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Approximate Solutions of Exact Renormalization Group Equations

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Abstract

We study exact renormalization group equations in the framework of the effective average action. We present analytical approximate solutions for the scale dependence of the potential in a variety of models. These solutions display a rich spectrum of physical behaviour such as fixed points governing the universal behaviour near second order phase transitions, critical exponents, first order transitions (some of which are radiatively induced) and tricritical behaviour.

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Introduction: The solution of an exact renormalization group equation [1]-[7] is a particularly difficult task. The reason is that such an equation describes the scale dependence of an effective action, which is characterized by infinitely many couplings multiplying the invariants consistent with the symmetries of the model under consideration. As a result an exact renormalization group equation corresponds to infinitely many evolution equations for the couplings of the theory. The crucial step is developing efficient approximation schemes which can reduce the complexity of the problem while capturing the essential aspects of the physical system. Perturbative expansions have been used for proofs of perturbative renormalizability [3, 8], while the powerful ϵ-expansion [1, 9] has been employed for the study of fixed points governing second order phase transitions in three dimensions. More recently, evolution equations for truncated forms of the effective action have been solved through a combination of analytical and numerical methods. A full, detailed and transparent picture of second and first order phase transitions for a variety of models has emerged [10]-[13]. Also, numerical solutions for the fixed point potential of three-dimensional scalar theories have been computed in ref. [14]. Fully analytical solutions have not been obtained, with the exception of ref. [11], where an exact solution for the three-dimensional $O(N)$-symmetric scalar theory in the large N limit is given.

In this letter we present analytical approximate solutions of evolution equations for truncated forms of the effective action. We work in the framework of the effective average action Γ_k [5, 15], which results from the integration of quantum fluctuations with characteristic momenta $q^2 \geq k^2$. The effective average action Γ_k interpolates between the classical action S for k equal to the ultraviolet cutoff Λ of the theory (no integration of modes) and the effective action Γ for $k = 0$ (all the modes are integrated). Its dependence on k is given by an exact renormalization group equation with the typical form ($t = \ln(k/\Lambda)$)

$$\frac{\partial}{\partial t}\Gamma_k = \frac{1}{2}\mathrm{Tr}\Big\{(\Gamma_k^{(2)} + R_k)^{-1}\frac{\partial}{\partial t}R_k\Big\}. \tag{1}$$

Here $\Gamma_k^{(2)}$ is the second functional derivative with respect to the fields, R_k is the effective infrared cutoff which prevents the integration of modes with $q^2 \leq k^2$, and the trace implies integration over all Fourier modes of the fields. We work with an approximation which neglects the effects of wave function renormalization. For the models that we shall consider in four and three dimensions, the anomalous dimensions are small (a few percent). As a result, wave function renormalization effects generate only small quantitative corrections without affecting the qualitative behaviour. Therefore, only a classical kinetic term in the effective average action is kept, which takes, for an $O(N)$-symmetric scalar theory, the following form

$$\Gamma_k = \int d^dx\Big\{U_k(\phi) + \frac{1}{2}\partial^\mu\phi_a\partial_\mu\phi^a\Big\}, \tag{2}$$

and all invariants which involve more derivatives of the fields are neglected. With this approximation, eq. (1) can be turned into an evolution equation for the potential U_k. First we shall discuss the $O(N)$-symmetric scalar theory in three dimensions. The fixed point solution which governs the second order phase transition will be identified. We shall also show that, for a certain parameter range, the theory has a first order phase transition.

The two regions in parameter space are separated by a tricritical line. Finally we shall discuss the Abelian Higgs model in four dimensions, for which the radiatively induced first order transition will be reproduced.

Scalar theory in three dimensions: We first consider the evolution equation describing the dependence of the effective average potential U_k on the scale k in arbitrary dimensions d, for an $O(N)$-symmetric scalar theory. The evolution equation reads [5]

$$\frac{\partial U_k(\rho)}{\partial t} = v_d \int_0^\infty dx\, x^{\frac{d}{2}-1} \frac{\partial P}{\partial t} \left\{ \frac{N-1}{P+U_k'(\rho)} + \frac{1}{P+U_k'(\rho)+2U_k''(\rho)\rho} \right\}. \tag{3}$$

Here $\rho = \frac{1}{2}\phi^a\phi_a,\ a = 1...N$, and primes denote derivatives with respect to ρ. The variable x denotes momentum squared $x = q^2$, and

$$v_d^{-1} = 2^{d+1}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right). \tag{4}$$

The inverse average propagator

$$P(x) = \frac{x}{1-\exp\left(-\frac{x}{k^2}\right)} \tag{5}$$

contains an effective infrared cutoff for the modes with $x < k^2$. For $x/k^2 \to 0$ we have $P(x) \to k^2$. Up to effects from the wave function renormalization (which have been neglected as we explained in the introduction) eq. (3) is an exact non-perturbative evolution equation [5]. It is easy to recognize the first term in the r.h.s of eq. (3) as the contribution of the $N-1$ Goldstone modes (U_k' vanishes at the minimum). The second term is related to the radial mode. After performing the momentum integration the evolution equation (3) becomes a partial differential equation for U_k with independent variables ρ and t. The effective average potential interpolates between the classical potential V for $k = \Lambda$ (with Λ the ultraviolet cutoff) and the effective potential U for $k = 0$ [5]. As a result the solution of eq. (3) with the initial condition $U_\Lambda(\rho) = V(\rho)$ uniquely determines, for $k \to 0$, the effective 1PI vertices at zero momentum for the renormalized theory. In order to write eq. (3) in a scale invariant form it is convenient to define the variables

$$\begin{aligned} \tilde{\rho} =& k^{2-d}\rho \\ u_k(\tilde{\rho}) =& k^{-d}U_k(\rho). \end{aligned} \tag{6}$$

In terms of these eq. (3) can be written as (for the details see ref. [11])

$$\frac{\partial u'}{\partial t} = -2u' + (d-2)\tilde{\rho}u'' - 2l_1^d v_d(N-1)u''s_1^d(u') - 2l_1^d v_d(3u''+2\tilde{\rho}u''')s_1^d(u'+2\tilde{\rho}u''), \tag{7}$$

with

$$l_1^d = \Gamma\left(\frac{d}{2}\right). \tag{8}$$

Primes on u denote derivatives with respect to $\tilde{\rho}$ and we omit the subscript k in u_k from now on. The functions $s_1^d(w)$ introduce "threshold" behaviour in the evolution equation, which results in the decoupling of heavy modes. They approach unity for vanishing argument, and vanish for large arguments. They do not have a simple analytical form, but are well approximated [3] for our purposes by

$$s_1^d(w) = (1+w)^{-2}. \tag{9}$$

Even with this approximation the evolution equation (7) remains a non-linear partial differential equation which seems very difficult to solve. The problems arise from the contribution from the radial mode to the r.h.s. of eq. (7) (the last term). An enormous simplification is achieved, however, if this contribution is replaced by $-6l_1^d v_d u'' s_1^d(u')$. From the physical point of view this approximation substitutes for the effects of the radial mode the effects of additional Goldstone modes. It is justified in the large N limit and near the origin of the potential ($\tilde{\rho} = 0$). The simplified evolution equation now reads

$$\frac{\partial u'}{\partial t} - (d-2)\tilde{\rho}\frac{\partial u'}{\partial \tilde{\rho}} + 2l_1^d v_d (N+2)\frac{1}{(1+u')^2}\frac{\partial u'}{\partial \tilde{\rho}} + 2u' = 0. \tag{10}$$

It is first order in both independent variables and can easily be solved with the method of characteristics.

We are interested in the behaviour of the theory in three dimensions, where a non-trivial fixed point structure arises. The most general solution of the partial differential equation (10) for $d = 3$ is given by the relations

$$\frac{\tilde{\rho}}{\sqrt{u'}} - \frac{C}{\sqrt{u'}} - \frac{C}{2}\frac{\sqrt{u'}}{1+u'} + \frac{3}{2}C\arctan\left(\frac{1}{\sqrt{u'}}\right) = F\left(u'e^{2t}\right) \qquad \text{for} \quad u' > 0 \tag{11}$$

$$\frac{\tilde{\rho}}{\sqrt{-u'}} - \frac{C}{\sqrt{-u'}} + \frac{C}{2}\frac{\sqrt{-u'}}{1+u'} - \frac{3}{4}C\ln\left(\frac{1-\sqrt{-u'}}{1+\sqrt{-u'}}\right) = F\left(u'e^{2t}\right) \quad \text{for} \quad u' < 0, \tag{12}$$

with

$$C = 2v_3(N+2)l_1^3 = \frac{N+2}{8\pi^{\frac{3}{2}}}. \tag{13}$$

The function F is undetermined until initial conditions are specified. For $t = 0$ ($k = \Lambda$), U_k coincides with the classical potential V. The initial condition, therefore, reads

$$u'(\tilde{\rho}, t = 0) = \Lambda^{-2}V'(\rho). \tag{14}$$

This uniquely specifies F and we obtain

$$\frac{\tilde{\rho}}{\sqrt{u'}} - \frac{C}{\sqrt{u'}} - \frac{C}{2}\frac{\sqrt{u'}}{1+u'} + \frac{3}{2}C\arctan\left(\frac{1}{\sqrt{u'}}\right) =$$

[3] The choice of the average propagator in eq. (5), which is reflected in the form of s_1^d is arbitrary within some general conditions [5]. It may be possible to find a form of P so that eq. (9) becomes exact, with a suitably chosen value of l_1^d.

$$\frac{G\left(u'e^{2t}\right)}{\sqrt{u'}e^t} - \frac{C}{\sqrt{u'}e^t} - \frac{C}{2}\frac{\sqrt{u'}e^t}{1+u'e^{2t}} + \frac{3}{2}C\arctan\left(\frac{1}{\sqrt{u'}e^t}\right) \qquad \text{for} \quad u' > 0 \tag{15}$$

$$\frac{\tilde{\rho}}{\sqrt{-u'}} - \frac{C}{\sqrt{-u'}} + \frac{C}{2}\frac{\sqrt{-u'}}{1+u'} - \frac{3}{4}C\ln\left(\frac{1-\sqrt{-u'}}{1+\sqrt{-u'}}\right) =$$
$$\frac{G\left(u'e^{2t}\right)}{\sqrt{-u'}e^t} - \frac{C}{\sqrt{-u'}e^t} + \frac{C}{2}\frac{\sqrt{-u'}e^t}{1+u'e^{2t}} - \frac{3}{4}C\ln\left(\frac{1-\sqrt{-u'}e^t}{1+\sqrt{-u'}e^t}\right) \text{ for } \quad u' < 0, \tag{16}$$

with the function G determined by inverting eq. (14) and solving for $\tilde{\rho}$ in terms of u'

$$G(u') = \tilde{\rho}(u')|_{t=0}. \tag{17}$$

I) *Classical ϕ^4 theory:* Let us first consider a theory with a quartic classical potential. The initial condition can be written as

$$u'(\tilde{\rho}, t=0) = \lambda_\Lambda(\tilde{\rho} - \kappa_\Lambda), \tag{18}$$

with

$$\kappa_\Lambda = \frac{\rho_{0\Lambda}}{\Lambda}, \qquad \lambda_\Lambda = \frac{\bar{\lambda}_\Lambda}{\Lambda} \tag{19}$$

the rescaled (dimensionless) minimum of the potential and quartic coupling respectively. The function G in eqs. (15), (16) is now given by

$$G(x) = \kappa_\Lambda + \frac{x}{\lambda_\Lambda}. \tag{20}$$

The typical form of the effective average potential $U_k(\rho)$ at different scales k, as given by eqs. (15), (16), is presented in fig. 1. The theory at the ultraviolet cutoff is defined in the regime with spontaneous symmetry breaking, with the minimum of the potential at $\rho_{0\Lambda} = \kappa_\Lambda \Lambda \neq 0$. At lower scales k the minimum of the potential moves continuously closer to zero, with no secondary minimum ever developing. We expect a second order phase transition (in dependence to κ_Λ) for the renormalized theory at $k = 0$. Eqs. (15), (16) contain all the qualitative information for the non-trivial behaviour of the three-dimensional theory, even though the approximations leading to eq. (10) do not permit quantitative accuracy in all respects. There is a critical value for the minimum of the classical potential

$$\kappa_\Lambda = \kappa_{cr} = C, \tag{21}$$

for which a scale invariant (fixed point) solution is approached in the limit $t \to -\infty$ ($k \to 0$). This solution (which corresponds to the Wilson-Fisher fixed point) is given by the relations

$$\frac{\tilde{\rho}}{\sqrt{u'_\star}} - \frac{C}{\sqrt{u'_\star}} - \frac{C}{2}\frac{\sqrt{u'_\star}}{1+u'_\star} + \frac{3}{2}C\arctan\left(\frac{1}{\sqrt{u'_\star}}\right) = \frac{3\pi}{4}C \qquad \text{for} \quad u'_\star > 0 \tag{22}$$

$$\frac{\tilde{\rho}}{\sqrt{-u'_\star}} - \frac{C}{\sqrt{-u'_\star}} + \frac{C}{2}\frac{\sqrt{-u'_\star}}{1+u'_\star} - \frac{3}{4}C\ln\left(\frac{1-\sqrt{-u'_\star}}{1+\sqrt{-u'_\star}}\right) = 0 \qquad \text{for} \quad u'_\star < 0. \tag{23}$$

Eqs. (22), (23) describe a potential u which has a minimum at a constant value

$$\kappa(k) = \kappa_\star = C. \tag{24}$$

This leads to a potential $U_k(\rho)$ with a minimum at $\rho_0(k) = k\kappa_\star \to 0$ for $k \to 0$, which corresponds to the phase transition between the spontaneously broken and the symmetric phase. (The values for κ_{cr} and $\kappa_\star$ coincide, but this is accidental.) For the second and third $\tilde{\rho}$-derivative of u at the minimum $\lambda = u''(\kappa)$, $\sigma = u'''(\kappa)$ we find

$$\lambda(k) = \lambda_\star = \frac{1}{2C} \tag{25}$$

$$\sigma(k) = \sigma_\star = \frac{1}{4C^2}, \tag{26}$$

and similar fixed point values for the higher derivarives of u. For $1 \ll \tilde{\rho}/C \ll (3\pi/4)e^{-t}$ the rescaled potential u has the form

$$u'_\star(\tilde{\rho}) = \left(\frac{4}{3\pi C}\right)^2 \tilde{\rho}^2. \tag{27}$$

Notice that the region of validity of eq. (27) extends to infinite $\tilde{\rho}$ for $t \to -\infty$. From eq. (27) with $t \to -\infty (k \to 0)$ we obtain for the effective potential at the phase transition

$$U_\star(\rho) = \frac{1}{3}\left(\frac{4}{3\pi C}\right)^2 \rho^3. \tag{28}$$

Through eqs. (15), (16) we can also study solutions which deviate slightly from the scale invariant one. For this purpose we define a classical potential with a minimum

$$\kappa_\Lambda = \kappa_{cr} + \delta\kappa_\Lambda, \tag{29}$$

with $|\delta\kappa_\Lambda| \ll 1$. We find for the minimum of the potential

$$\kappa(k) = \kappa_\star + \delta\kappa_\Lambda e^{-t}, \tag{30}$$

and for λ

$$\lambda(k) = \frac{\lambda_\star}{1 + \left(\frac{\lambda_\star}{\lambda_\Lambda} - 1\right) e^t}. \tag{31}$$

Eq. (30) indicates that the minimum of u stays close to the fixed point value $\kappa_\star$ given by eq. (24), for a very long "time" $|t| < -\ln|\delta\kappa_\Lambda|$. For $|t| > -\ln|\delta\kappa_\Lambda|$ it deviates from the fixed point, either towards the phase with spontaneous symmetry breaking (for $\delta\kappa_\Lambda > 0$), or the symmetric one (for $\delta\kappa_\Lambda < 0$). Eq. (31) implies an attractive fixed point for λ, with a value given by eq. (25). Similarly the higher derivatives are attracted to their fixed point values. The full phase diagram corresponds to a second order phase transition. For $\delta\kappa_\Lambda > 0$ the system ends up in the phase with spontaneous symmetry breaking, with

$$\rho_0 = \lim_{k\to 0} \rho_0(k) = \lim_{k\to 0} k\kappa(k) = \delta\kappa_\Lambda \Lambda. \tag{32}$$

In this phase the renormalized quartic coupling approaches zero linearly with k

$$\lambda_R = \lim_{k\to 0} k\lambda(k) = \lim_{k\to 0} k\lambda_\star = 0. \tag{33}$$

The fluctuations of the Goldstone bosons lead to an infrared free theory in the phase with spontaneous symmetry breaking. For $\delta\kappa_\Lambda < 0$, $\kappa(k)$ becomes zero at a scale

$$t_s = -\ln\left(\frac{\kappa_\star}{|\delta\kappa_\Lambda|}\right) \tag{34}$$

and the system ends up in the symmetric regime ($\rho_0 = 0$). From eq. (15), in the limit $t \to -\infty$, with $u', u'', u''' \to \infty$, so that $u'e^{2t} \sim |\delta\kappa_\Lambda|^2$, $u''e^t \sim |\delta\kappa_\Lambda|$, $u''' \sim 1$, we find

$$U(\rho) = U_0(\rho) = \left(\frac{4}{3\pi C}\right)^2 \left[|\delta\kappa_\Lambda|^2\Lambda^2\rho + |\delta\kappa_\Lambda|\Lambda\rho^2 + \frac{1}{3}\rho^3\right]. \tag{35}$$

Notice how every reference to the classical theory has disappeared in the above expression. The effective potential of the critical theory is determined uniquely in terms of $\delta\kappa_\Lambda$, which measures the distance from the phase transition. The above results are essentially identical to those obtained through the study of the evolution equation in the large N limit [11, 16] (with a redefinition of the constant C in eq. (13)). In particular the values for the critical exponents β, ν, describing the behaviour of the system very close to the phase transition, correspond to the large N limit of the model (for details see section 6 of ref. [11])

$$\begin{aligned}
\beta &= \lim_{\delta\kappa_\Lambda\to 0^+} \frac{d\left(\ln\sqrt{\rho_0}\right)}{d(\ln\delta\kappa_\Lambda)} = 0.5 \\
\nu &= \lim_{\delta\kappa_\Lambda\to 0^-} \frac{d\left(\ln m_R\right)}{d(\ln|\delta\kappa_\Lambda|)} = \frac{d\left(\ln\sqrt{U'(0)}\right)}{d(\ln|\delta\kappa_\Lambda|)} = 1.
\end{aligned} \tag{36}$$

This is expected, since the replacement of the contribution from the radial mode to the evolution equation by a contribution involving additional Goldstone modes is a valid approximation in the large N limit. However, the purpose of our discussion was to obtain an insight into the qualitative behaviour of the theory in the context of a simplified analytical framework. From this point of view, all the essential non-trivial behaviour of the theory is incorporated in eqs. (15), (16). We also point out that, for a theory with spontaneous symmetry breaking , we can use eq. (16) in order to study the "inner" part of the potential. In particular, for $\tilde{\rho} = 0$ and $t \to -\infty$ eq. (16) predicts a potential u which asymptotically behaves as

$$\lim_{t\to -\infty} u'(0) = -1. \tag{37}$$

This leads to an effective average potential U_k which becomes convex with

$$\lim_{k\to 0} U_k'(0) = -k^2, \tag{38}$$

in agreement with the detailed study of ref. [17].

II) *Classical ϕ^6 theory:* As a second example we consider a theory defined through a classical potential with a ρ^3 (ϕ^6) term

$$u'(\tilde{\rho}, t=0) = \lambda_\Lambda(\tilde{\rho} - \kappa_\Lambda) + \frac{\sigma_\Lambda}{2}(\tilde{\rho} - \kappa_\Lambda)^2, \tag{39}$$

where κ_Λ, λ_Λ are defined in eq. (19) and the coupling σ_Λ is dimensionless in $d = 3$. The function G in eqs. (15), (16) is now given by

$$G(x) = \kappa_\Lambda + \frac{-\lambda_\Lambda + \sqrt{\lambda_\Lambda^2 + 2\sigma_\Lambda x}}{\sigma_\Lambda} \qquad \text{for} \quad u'' > 0 \tag{40}$$

$$G(x) = \kappa_\Lambda + \frac{-\lambda_\Lambda - \sqrt{\lambda_\Lambda^2 + 2\sigma_\Lambda x}}{\sigma_\Lambda} \qquad \text{for} \quad u'' < 0. \tag{41}$$

We distinguish two regions in parameter space which result in two different types of behaviour for the theory:
(a) For $\kappa_\Lambda < 2\lambda_\Lambda/\sigma_\Lambda$ the classical potential has only one minimum at $\rho_{0\Lambda} = \kappa_\Lambda \Lambda$. Near this minimum the initial condition of eq. (39) is very well approximated by eq. (18). As a result, for κ_Λ near the critical value of eq. (21), the critical theory has exactly the same behaviour as for a quartic classical potential. The running potential first approaches the fixed point solution of eqs. (22), (23) (notice that $\kappa_\star < 2\lambda_\star/\sigma_\star$), and subsequently evolves towards the phase with spontaneous symmetry breaking or the symmetric one. The behaviour of the critical theory for $k = 0$ is determined only by the distance from the phase transition (as measured by $\delta\kappa_\Lambda$), without any memory of the details of the classical theory. This is a manifestation of universality, typical of second order phase transitions.
(b) For $\kappa_\Lambda > 2\lambda_\Lambda/\sigma_\Lambda$ the classical potential has two minima, one at the origin and one at $\rho_{0\Lambda} = \kappa_\Lambda \Lambda$. The minimum at the origin is deeper for $\kappa_\Lambda > 3\lambda_\Lambda/\sigma_\Lambda$. An example of the evolution of the effective average potential for such a theory is given in fig. 2. The minimum of the potential at non-zero ρ moves towards the origin for decreasing scale k. In the same time the positive curvature at the origin decreases. The combined effect is (very crudely) similar to the whole potential being shifted to the left of the graph. As a result the minimum at the origin becomes shallower. For a certain range of the parameter space (for small enough κ_Λ, such as chosen for fig. 2) the minimum away from the origin becomes the absolute minimum of the potential at some point during the evolution. This results in a discontinuity in the running order parameter. Finally the absolute minimum of the potential settles down at some non-zero ρ_0. For even larger κ_Λ the minimum at the origin is deep enough for the evolution to stop while this minimum is still the absolute minimum of the potential. When the minimum of the renormalized potential ρ_0 (which is obtained at the end of the evolution) is considered as a function of κ_Λ, a discontinuity is observed in the function $\rho_0(\kappa_\Lambda)$. This indicates a first order phase transition. Unfortunately, an exact quantitative determination of the region in parameter space which results in first order transitions is not possible within the approximations we have used. The reason for this is the omission of the term $2\tilde{\rho}u''$ in the "threshold" function for the radial mode. As

a result our approximation is not adequate for dealing with the shape of the barrier in the limit $k \to 0$, where the theshold function for the radial mode becomes important. Also the approach to convexity cannot be reliably discussed (in contrast to the case of a classical ϕ^4 potential). If the shape of the barrier cannot be reliably determined the relative depth of the two minima cannot be calculated, and our discussion is valid only at the qualitative level.

However, more information can be extracted from our results. As long as we concentrate on regions of the potential away from the top of the barrier the solution given by eqs. (15), (40), (41) is reliable. This means that we can study the potential around its two minima. We are interested in the limit $t \to -\infty$ ($k \to 0$), with $U' = u' e^{2t}$, $\rho = \tilde{\rho} e^t$ approaching finite values. The form of the potential near the minimum away from the origin is determined by eqs. (15), (40). We find

$$\frac{\rho}{\Lambda} - \kappa_\Lambda + C = \frac{-\lambda_\Lambda + \sqrt{\lambda_\Lambda^2 + 2\sigma_\Lambda \frac{U'}{\Lambda^2}}}{\sigma_\Lambda} - \frac{C}{2} \frac{\frac{U'}{\Lambda^2}}{1 + \frac{U'}{\Lambda^2}} + \frac{3}{2} C \sqrt{\frac{U'}{\Lambda^2}} \arctan\left(\frac{1}{\sqrt{\frac{U'}{\Lambda^2}}}\right). \tag{42}$$

The minimum ρ_0 (where $U'(\rho_0) = 0$) is located at $\rho_0 = (\kappa_\Lambda - C)\Lambda = \delta\kappa_\Lambda \Lambda$. This requires $\delta\kappa_\Lambda \geq 0$. Eqs. (15), (41) describe the form of the potential around the minimum at the origin. Similarly as above we find

$$\frac{\rho}{\Lambda} - \kappa_\Lambda + C = \frac{-\lambda_\Lambda - \sqrt{\lambda_\Lambda^2 + 2\sigma_\Lambda \frac{U'}{\Lambda^2}}}{\sigma_\Lambda} - \frac{C}{2} \frac{\frac{U'}{\Lambda^2}}{1 + \frac{U'}{\Lambda^2}} + \frac{3}{2} C \sqrt{\frac{U'}{\Lambda^2}} \arctan\left(\frac{1}{\sqrt{\frac{U'}{\Lambda^2}}}\right). \tag{43}$$

In the parameter range $\kappa_\Lambda - 2\lambda_\Lambda/\sigma_\Lambda, 2\lambda_\Lambda/\sigma_\Lambda \gg C$ the above solution reproduces the classical potential, with a large positive curvature $U'(0)/\Lambda^2$ at the origin. This is due to the fact that the fluctuations which renormalize the potential around the origin are massive, with their masses acting as an effective infrared cutoff. For the above parameter range these masses are of the order of the ultraviolet cutoff Λ and no renormalization of the potential takes place. This is in contrast with the form of the potential near the minimum away from the origin ρ_0. The presence of the Goldstone modes in this region always results in strong renormalization. There is a range of parameters for which the curvature at the origin becomes zero. It is given by the relation

$$\kappa_\Lambda = C + 2\lambda_\Lambda/\sigma_\Lambda. \tag{44}$$

For this range the minimum at the origin disappears and the potential has only one minimum at $\rho_0 = \delta\kappa_\Lambda \Lambda = (\kappa_\Lambda - C)\Lambda$. The above condition does not determine precisely the first order phase transition, as this takes place when the two minima are degenerate, and not when the minimum at the origin disappears. However, it provides a good estimate of its location. The discontinuity in the order parameter is expected to be $\mathcal{O}(\delta\kappa_\Lambda)$. Weakly first order transitions are obtained for $\lambda_\Lambda \to 0$. We should emphasize that eq. (43) is not valid for arbitrarily small U'/Λ^2. This would correspond to a range of the potential near the top of the disappearing barrier, where we know that our approximation fails. This

is another reason why eq. (44) is only indicative of the location of the first order phase transition.

We have identified two critical surfaces in parameter space. We saw in (a) that the surface $\kappa_\Lambda = C$ corresponds to second order phase transitions. Also in (b) we argued that the surface $\kappa_\Lambda = C + 2\lambda_\Lambda/\sigma_\Lambda$ corresponds to first order transitions. As a result we expect tricritical behaviour to characterize their intersection, which is given by the line $\kappa_\Lambda = C$, $\lambda_\Lambda = 0$. This is confirmed if we approach this line close to the critical surface $\kappa_\Lambda = C$. More specifically we consider a theory with $0 < -\delta\kappa_\Lambda = -\kappa_\Lambda + C \ll 1$ and $\lambda_\Lambda \ll 1$. For this choice of parameters the renormalized theory is in the symmetric phase very close to the second order phase transition. The form of the potential is given by eq. (42) with $U'/\Lambda^2 \ll 1$

$$\frac{\rho}{\Lambda} + |\delta\kappa_\Lambda| = \frac{1}{\lambda_\Lambda}\frac{U'}{\Lambda^2} + \frac{3\pi}{4}C\sqrt{\frac{U'}{\Lambda^2}}. \tag{45}$$

For $|\delta\kappa_\Lambda| \ll \lambda_\Lambda$ the potential has the universal form of eq. (35). The initial point of the evolution is sufficiently close to the critical surface for the flows to approach the Wilson-Fisher critical point before deviating towards the symmetric phase. The critical exponent ν takes the large N value $\nu = 1$ according to eq. (36). In the opposite limit $|\delta\kappa_\Lambda| \gg \lambda_\Lambda$ the potential near the origin is given by

$$U(\rho) = \lambda_\Lambda \left(|\delta\kappa_\Lambda|\Lambda^2\rho + \frac{1}{2}\Lambda\rho^2 \right) \tag{46}$$

and the exponent ν takes its mean field value $\nu = 0.5$. A continuous transition from one type of behaviour to the other (a crossover curve) connects the two parameter regions. Clearly, the line $\kappa_\Lambda = C$, $\lambda_\Lambda = 0$ gives tricritical behaviour with mean field exponents.

The Abelian Higgs model in four dimensions: We now turn to gauge theories, for which exact renormalization equations have also been obtained [6]. As an example we discuss the Abelian Higgs model with one complex scalar, in four dimensions. The evolution equation can be written in the form (for the details see ref. [13])

$$\frac{\partial u'}{\partial t} = -2u' + 2\tilde{\rho}u'' - 2l_1^4 v_4 u'' s_1^4(u') - 2l_1^4 v_4(3u'' + 2\tilde{\rho}u''')s_1^4(u' + 2\tilde{\rho}u'') - 12 l_1^4 v_4 e^2 s_1^4(2e^2\tilde{\rho}), \tag{47}$$

with v_4 given by eq. (4) and l_1^4 by eq. (8). We have again neglected the small wave function renormalization effects for the scalar field. We recognize the contributions of the Goldstone and radial mode. The last term in eq. (47) is the contribution of the gauge field. It involves the gauge coupling e^2, whose evolution can be computed independently [6, 13]. Since the resulting running for e^2 is only logarithmic in $d = 4$, it is a good approximation to neglect it in the following. We assume the form of eq. (9) for the "threshold" functions s_1^4. The contribution of the radial mode introduces higher derivatives in the evolution equation, making an explicit solution impossible. We shall again resort to the replacement of this contribution by $-6l_1^4 v_4 u'' s_1^4(u')$, as in the first part of the paper. The resulting partial differential equation is first order and can be solved with the method of characteristics.

We have not managed to obtain an analytical solution in closed form, even though a numerical solution is possible. For this reason we make an additional approximation which is not crucial for the physical behaviour that we are interested in (see below). We shall set $s_1^4(u') = 1$ in the contributions of the scalar field, while maintaining the full "threshold" function in the contribution of the gauge field. As a result we cannot see the decoupling of the scalar modes or the approach to convexity for the effective potential. However, we preserve the full effect of the gauge field on the form of the potential. We thus finally arrive to the following evolution equation

$$\frac{\partial u'}{\partial t} - 2\tilde{\rho}\frac{\partial u'}{\partial \tilde{\rho}} + B\frac{\partial u'}{\partial \tilde{\rho}} + \frac{De^2}{(1+2e^2\tilde{\rho})^2} + 2u' = 0, \tag{48}$$

with

$$\begin{aligned} B =& 8l_1^4 v_4 = \frac{1}{4\pi^2} \\ D =& 12 l_1^4 v_4 = \frac{3}{8\pi^2}. \end{aligned} \tag{49}$$

The most general solution of eq. (48) is given by

$$\begin{aligned} &\frac{u'}{2\tilde{\rho}-B} + \frac{De^2}{2(Be^2+1)}\frac{1}{(2e^2\tilde{\rho}+1)(2\tilde{\rho}-B)} + \frac{De^4}{(Be^2+1)^2}\frac{1}{2e^2\tilde{\rho}+1} \\ &\quad - \frac{De^4}{(Be^2+1)^3}\ln\left(\frac{2e^2\tilde{\rho}+1}{|2\tilde{\rho}-B|}\right) = F\left((2\tilde{\rho}-B)e^{2t}\right). \end{aligned} \tag{50}$$

The function F is determined throught the initial condition for the potential. Assuming a quartic classical potential given by eqs. (18), (19) we find

$$\begin{aligned} F(x) =& \lambda_\Lambda \frac{x+B}{2x} - \lambda_\Lambda \kappa_\Lambda \frac{1}{x} + \frac{De^2}{2(Be^2+1)}\frac{1}{x\left[e^2(x+B)+1\right]} \\ &+ \frac{De^4}{(Be^2+1)^2}\frac{1}{e^2(x+B)+1} - \frac{De^4}{(Be^2+1)^3}\ln\left(\frac{e^2(x+B)+1}{|x|}\right). \end{aligned} \tag{51}$$

In fig. 3 we plot the potential which results from eqs. (50), (51) for a certain choice of the parameters of the theory. Initially the effective average potential has only one minimum at a non-zero value of ρ. As k is lowered a second minimum appears around zero, which eventually becomes the absolute minimum of the potential. The discontinuity in the expectation value signals the presence of a first order phase transition. The development of the minimum around zero is caused by the logarithmic terms in eqs. (50), (51). The situation is typical of a Coleman-Weinberg phase transition triggered by radiative corrections [18]. The effective potential $U = U_0$ can be calculated from eqs. (50), (51) in the limit $t \to -\infty$, with $u', \tilde{\rho} \to \infty$, so that $u'e^{2t} \sim 1$, $\tilde{\rho}e^{2t} \sim 1$. We find

$$\frac{U'(\rho)}{\Lambda^2} = \lambda_\Lambda\left(\frac{\rho}{\Lambda^2} - \kappa_\Lambda\right) + \lambda_\Lambda\frac{B}{2} + \frac{De^2}{2(Be^2+1)}\frac{1}{e^2\left(2\frac{\rho}{\Lambda^2}+B\right)+1}$$

$$+\frac{2De^4}{(Be^2+1)^2}\frac{\frac{\rho}{\Lambda^2}}{e^2\left(2\frac{\rho}{\Lambda^2}+B\right)+1}+\frac{2De^4}{(Be^2+1)^3}\frac{\rho}{\Lambda^2}\ln\left(\frac{2e^2\frac{\rho}{\Lambda^2}}{e^2\left(2\frac{\rho}{\Lambda^2}+B\right)+1}\right). \quad (52)$$

Without the logarithmic term the phase transition in dependence to κ_Λ would have been second order. The presence of the last term in eq. (52) results in the development of a barrier near a secondary minimum at the origin. This leads to a weakly first order transition, with a discontinuity for the expectation value much smaller than the minimum of the classical potential. (For a detailed discussion of the Coleman-Weinberg transition using the full evolution equation see ref. [13].) The effective potential of eq. (52) is not convex. As we have mentioned already, the reason for this is the approximation of the "threshold" function by a constant in the evolution equation.

Conclusions: In this letter we presented analytical approximate solutions of the exact renormalization group equation for the effective average action. We neglected the effects of wave function renormalization and approximated the action by the potential and a standard kinetic term. This approximation is justified by the smallness of the anomalous dimension for the models that we considered. We solved the evolution equation for the potential as a function of the field and the running scale k. In order to achieve this we resorted to additional approximations, which, however, do not affect the qualitative behaviour of the solutions.

a) For the $O(N)$-symmetric scalar theory in three dimensions we distinguish two types of behaviour: I) For a classical ϕ^4 potential given by eq. (18) the renormalized theory has a second order phase transition in dependence on κ_Λ. The universal behaviour near the transition is governed by the Wilson-Fisher fixed point and can be parametrized by critical exponents. Due to our approximations, the valued we obtained for these exponents correspond to the large N limit of the theory. II) For a classical ϕ^6 potential given by eq. (39) there is a parameter range for which the renormalized theory has a second order phase transition in dependence on κ_Λ, with universal critical behaviour. For another parameter range the theory has a first order phase transition. The two regions are separated by a tricritical line (at $\lambda_\Lambda = 0$) which displays tricritical behaviour with mean field exponents.
b) For the Abelian Higgs model in four dimensions we reproduced the Coleman-Weinberg first order phase transition which is triggered by radiative corrections.

Our results on the universal behaviour of the three-dimensional scalar theory and the four-dimensional Abelian Higgs model are in perfect agreement with refs. [10, 11, 13] and provide an additional argument for the validity and applicability of the method of the effective average action in a wide range of problems. The most important aspect of our solutions, however, is that they are fully analytical. They encode all the relevant qualitative information for the dependence of the potential on the field and the running scale k. They provide a concise, transparent picture of the physical system, with which numerical results can be easily compared.

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Figure captions

Fig. 1 The effective average potential $U_k(\rho)$ at different scales for a scalar model with $N = 4$ and a classical potential given by eq. (18) with $\kappa_\Lambda = 0.162$ and $\lambda_\Lambda = 0.1$ ($d = 3$).

Fig. 2 The effective average potential $U_k(\rho)$ at different scales for a scalar model with $N = 4$ and a classical potential given by eq. (39) with $\kappa_\Lambda = 0.202$, $\lambda_\Lambda = 0.1$ and $\sigma_\Lambda = 2.228$ ($d = 3$).

Fig. 3 The effective average potential $U_k(\rho)$ at different scales for an Abelian Higgs model with one complex scalar. The classical potential is given by eq. (18) with $\kappa_\Lambda = 1$ and $\lambda_\Lambda = 0.003$, and $e^2 = 0.185$ ($d = 4$).

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Fig. 1

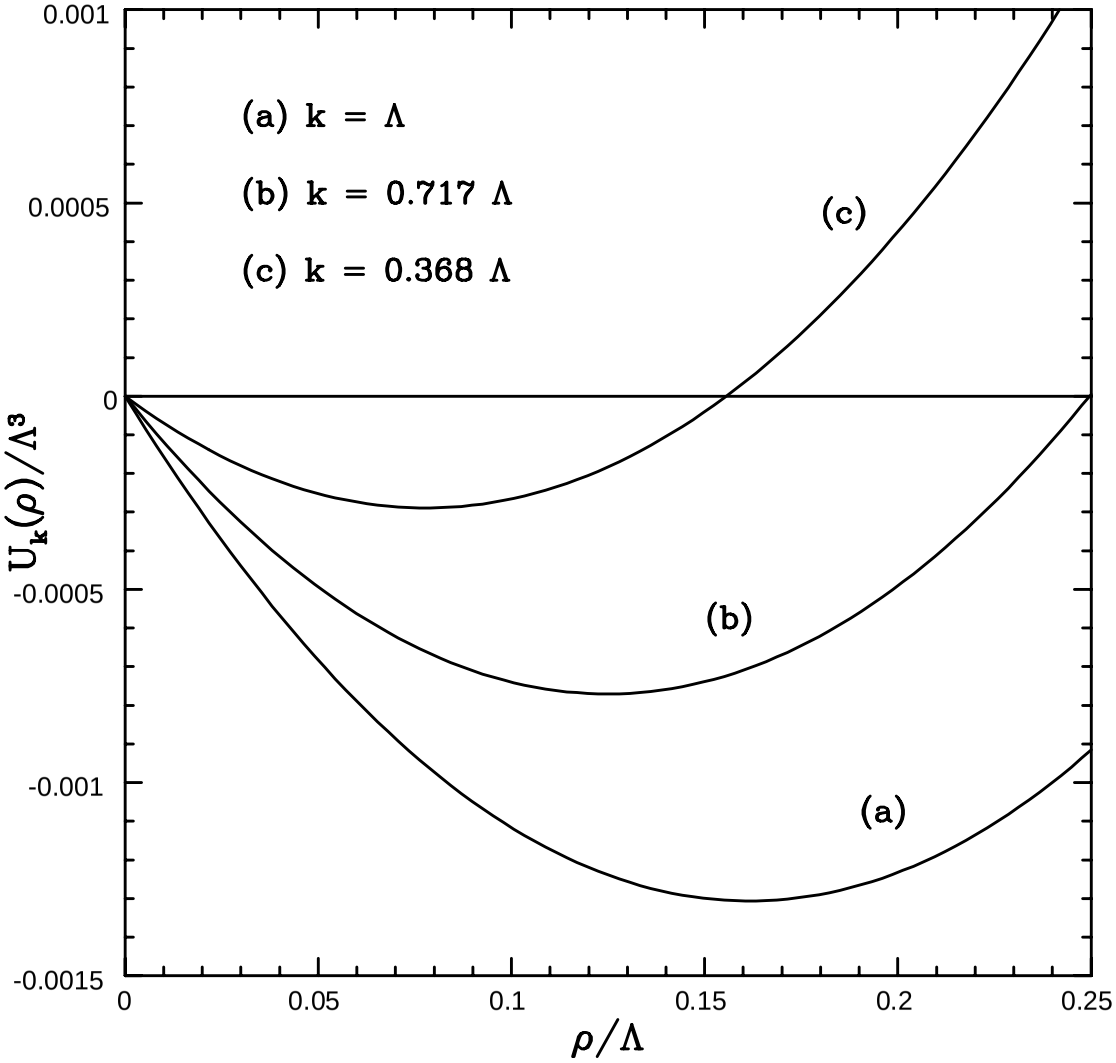

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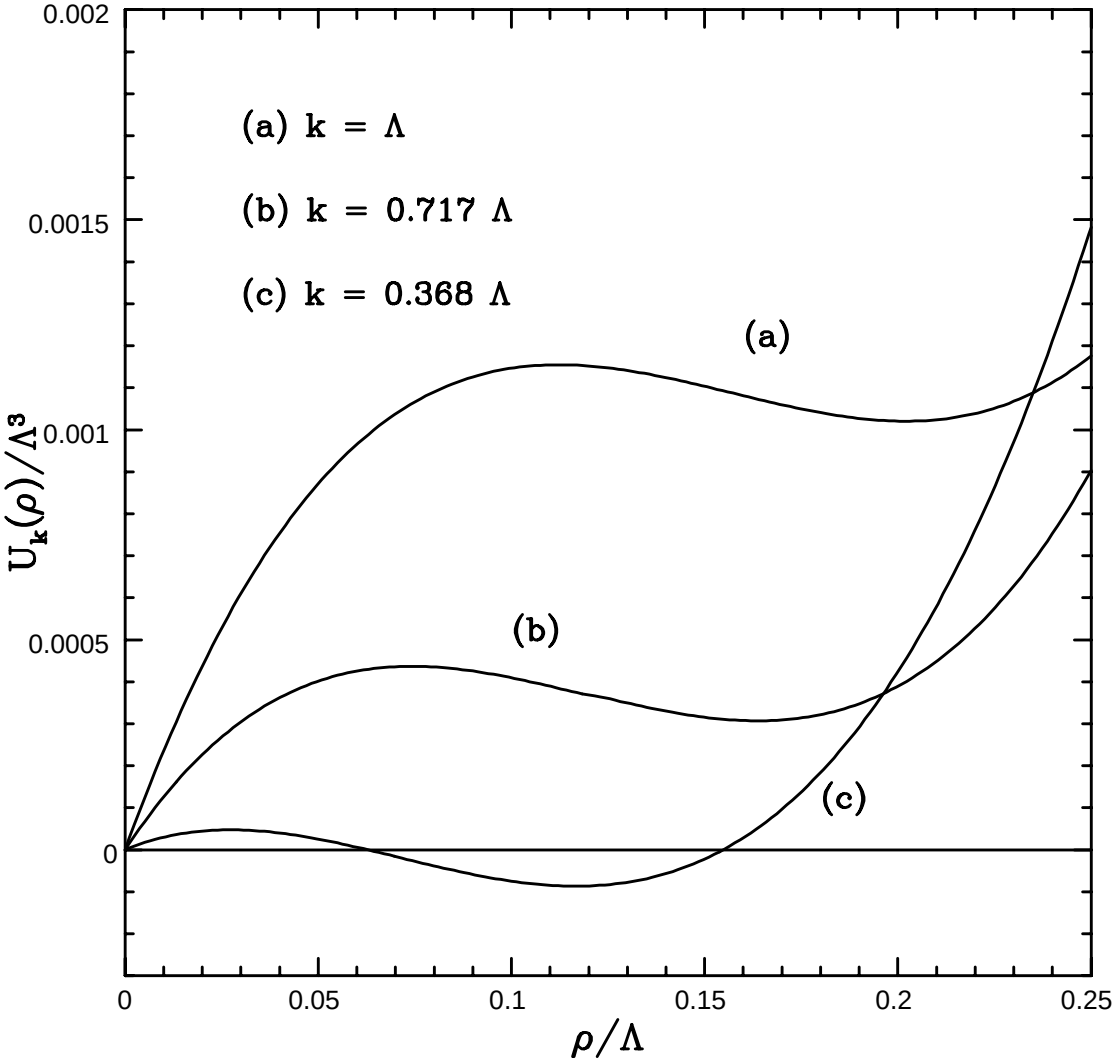
Fig. 2


(a) k = Λ
(b) k = 0.717 Λ
(c) k = 0.368 Λ
(a)
(b)
(c)
0.002
0.0015
0.001
0.0005
0
0
0.05
0.1
0.15
0.2
0.25
$U_k(\rho)/\Lambda^3$
ρ/Λ

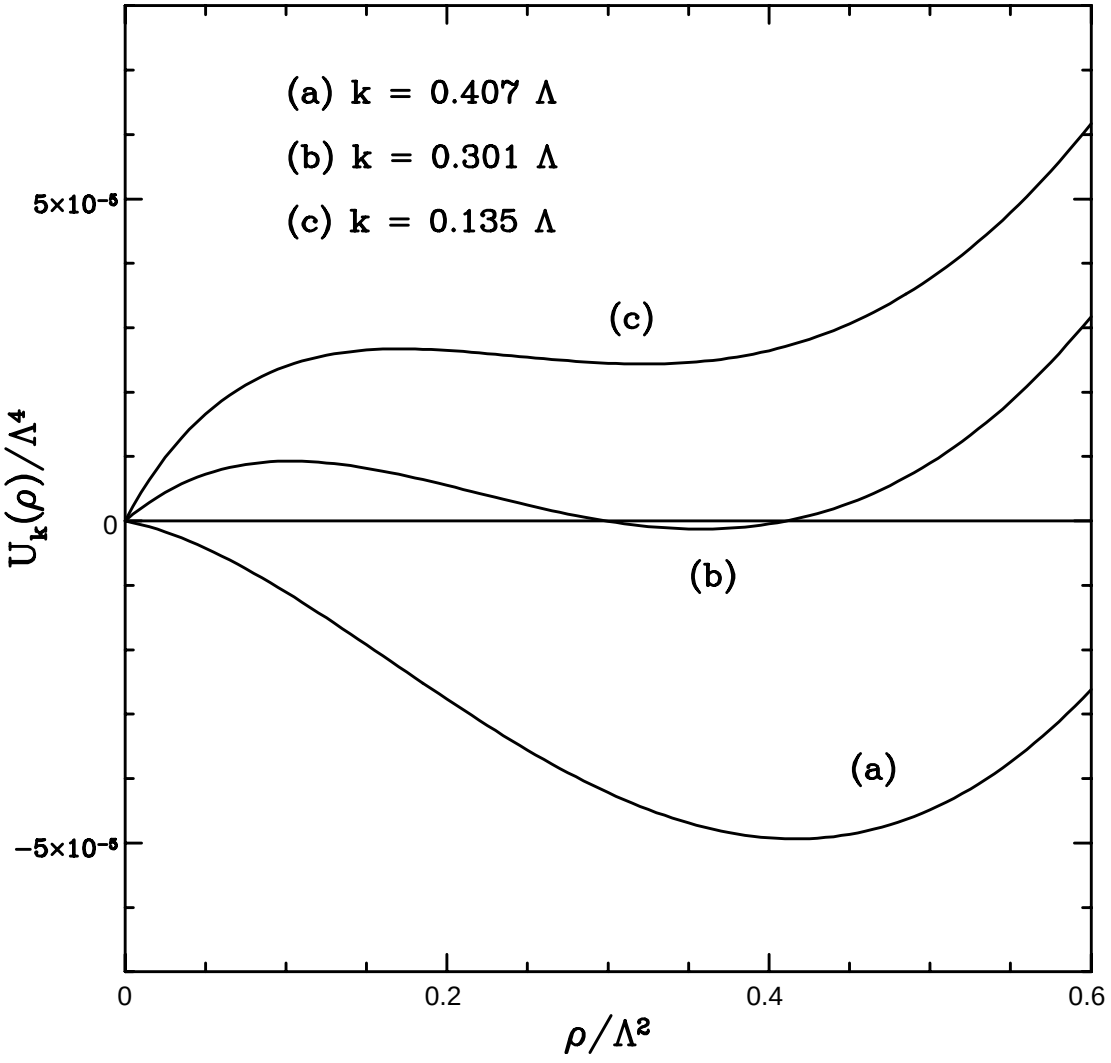



Fig. 3