{k^2+m^2}\left(k^2+\frac{3}{2}m^2\right)G_1\right] \tag{50}
$$

where

$$
G_{1,2} \equiv \left(\frac{m}{\mathcal{K}a}\right)^2 Z_{1,2}\left(\frac{m}{\mathcal{K}a}\right) e^{\mathrm{i}\left(-\sqrt{k^2+m^2}\,T(t)+kx^3\right)}\,. \tag{51}
$$

The function ζ being now known, the induced metric $\gamma_{\mu\nu}$ on the brane, for $m \neq 0$, is also completely known in closed form via equations (47-51) in terms of two (or three) arbitrary functions $e(k,m)$, $b_m(k,m)$ (and, should the occasion arise, $a_m(k,m)$). For the zero mode $m = 0$, e_{13}, e_{23} and $e_{33} \equiv e$ are arbitrary functions of k and the above expression for ζ becomes, since $a_m = 0, b_m = 1$ and $z^2 N_2(z) \to -4/\pi$ as $z \to 0$,

$$
\zeta = -\frac{2\mathrm{i}e}{\pi k\mathcal{K}}\,Ha\,e^{-\mathrm{i}k(T-x^3)}\,. \tag{52}
$$

What remains to be determined is the scalar field perturbation χ which must be extracted from the trace equation (45) and the Klein-Gordon equation (46). One can proceed as follows. Inserting the expressions for ζ and the induced metric obtained above one can write (45-46) as

$$\dot{\Phi}\dot{\chi} + \chi\frac{dV}{d\Phi} = F_1(t) \quad , \quad \ddot{\chi} + 3H\dot{\chi} + \chi\left(\frac{k^2}{a^2} + \frac{d^2V}{d\Phi^2}\right) = F_2(t) \tag{53}$$

where F_1 and F_2 are known function of t. Hence

$$\chi = \frac{1}{6H\frac{dV}{d\Phi} - \dot{\Phi}\frac{k^2}{a^2}}(\dot{F}_1 + 6HF_1 - \dot{\Phi}F_2)\,. \tag{54}$$

In the case of the zero mode $m = 0$ one obtains

$$\chi = \frac{2\mathrm{i}e}{\pi k\mathcal{K}}\, a\dot{\Phi}\sqrt{\mathcal{K}^2 + H^2}\, e^{-\mathrm{i}k(T-x^3)} \tag{55}$$

and it can be checked that the expression found is indeed a solution of (53). For massive modes $m \neq 0$ the algebra is more involved. Since our purpose in this paper was just to write the Lanczos-Darmois-Israel equations in such a way as to be able to try and solve them we shall present elsewhere the $m \neq 0$ case as well as a comparison with the results of ordinary chaotic inflation.

ACKNOWLEDGEMENTS

We warmly thank David Langlois for arousing our interest in brane cosmologies and discussing with us the subtleties of gaussian normal coordinates, as well as Thibaut Damour for comments on the restrictive aspect of the junction conditions. We also acknowledge discussions with the participants of the UK Cosmology meeting held in Durham in September 2000, in particular Anne Davis, Jaume Garriga, Ruth Gregory and David Wands.

APPENDIX A:

It can be useful, in particular when considering global properties of the brane or boundary conditions on the bulk perturbations, to embed AdS_5 in a higher, six dimensional, flat space. It is known [31] that the surface defined by

$$(y^0)^2 + \delta_{ij}y^iy^j - (y^4)^2 - z^2 = -\frac{1}{\mathcal{K}^2} \tag{A1}$$

in the six dimensional flat space with metric

$$ds^2|_6 = (dy^0)^2 + \delta_{ij}dy^idy^j - (dy^4)^2 - dz^2 \tag{A2}$$

is AdS_5. This space contains closed timelike curves: the circles $(y^4)^2 + z^2 =$ constant. One goes round this difficulty by introducing an integer "winding number" which increases by 1 each time one goes round the circle. One thus obtains the AdS_5 universal covering space [26].

The intersections of the planes $y^0 =$ constant (or $y^i =$ constant) with the surface (A1) are four dimensional simply connected hyperboloids of smallest radius $\sqrt{\mathcal{K}^{-2} + (y^0)^2}$. The sections $y^4 =$ constant (or $z =$ constant) are either four dimensional simply connected hyperboloids, or cones, or else doubly connected hyperboloids, depending on whether $(y^4\mathcal{K})^2 < 1$, $(y^4\mathcal{K})^2 = 1$ or $(y^4\mathcal{K})^2 > 1$.

If one parametrises the surface (A1) by the coordinates X^A such that [25], [26]

$$\begin{aligned} y^0 &= \frac{1}{2X^4}\left[\mathcal{K}^{-2} + (X^0)^2 - \delta_{ij}X^iX^j - (X^4)^2\right] \\ y^i &= \frac{X^i}{\mathcal{K}X^4} \\ y^4 &= \frac{X^0}{\mathcal{K}X^4} \\ z &= \frac{1}{2X^4}\left[\mathcal{K}^{-2} - (X^0)^2 + \delta_{ij}X^iX^j + (X^4)^2\right] \end{aligned} \tag{A3}$$

its induced metric is conformally minkowskian:

$$ds^2|_5 = \frac{1}{(\mathcal{K}X^4)^2}\,\eta_{AB}\,dX^AdX^B. \tag{A4}$$

Note that the plane $X^4 = \infty \Leftrightarrow y^0 + z = 0$ is a coordinate singularity.

A de Sitter brane (such that $A(\eta) = \eta$, see equations (2-5)) is the intersection of the surface (A1) with the plane $y^4 = -\sqrt{2}/\mathcal{K}$ which is the familiar four dimensional simply connected hyperboloid of smallest radius $1/\mathcal{K}$ embedded in a five dimensional Minkowski flat space [32]. The Minkowski brane $A(\eta) = 1/\mathcal{K}$ is the intersection of the surface (A1) with the plane $y^0 + z = 1/\mathcal{K}$. The Randall-Sundrum [1] spacetime is obtained by keeping only the region between the brane and the coordinate singularity $X^4 = +\infty$ [10].

If one now parametrises the surface (A1) by the coordinates $(\tau, r, \chi, \theta, \phi)$ such that

$$\begin{aligned} y^0 &= r\sin\chi\sin\theta\sin\phi \quad ; \quad y^1 = r\sin\chi\sin\theta\cos\phi \\ y^2 &= r\sin\chi\cos\theta \quad ; \quad y^3 = r\cos\chi \\ y^4 &= \sqrt{1+r^2}\sin\tau \quad ; \quad z = \sqrt{1+r^2}\cos\tau \end{aligned} \tag{A5}$$

the induced metric is Schwarzschild-like

$$ds^2|_5 = -(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + r^2(d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\phi^2). \tag{A6}$$

By letting the coordinate τ vary from $-\infty$ to $+\infty$ one covers the AdS_5 universal covering space without having to introduce the winding number.

The Schwarschild-like coordinates $(\tau, r, \chi, \theta, \phi)$ are therefore best suited to study the asymptotic properties of quantum fields or classical gravitational waves in AdS_5 [33] and care must be exercised when one uses the technically simpler conformally minkowskian coordinates X^A. The Schwarzschild-like coordinates are also well suited to the study of Robertson-Walker branes

with *closed* spatial sections (they are simply defined by $r = a(\eta),\ \tau = t(\eta)$ with $t(\eta)$ chosen so that η is conformal time [16]).

On the other hand the conformally minkowskian coordinates X^A are better suited to the study, to which we confine ourselves here, of Robertson-Walker branes with *flat* spatial sections as well as that of the Randall-Sundrum minkowskian brane.

APPENDIX B:

We show here explicitly the effect of a coordinate change in the bulk on the induced metric of the brane and on its extrinsic curvature.

Let us consider the infinitesimal change of coordinates in the bulk $\tilde{X}^A \to X^A = \tilde{X}^A - \epsilon^A$, $\epsilon^A(X^C)$ being five arbitrary functions of the coordinates X^C, *without* changing accordingly the equation for the brane that we still define as in section IV by $X^A = \bar{X}^A(x^\mu)$. Then the induced perturbation of the bulk metric is just the Lie derivative

$$h_{AB} = -2\eta_{AB}\,\frac{\epsilon^4}{X^4} + \eta_{AC}\partial_B\epsilon^C + \eta_{BC}\partial_A\epsilon^C. \tag{B1}$$

The corresponding change of the induced metric of the brane is obtained from equation (21) and reads

$$\begin{aligned}
\gamma^{(b)}_{\eta\eta} &= \frac{2\epsilon^4|_{\bar\Sigma}}{A} - 2T'\partial_\eta\epsilon^0|_{\bar\Sigma} + 2A'\partial_\eta\epsilon^4|_{\bar\Sigma} \\
\gamma^{(b)}_{\eta i} &= -T'\partial_i\epsilon^0|_{\bar\Sigma} + A'\partial_i\epsilon^4|_{\bar\Sigma} + \delta_{ij}\partial_\eta\epsilon^j|_{\bar\Sigma} \\
\gamma^{(b)}_{ij} &= -2\delta_{ij}\,\frac{\epsilon^4|_{\bar\Sigma}}{A} + \delta_{jk}\partial_i\epsilon^k|_{\bar\Sigma} + \delta_{ik}\partial_j\epsilon^k|_{\bar\Sigma}\,.
\end{aligned} \tag{B2}$$

As for the change of its extrinsic curvature it is given by (22) and reads

$$\begin{aligned}
\delta^{(b)}K_{ij} = &-A'\partial_{ij}\epsilon^0|_{\bar\Sigma} + T'\partial_{ij}\epsilon^4|_{\bar\Sigma} + \frac{T'}{A}\left(\delta_{jk}\partial_i\epsilon^k|_{\bar\Sigma} + \delta_{ik}\partial_j\epsilon^k|_{\bar\Sigma}\right) + \\
&+ \frac{\delta_{ij}}{A}\left(A'T'\partial_\eta\epsilon^4|_{\bar\Sigma} - A'^2\partial_\eta\epsilon^0|_{\bar\Sigma} - \frac{2T'}{A}\epsilon^4|_{\bar\Sigma}\right).
\end{aligned} \tag{B3}$$

We can now decompose, as we did in section III, $\epsilon^A|_{\bar\Sigma}$ along the tangent and normal vectors to the brane as

$$\epsilon^A|_{\bar\Sigma} = \xi^\lambda\bar{V}^A_\lambda + \zeta\bar{n}^A. \tag{B4}$$

It is then easy to see that if $\zeta(x^\mu) = 0$, so that $\epsilon^A|_{\bar\Sigma} = (T'\xi^\eta, \xi^i, A'\xi^\eta)$, then $\gamma^{(b)}_{\mu\nu}$ and $\delta^{(b)}K_{ij}$ as given by (B2) and (B3) are just the Lie derivatives of the brane metric and its extrinsic curvature with respect to the vector field ξ^μ, and therefore describe the change in the components of these tensors under the coordinate shift $\tilde{x}^\mu \to x^\mu = \tilde{x}^\mu - \xi^\mu$ on the brane. This result is geometrically obvious. Indeed $\zeta = 0$ means that the coordinate change in the bulk is such that the grid

is moved parallely to the surface $X^A = \bar{X}^A(x^\mu)$ which defines the brane. Hence the brane is geometrically unperturbed by this operation.

On the other hand if $\xi^\lambda = 0$, so that $\epsilon^A|_{\bar{\Sigma}} = \mathcal{K}A(A'\zeta, 0, 0, 0, T'\zeta)$, the expressions (B2) and (B3) for $\gamma^{(b)}_{\mu\nu}$ and $\delta^{(b)}K_{ij}$ reduce to

$$\begin{aligned} \gamma^{(b)}_{\mu\nu} &= -2(\mathcal{K}A)^2\zeta\bar{K}_{\mu\nu} \\ \delta^{(b)}K_{ij} &= \partial_{ij}\zeta + \delta_{ij}\left(\frac{A'\zeta'}{A} - \frac{2\zeta}{A^2} - \frac{A'^2\zeta}{A^2}\right). \end{aligned} \tag{B5}$$

As expected they are identical to the perturbations due to a change of the position of the brane studied in section III and given by equations (16) and (18). (For related views on the relationship between gauge and brane bending effects, see e.g. [21], [27], [23], [28].)

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