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Analytical Solutions of Exact Renormalization Group Equations

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Abstract

We study exact renormalization group equations in the framework of the effective average action. We present analytical solutions for the scale dependence of the potential in a variety of models. These solutions display a rich spectrum of physical behaviour such as fixed points governing the universal behaviour near second order phase transitions, critical exponents, first order transitions (some of which are radiatively induced) and tricritical behaviour.

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1. Introduction

The solution of an exact renormalization group equation [1]-[10] is a particularly difficult task. The reason is that such an equation describes the scale dependence of an effective action, which is characterized by infinitely many couplings multiplying the invariants consistent with the symmetries of the model under consideration. As a result an exact renormalization group equation corresponds to infinitely many evolution equations for the couplings of the theory. The crucial step is developing efficient approximation schemes which can reduce the complexity of the problem while capturing the essential aspects of the physical system. Perturbative expansions have been used for proofs of perturbative renormalizability [5, 11], while the powerful ϵ-expansion [1, 12] has been employed for the study of fixed points governing second order phase transitions in three dimensions. In many cases evolution equations for truncated forms of the effective action have been solved through a combination of analytical and numerical methods (see the review [7] and references therein, [6, 13], [14]-[20]. Also, numerical solutions for the fixed point potential of three-dimensional scalar theories have been computed in ref. [21, 22].

In this paper we present analytical solutions of equations which describe the evolution of the running effective potential. We are interested in describing the evolution not only near the possible fixed points, but also far away from them. This allows us to study the influence of the initial conditions (the parameters of the classical theory) on the renormalized theory. We can also investigate theories with first order phase transitions, which do not exhibit universal behaviour. The biggest part of this work concerns the $O(N)$-symmetric scalar theory in the large N limit, the spherical model [23]. We obtain analytical solutions of the evolution equation which describes the dependence of the running effective potential U_k on an effective infrared cutoff k. This equation, and our solutions, are exact in the large N limit. In three dimensions, we show that for a certain range of classical parameters the evolution of the critical theory leads to an infrared attractive fixed point (the Wilson-Fisher fixed point). This results in a second order phase transition with universal behaviour. For a different parameter range the theory has a first order phase transition. A tricritical line separates the two regimes. In four dimensions we describe how the renormalized quartic coupling evolves to zero as the infrared cutoff is removed. Finally in two dimensions we explicitly demonstrate the absence of symmetry breaking and a phase transition for this model. The incorporation of physical phenomena of such a wide range is the most prominent and promising characteristic of our analytical solutions. We subsequently study an approximate evolution equation for the running effective potential in the context of the Abelian Higgs model in four dimensions. We demonstrate how its solution predicts a first order phase transition driven by radiative corrections (the Coleman-Weinberg mechanism). Parts of our solutions for the $O(N)$-symmetric scalar theory have been obtained previously [14, 7]. However, the starting point for our derivation is a recently proposed renormalization group equation [8], and our results demonstrate the complete agreement of its physical predictions with those of other renormalization group equations [2, 4, 6]. We also give a detailed discussion of the full range of the running effective potential (including its non-convex part), and consider

a variety of regions for the classical parameters of the theory.

We work in the framework of the effective average action Γ_k [8, 15], which has been used for the study of second and first order phase transitions for a variety of four-dimensional field theories at non-zero temperature [16]-[19]. Also, in ref. [17] an approximate solution of the evolution equation for the potential for the three-dimensional $O(N)$-symmetric scalar theory in the large N limit has been given. The effective average action Γ_k results from the integration of quantum fluctuations with characteristic momenta $q^2 \geq k^2$. It interpolates between the classical action S for k equal to the ultraviolet cutoff Λ of the theory (no integration of modes) and the effective action Γ for $k = 0$ (all the modes are integrated). Its dependence on k is given by an exact renormalization group equation with the typical form ($t = \ln(k/\Lambda)$)

$$\frac{\partial}{\partial t}\Gamma_k = \frac{1}{2}\mathrm{Tr}\Big\{(\Gamma_k^{(2)} + R_k)^{-1}\frac{\partial}{\partial t}R_k\Big\}. \tag{1.1}$$

Here $\Gamma_k^{(2)}$ is the second functional derivative with respect to the fields, R_k is the effective infrared cutoff which prevents the integration of modes with $q^2 \leq k^2$, and the trace implies integration over all Fourier modes of the fields. We work with an approximation which neglects the effects of wavefunction renormalization. Therefore, only a classical kinetic term in the effective average action is kept, which takes, for an $O(N)$-symmetric scalar theory, the form

$$\Gamma_k = \int d^dx\Big\{U_k(\phi) + \frac{1}{2}\partial^\mu\phi_a\partial_\mu\phi^a\Big\}, \tag{1.2}$$

and all invariants which involve more derivatives of the fields are neglected. With this approximation, eq. (**1**.1) can be turned into an evolution equation for the potential U_k. First we shall discuss the $O(N)$-symmetric scalar theory in the large N limit. In three dimensions the fixed point solution which governs the second order phase transition will be identified. We shall also show that, for a certain parameter range, the theory has a first order phase transition. The two regions in parameter space are separated by a tricritical line. We shall also study the same theory in four and two dimensions. Finally we shall discuss the Abelian Higgs model in four dimensions, for which the radiatively induced first order transition will be reproduced.

2. The $O(N)$-symmetric scalar theory

We first consider the evolution equation describing the dependence of the effective average potential U_k on the scale k in arbitrary dimensions d, for an $O(N)$-symmetric scalar theory. As we have mentioned in the introduction, we neglect the effects of wavefunction renormalization. The evolution equation reads ($t = \ln(k/\Lambda)$) [8]

$$\frac{\partial U_k(\rho)}{\partial t} = v_d \int_0^\infty dx\, x^{\frac{d}{2}-1}\,\frac{\partial P}{\partial t}\left\{\frac{N-1}{P + U_k'(\rho)} + \frac{1}{P + U_k'(\rho) + 2U_k''(\rho)\rho}\right\}. \tag{2.1}$$

Here $\rho = \frac{1}{2}\phi^a\phi_a,\ a = 1...N$, and primes denote derivatives with respect to ρ. The variable x denotes momentum squared $x = q^2$, and

$$v_d^{-1} = 2^{d+1}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right). \tag{2.2}$$

The inverse average propagator

$$P(x) = \frac{x}{1 - f_k^2(x)} \tag{2.3}$$

incorporates an effective infrared cutoff k, such that modes with $x = q^2 \ll k^2$ do not propagate. The "cutoff" function $f_k^2(x)$ is given by

$$f_k^2(x) = \exp\left\{-2a\left(\frac{x}{k^2}\right)^b\right\} \tag{2.4}$$

and can be made sharper or smoother by an appropriate choice of the two free parameters a, b. Up to effects from the wavefunction renormalization eq. (**2.**1) is an exact non-perturbative evolution equation [8]. It is easy to recognize the first term in the r.h.s. of eq. (**2.**1) as the contribution of the $N-1$ Goldstone modes (U_k' vanishes at the minimum). The second term is related to the radial mode. After performing the momentum integration the evolution equation (**2.**1) becomes a partial differential equation for U_k with independent variables ρ and t. The effective average potential interpolates between the classical potential V for $k = \Lambda$ (with Λ the ultraviolet cutoff) and the effective potential U for $k = 0$ [8]. As a result the solution of eq. (**2.**1) with the initial condition $U_\Lambda(\rho) = V(\rho)$ uniquely determines, for $k \to 0$, the effective 1PI vertices at zero momentum for the renormalized theory. In order to write eq. (**2.**1) in a scale invariant form it is convenient to define the variables

$$\begin{aligned} \tilde{\rho} =& k^{2-d}\rho \\ u_k(\tilde{\rho}) =& k^{-d}U_k(\rho). \end{aligned} \tag{2.5}$$

In terms of these eq. (**2.**1) can be written as (for the details see refs. [8, 15, 16, 17])

$$\frac{\partial u'}{\partial t} = -2u' + (d-2)\tilde{\rho}u'' + v_d(N-1)u''L_1^d(u') + v_d(3u'' + 2\tilde{\rho}u''')L_1^d(u' + 2\tilde{\rho}u''). \tag{2.6}$$

Primes on u denote derivatives with respect to $\tilde{\rho}$ and we omit the subscript k in u_k from now on. The functions $L_1^d(w)$ are given by

$$L_1^d(w) = -2(2a)^{\frac{2-d}{2b}}\int_0^\infty dy\ y^{\frac{d-2}{2b}}e^{-y}\left[1 + \left(\frac{2a}{y}\right)^{\frac{1}{b}}\left(1 - e^{-y}\right)w\right]^{-2} \tag{2.7}$$

(where we have used the variable $y = 2a\left(x/k^2\right)^b$). They introduce "threshold" behaviour in the evolution equation, which results in the decoupling of heavy modes. They approach

a constant for vanishing argument, and vanish for large arguments. For general a and b they do not have a simple analytical form. However, in the sharp cutoff limit $b \to \infty$ they are given by the simple expression

$$L_1^d(w) = \; -\frac{2}{1+w}. \tag{2.8}$$

The resulting evolution equation (with an appropriate rescaling of the fields) is identical to the one introduced in ref. [2] and studied in refs. [2, 6, 22]. Also, in the large N limit (see below) it is the same as the evolution equation [7] which results from the exact renormalization group equation of ref. [4], for which solutions are given in ref. [14]. Our derivation establishes the agreement of the physical predictions resulting from the formalism of the effective average action with those of previous renormalization group formulations. We should point out that the use of a sharp cutoff renders the effective average action Γ_k non-local. This complication, however, does not affect our analysis, as we have neglected the effect of the higher derivative terms on the evolution equation for the potential. For studies which take into account the effects of the wavefunction renormalization [17] the use of a smooth cutoff is necessary.

Even with the simple form of eq. (**2.**8) for the "threshold" functions, the evolution equation (**2.**6) remains a non-linear partial differential equation which seems very difficult to solve exactly. [3] The problems arise from the contribution from the radial mode to the r.h.s. of eq. (**2.**6) (the last term). An enormous simplification is achieved, however, if this contribution is neglected. This is justified in the large N limit, in which the contributions of the Goldstone modes dominate the evolution, and the contribution of the radial mode becomes a subleading effect. Another important simplification results from the fact that the anomalous dimension of the field is zero to leading order in $1/N$, in four and three dimensions which are of most interest [24]. As a result, our approximation of neglecting the wavefunction renormalization is justified in this limit and our solution of eq. (**2.**6) becomes exact. We should point out that the contributions of the radial mode cannot be neglected even in the large N limit in some cases. For certain forms of the classical potential, the approach to convexity for the effective potential depends crucially on the radial mode. An example of this behaviour will be given in the following. For $N \gg 1$ the simplified evolution equation reads

$$\frac{\partial u'}{\partial t} - (d-2)\tilde{\rho}\frac{\partial u'}{\partial \tilde{\rho}} + \frac{NC}{1+u'}\frac{\partial u'}{\partial \tilde{\rho}} + 2u' = 0, \tag{2.9}$$

with $C = 2v_d$. It is first order in both independent variables and can be solved with the method of characteristics.

a) Three dimensions

We are interested in the behaviour of the theory in three dimensions, where a non-trivial fixed point structure arises. The most general solution of the partial differential equation

[3] See ref. [20] for a numerical solution of eq. (**2.**6).

(**2.**9) for $d = 3$ is given by the relations

$$\frac{\tilde{\rho}}{\sqrt{u'}} - \frac{NC}{\sqrt{u'}} + NC \arctan\left(\frac{1}{\sqrt{u'}}\right) = F\left(u' e^{2t}\right) \qquad \text{for} \quad u' > 0 \tag{2.10}$$

$$\frac{\tilde{\rho}}{\sqrt{-u'}} - \frac{NC}{\sqrt{-u'}} - \frac{1}{2} NC \ln\left(\frac{1-\sqrt{-u'}}{1+\sqrt{-u'}}\right) = F\left(u' e^{2t}\right) \qquad \text{for} \quad u' < 0, \tag{2.11}$$

with

$$C = 2v_3 = \frac{1}{4\pi^2}. \tag{2.12}$$

The function F is undetermined until initial conditions are specified. For $t = 0$ ($k = \Lambda$), U_k coincides with the classical potential V. The initial condition, therefore, reads

$$u'(\tilde{\rho}, t = 0) = \Lambda^{-2} V'(\rho). \tag{2.13}$$

This uniquely specifies F and we obtain

$$\begin{aligned} &\frac{\tilde{\rho}}{\sqrt{u'}} - \frac{NC}{\sqrt{u'}} + NC \arctan\left(\frac{1}{\sqrt{u'}}\right) = \\ &\qquad \frac{G\left(u' e^{2t}\right)}{\sqrt{u'} e^t} - \frac{NC}{\sqrt{u'} e^t} + NC \arctan\left(\frac{1}{\sqrt{u'} e^t}\right) \qquad \text{for} \quad u' > 0 \end{aligned} \tag{2.14}$$

$$\begin{aligned} &\frac{\tilde{\rho}}{\sqrt{-u'}} - \frac{NC}{\sqrt{-u'}} - \frac{1}{2} NC \ln\left(\frac{1-\sqrt{-u'}}{1+\sqrt{-u'}}\right) = \\ &\qquad \frac{G\left(u' e^{2t}\right)}{\sqrt{-u'} e^t} - \frac{NC}{\sqrt{-u'} e^t} - \frac{1}{2} NC \ln\left(\frac{1-\sqrt{-u'} e^t}{1+\sqrt{-u'} e^t}\right) \qquad \text{for} \quad u' < 0, \end{aligned} \tag{2.15}$$

with the function G determined by inverting eq. (**2.**13) and solving for $\tilde{\rho}$ in terms of u'

$$G(u') = \tilde{\rho}(u')|_{t=0}. \tag{2.16}$$

We are also interested in the scale invariant (independent of t) solutions of eq. (**2.**9). These solutions correspond to the possible fixed points of the evolution and are obtained by setting the first term in the l.h.s. of eq. (**2.**9) equal to zero. We can easily identify the "trivial" solution $u'_\star = 0$, which is independent of the number of dimensions. For $d = 3$ the only other solution which remains finite for finite $\tilde{\rho}$ is given by the relations

$$\frac{\tilde{\rho}}{\sqrt{u'_\star}} - \frac{NC}{\sqrt{u'_\star}} + NC \arctan\left(\frac{1}{\sqrt{u'_\star}}\right) = \frac{\pi}{2} NC \qquad \text{for} \quad u'_\star > 0 \tag{2.17}$$

$$\frac{\tilde{\rho}}{\sqrt{-u'_\star}} - \frac{NC}{\sqrt{-u'_\star}} - \frac{1}{2} NC \ln\left(\frac{1-\sqrt{-u'_\star}}{1+\sqrt{-u'_\star}}\right) = 0 \qquad \text{for} \quad u'_\star < 0. \tag{2.18}$$

This solution corresponds to the Wilson-Fisher fixed point. A large part of the above results has been obtained in the past [14, 7]. However, we have explicitly presented here

the solution for the non-convex part of the potential. We are also mainly interested in studying the various phase transitions which result for different ranges of classical parameters of the theory.

I) *Classical ϕ^4 theory:* Let us first consider a theory with a quartic classical potential. The initial condition can be written as

$$u'(\tilde{\rho}, t=0) = \lambda_\Lambda(\tilde{\rho} - \kappa_\Lambda), \tag{2.19}$$

with

$$\kappa_\Lambda = \frac{\rho_{0\Lambda}}{\Lambda}, \qquad \lambda_\Lambda = \frac{\bar{\lambda}_\Lambda}{\Lambda} \tag{2.20}$$

the rescaled (dimensionless) minimum of the potential and quartic coupling respectively. The function G in eqs. (**2.**14), (**2.**15) is now given by

$$G(x) = \kappa_\Lambda + \frac{x}{\lambda_\Lambda}. \tag{2.21}$$

The typical form of the effective average potential $U_k(\rho)$ at different scales k, as given by eqs. (**2.**14), (**2.**15), is presented in fig. 1. The theory at the ultraviolet cutoff is defined in the regime with spontaneous symmetry breaking, with the minimum of the potential at $\rho_{0\Lambda} = \kappa_\Lambda \Lambda \neq 0$. At lower scales k the minimum of the potential moves continuously closer to zero, with no secondary minimum ever developing. We expect a second order phase transition (in dependence to κ_Λ) for the renormalized theory at $k = 0$. There is a critical value for the minimum of the classical potential

$$\kappa_\Lambda = \kappa_{cr} = NC, \tag{2.22}$$

for which the scale invariant (fixed point) solution of eqs. (**2.**17), (**2.**18) is approached in the limit $t \to -\infty$ ($k \to 0$). Eqs. (**2.**17), (**2.**18) describe a potential u which has a minimum at a constant value

$$\kappa(k) = \kappa_\star = NC. \tag{2.23}$$

This leads to a potential $U_k(\rho)$ with a minimum at $\rho_0(k) = k\kappa_\star \to 0$ for $k \to 0$, which corresponds to the phase transition between the spontaneously broken and the symmetric phase. (The values for κ_{cr} and $\kappa_\star$ coincide, but this is accidental.) For the second and third $\tilde{\rho}$-derivative of u at the minimum $\lambda = u''(\kappa)$, $\sigma = u'''(\kappa)$ we find

$$\lambda(k) = \lambda_\star = \frac{1}{NC} \tag{2.24}$$

$$\sigma(k) = \sigma_\star = \frac{2}{3(NC)^2}, \tag{2.25}$$

and similar fixed point values for the higher derivarives of u. For $1 \ll \tilde{\rho}/NC \ll (\pi/2)e^{-t}$ the rescaled potential u has the form

$$u'_\star(\tilde{\rho}) = \left(\frac{2}{\pi NC}\right)^2 \tilde{\rho}^2. \tag{2.26}$$

Notice that the region of validity of eq. (**2.**26) extends to infinite $\tilde{\rho}$ for $t \to -\infty$. From eq. (**2.**26) with $t \to -\infty (k \to 0)$ we obtain for the effective potential at the phase transition

$$U_\star(\rho) = \frac{1}{3}\left(\frac{2}{\pi N C}\right)^2 \rho^3. \tag{2.27}$$

Through eqs. (**2.**14), (**2.**15) we can also study solutions which deviate slightly from the scale invariant one. For this purpose we define a classical potential with a minimum

$$\kappa_\Lambda = \kappa_{cr} + \delta\kappa_\Lambda, \tag{2.28}$$

with $|\delta\kappa_\Lambda| \ll NC$. A typical solution is depicted in fig. 2. The potential starts its evolution with its classical form at $k = \Lambda$. It subsequently evolves towards the scale invariant solution given by eqs. (**2.**17), (**2.**18). It stays very close this solution for a long "time" t, which can be rendered arbitrarily long for sufficiently small $|\delta\kappa_\Lambda|$. Eventually it deviates towards the phase with spontaneous symmetry breaking or the symmetric one. In fig. 2 $\delta\kappa_\Lambda$ was chosen negative, so that the evolution for $k \to 0$ leads to the symmetric phase. The curvature of the potential at the origin becomes positive and $u'(0)$ diverges so that the renormalized mass term $U_k'(0) = k^2 u'(0)$ reaches a constant value. The evolution of the minimum of the potential is given by

$$\kappa(k) = \kappa_\star + \delta\kappa_\Lambda e^{-t}. \tag{2.29}$$

Also for λ we find

$$\lambda(k) = \frac{\lambda_\star}{1 + \left(\frac{\lambda_\star}{\lambda_\Lambda} - 1\right) e^t}. \tag{2.30}$$

Eq. (**2.**29) indicates that the minimum of u stays close to the fixed point value $\kappa_\star$ given by eq. (**2.**23), for a very long "time" $|t| < -\ln|\delta\kappa_\Lambda|$. For $|t| > -\ln|\delta\kappa_\Lambda|$ it deviates from the fixed point, either towards the phase with spontaneous symmetry breaking (for $\delta\kappa_\Lambda > 0$), or the symmetric one (for $\delta\kappa_\Lambda < 0$). Eq. (**2.**30) implies an attractive fixed point for λ, with a value given by eq. (**2.**24). Similarly the higher derivatives are attracted to their fixed point values. The full phase diagram corresponds to a second order phase transition. For $\delta\kappa_\Lambda > 0$ the system ends up in the phase with spontaneous symmetry breaking, with

$$\rho_0 = \lim_{k\to 0} \rho_0(k) = \lim_{k\to 0} k\kappa(k) = \delta\kappa_\Lambda \Lambda. \tag{2.31}$$

In this phase the renormalized quartic coupling approaches zero linearly with k

$$\lambda_R = \lim_{k\to 0} k\lambda(k) = \lim_{k\to 0} k\lambda_\star = 0. \tag{2.32}$$

The fluctuations of the Goldstone bosons lead to an infrared free theory in the phase with spontaneous symmetry breaking. For $\delta\kappa_\Lambda < 0$, $\kappa(k)$ becomes zero at a scale

$$t_s = -\ln\left(\frac{\kappa_\star}{|\delta\kappa_\Lambda|}\right) \tag{2.33}$$

and the system ends up in the symmetric regime ($\rho_0 = 0$). From eq. (**2.**14), in the limit $t \to -\infty$, with $u', u'', u''' \to \infty$, so that $u'e^{2t} \sim |\delta\kappa_\Lambda|^2$, $u''e^t \sim |\delta\kappa_\Lambda|$, $u''' \sim 1$, we find

$$U(\rho) = U_0(\rho) = \left(\frac{2}{\pi NC}\right)^2 \left[|\delta\kappa_\Lambda|^2\Lambda^2\rho + |\delta\kappa_\Lambda|\Lambda\rho^2 + \frac{1}{3}\rho^3\right]. \tag{2.34}$$

Notice how every reference to the classical theory has disappeared in the above expression. The effective potential of the critical theory is determined uniquely in terms of $\delta\kappa_\Lambda$, which measures the distance from the phase transition. The above results are in exact agreement with refs. [23, 14, 7, 24, 17, 25]. In particular the values for the critical exponents β, ν, describing the behaviour of the system very close to the phase transition, correspond to the large N limit of the model (for details see section 10 of ref. [17])

$$\begin{aligned}\beta &= \lim_{\delta\kappa_\Lambda \to 0^+} \frac{d\left(\ln\sqrt{\rho_0}\right)}{d(\ln\delta\kappa_\Lambda)} = 0.5 \\ \nu &= \lim_{\delta\kappa_\Lambda \to 0^-} \frac{d\left(\ln m_R\right)}{d(\ln|\delta\kappa_\Lambda|)} = \frac{d\left(\ln\sqrt{U'(0)}\right)}{d(\ln|\delta\kappa_\Lambda|)} = 1.\end{aligned} \tag{2.35}$$

We also point out that, for a theory with spontaneous symmetry breaking , we can use eq. (**2.**15) in order to study the "inner" part of the potential. In particular, for $\tilde{\rho} = 0$ and $t \to -\infty$ eq. (**2.**15) predicts a potential u which asymptotically behaves as

$$\lim_{t \to -\infty} u'(0) = -1. \tag{2.36}$$

This leads to an effective average potential U_k which becomes convex with

$$\lim_{k \to 0} U_k'(0) = -k^2, \tag{2.37}$$

in agreement with the detailed study of ref. [28].

II) *Classical ϕ^6 theory:* As a second example we consider a theory defined through a classical potential with a ρ^3 (ϕ^6) term

$$u'(\tilde{\rho}, t = 0) = \lambda_\Lambda(\tilde{\rho} - \kappa_\Lambda) + \frac{\sigma_\Lambda}{2}(\tilde{\rho} - \kappa_\Lambda)^2, \tag{2.38}$$

where κ_Λ, λ_Λ are defined in eq. (**2.**20) and the coupling σ_Λ is dimensionless in $d = 3$. The function G in eqs. (**2.**14), (**2.**15) is now given by

$$G(x) = \kappa_\Lambda + \frac{-\lambda_\Lambda + \sqrt{\lambda_\Lambda^2 + 2\sigma_\Lambda x}}{\sigma_\Lambda} \qquad \text{for} \quad u'' > 0 \tag{2.39}$$

$$G(x) = \kappa_\Lambda + \frac{-\lambda_\Lambda - \sqrt{\lambda_\Lambda^2 + 2\sigma_\Lambda x}}{\sigma_\Lambda} \qquad \text{for} \quad u'' < 0. \tag{2.40}$$

We distinguish two regions in parameter space which result in two different types of behaviour for the theory:

(a) For $\kappa_\Lambda < 2\lambda_\Lambda/\sigma_\Lambda$ the classical potential has only one minimum at $\rho_{0\Lambda} = \kappa_\Lambda \Lambda$. Near this minimum the initial condition of eq. (**2.**38) is very well approximated by eq. (**2.**19). As a result, for κ_Λ near the critical value of eq. (**2.**22), the critical theory has exactly the same behaviour as for a quartic classical potential. The running potential first approaches the fixed point solution of eqs. (**2.**17), (**2.**18) (notice that $\kappa_\star < 2\lambda_\star/\sigma_\star$), and subsequently evolves towards the phase with spontaneous symmetry breaking or the symmetric one. The behaviour of the critical theory for $k = 0$ is determined only by the distance from the phase transition (as measured by $\delta\kappa_\Lambda$), without any memory of the details of the classical theory. This is a manifestation of universality, typical of second order phase transitions. For $\kappa_\Lambda > 2\lambda_\Lambda/\sigma_\Lambda$ the classical potential has two minima, one at the origin and one at $\rho_{0\Lambda} = \kappa_\Lambda \Lambda$. The minimum at the origin is less deep for $\kappa_\Lambda < 3\lambda_\Lambda/\sigma_\Lambda$. Again, for κ_Λ near κ_{cr} the scale invariant solution is approached. We demonstrate this type of behaviour in fig. 3 for a theory with $\kappa_\Lambda = \kappa_{cr}$. The classical potential includes a ϕ^6 term and has a (less deep) second minimum at the origin, but the universal scale invariant solution is again approached for $k \to 0$. Small deviations of κ_Λ from the critical value result in universal behaviour for the renormalized theory.
(b) The minimum of the classical potential at the origin is deeper for $\kappa_\Lambda > 3\lambda_\Lambda/\sigma_\Lambda$. An example of the evolution of the effective average potential $U_k(\rho)$ for such a theory is given in fig. 4. The minimum of the potential at non-zero ρ moves towards the origin for decreasing scale k. In the same time the positive curvature at the origin decreases. The combined effect is (very crudely) similar to the whole potential being shifted to the left of the graph. As a result the minimum at the origin becomes shallower. For a certain range of the parameter space (for small enough κ_Λ, such as chosen for fig. 4) the minimum away from the origin becomes the absolute minimum of the potential at some point during the evolution. This results in a discontinuity in the running order parameter. Finally the absolute minimum of the potential settles down at some non-zero ρ_0. For even larger κ_Λ the minimum at the origin is deep enough for the evolution to stop while this minimum is still the absolute minimum of the potential. When the minimum of the renormalized potential ρ_0 (which is obtained at the end of the evolution) is considered as a function of κ_Λ, a discontinuity is observed in the function $\rho_0(\kappa_\Lambda)$. This indicates a first order phase transition. Unfortunately, an exact quantitative determination of the region in parameter space which results in first order transitions is not possible within the approximations we have used. The reason for this is the omission of the "threshold" function for the radial mode which includes the term $2\tilde{\rho}u''$. As a result our approximation is not adequate for dealing with the shape of the barrier in the limit $k \to 0$, where the theshold function for the radial mode becomes important. Also the approach to convexity cannot be reliably discussed (in contrast to the case of a classical ϕ^4 potential). If the shape of the barrier cannot be reliably determined the relative depth of the two minima cannot be calculated, and our discussion is valid only at the qualitative level.

However, more information can be extracted from our results. As long as we concentrate on regions of the potential away from the top of the barrier the solution given by eqs. (**2.**14), (**2.**39), (**2.**40) is reliable. This means that we can study the potential around

its two minima. We are interested in the limit $t \to -\infty$ ($k \to 0$), with $U' = u' e^{2t}$, $\rho = \tilde{\rho} e^t$ approaching finite values. The form of the potential near the minimum away from the origin is determined by eqs. (**2.**14), (**2.**39). We find

$$\frac{\rho}{\Lambda} - \kappa_\Lambda + NC = \frac{-\lambda_\Lambda + \sqrt{\lambda_\Lambda^2 + 2\sigma_\Lambda \frac{U'}{\Lambda^2}}}{\sigma_\Lambda} + NC\sqrt{\frac{U'}{\Lambda^2}} \arctan\left(\frac{1}{\sqrt{\frac{U'}{\Lambda^2}}}\right). \tag{2.41}$$

The minimum ρ_0 (where $U'(\rho_0) = 0$) is located at $\rho_0 = (\kappa_\Lambda - NC)\Lambda = \delta\kappa_\Lambda \Lambda$. This requires $\delta\kappa_\Lambda \geq 0$. Eqs. (**2.**14), (**2.**40) describe the form of the potential around the minimum at the origin. Similarly as above we find

$$\frac{\rho}{\Lambda} - \kappa_\Lambda + NC = \frac{-\lambda_\Lambda - \sqrt{\lambda_\Lambda^2 + 2\sigma_\Lambda \frac{U'}{\Lambda^2}}}{\sigma_\Lambda} + NC\sqrt{\frac{U'}{\Lambda^2}} \arctan\left(\frac{1}{\sqrt{\frac{U'}{\Lambda^2}}}\right). \tag{2.42}$$

In the parameter range $\kappa_\Lambda - 2\lambda_\Lambda/\sigma_\Lambda, 2\lambda_\Lambda/\sigma_\Lambda \gg NC$ the above solution reproduces the classical potential, with a large positive curvature $U'(0)/\Lambda^2$ at the origin. This is due to the fact that the fluctuations which renormalize the potential around the origin are massive, with their masses acting as an effective infrared cutoff. For the above parameter range these masses are of the order of the ultraviolet cutoff Λ and no renormalization of the potential takes place. This is in contrast with the form of the potential near the minimum ρ_0 away from the origin. The presence of the Goldstone modes in this region always results in strong renormalization. There is a range of parameters for which the curvature at the origin becomes zero. It is given by the relation

$$\kappa_\Lambda = NC + 2\lambda_\Lambda/\sigma_\Lambda. \tag{2.43}$$

For this range the minimum at the origin disappears and the potential has only one minimum at $\rho_0 = \delta\kappa_\Lambda \Lambda = (\kappa_\Lambda - NC)\Lambda = 2\lambda_\Lambda/\sigma_\Lambda$. The above condition does not determine precisely the first order phase transition, as this takes place when the two minima are degenerate, and not when the minimum at the origin disappears. However, it provides a good estimate of its location. The discontinuity in the order parameter is expected to be $\mathcal{O}(\delta\kappa_\Lambda)$. Weakly first order transitions are obtained for $\lambda_\Lambda \to 0$. We should emphasize that eq. (**2.**42) is not valid for arbitrarily small U'/Λ^2. This would correspond to a range of the potential near the top of the disappearing barrier, where we know that our approximation fails. This is another reason why eq. (**2.**43) is only indicative of the location of the first order phase transition.

We have identified two critical surfaces in parameter space. We saw in (a) that the surface $\kappa_\Lambda = NC$ corresponds to second order phase transitions. Also in (b) we argued that the surface $\kappa_\Lambda = NC + 2\lambda_\Lambda/\sigma_\Lambda$ corresponds to first order transitions. As a result we expect tricritical behaviour to characterize their intersection, which is given by the line $\kappa_\Lambda = NC$, $\lambda_\Lambda = 0$. This is confirmed if we approach this line close to the critical surface $\kappa_\Lambda = NC$. More specifically we consider a theory with $0 < -\delta\kappa_\Lambda = -\kappa_\Lambda + NC \ll NC$ and $\lambda_\Lambda \ll 1/NC$. For this choice of parameters the renormalized theory is in the symmetric

phase very close to the second order phase transition. The form of the potential is given by eq. (**2.**41) with $U'/\Lambda^2 \ll 1$

$$\frac{\rho}{\Lambda} + |\delta\kappa_\Lambda| = \frac{1}{\lambda_\Lambda}\frac{U'}{\Lambda^2} + \frac{\pi}{2}NC\sqrt{\frac{U'}{\Lambda^2}}. \tag{2.44}$$

For $|\delta\kappa_\Lambda|/NC \ll \lambda_\Lambda NC$ the potential has the universal form of eq. (**2.**34). The initial point of the evolution is sufficiently close to the critical surface for the flows to approach the Wilson-Fisher critical point before deviating towards the symmetric phase. The critical exponent ν takes the large N value $\nu = 1$ according to eq. (**2.**35). In the opposite limit $|\delta\kappa_\Lambda|/NC \gg \lambda_\Lambda NC$ the potential near the origin is given by

$$U(\rho) = \lambda_\Lambda \left(|\delta\kappa_\Lambda|\Lambda^2\rho + \frac{1}{2}\Lambda\rho^2\right) \tag{2.45}$$

and the exponent ν takes its mean field value $\nu = 0.5$. A continuous transition from one type of behaviour to the other (a crossover curve) connects the two parameter regions. Clearly, the line $\kappa_\Lambda = NC$, $\lambda_\Lambda = 0$ gives tricritical behaviour with mean field exponents, in agreement with the analysis of ref. [26]. [4]

b) Four dimensions

In the previous subsection we explored the non-trivial fixed point structure of the three-dimensional scalar theory. We now look for similar structure in four dimensions. The most general solution of the partial differential equation (**2.**9) for $d = 4$ is given by the relation

$$\frac{\tilde{\rho}}{u'} - \frac{NC}{2}\frac{1}{u'} + \frac{NC}{2}\ln\left(\frac{1+u'}{|u'|}\right) = F\left(u'e^{2t}\right), \tag{2.46}$$

where now

$$C = 2v_4 = \frac{1}{16\pi^2}. \tag{2.47}$$

We consider a quartic classical potential according to eq. (**2.**19), with

$$\kappa_\Lambda = \frac{\rho_{0\Lambda}}{\Lambda^2}, \qquad \lambda_\Lambda = \bar{\lambda}_\Lambda. \tag{2.48}$$

This uniquely specifies F and we obtain

$$\frac{\tilde{\rho}}{u'} - \frac{NC}{2}\frac{1}{u'} + \frac{NC}{2}\ln\left(\frac{1+u'}{|u'|}\right) = \frac{1}{\lambda_\Lambda} + \left(\kappa_\Lambda - \frac{NC}{2}\right)\frac{1}{u'e^{2t}} + \frac{NC}{2}\ln\left(\frac{1+u'e^{2t}}{|u'|e^{2t}}\right). \tag{2.49}$$

There is again a critical value for the minimum of the classical potential

$$\kappa_\Lambda = \kappa_{cr} = \frac{NC}{2} \tag{2.50}$$

[4] The Bardeen-Moshe-Bander phenomenon [27] was not considered in our discussion.

which separates two possible phases. The scale invariant solution is obtained for the above value of κ_Λ in the limit $t \to -\infty$. The resulting solution is the "trivial" one

$$u'_\star = 0. \tag{2.51}$$

As a result, no interesting universal (independent of the classical parameters) structure can be obtained for this theory in four dimensions.

We can also investigate solutions which deviate slightly from the critical one, by considering a classical potential with a minimum given by eq. (**2**.28). The renormalized potential in the limit $t \to -\infty$, with $u' \to \infty$ and finite $U' = u'e^{2t}$, is given by the relation

$$\frac{U'}{\Lambda^2} + \frac{NC}{2}\lambda_\Lambda \frac{U'}{\Lambda^2} \ln\left(\frac{1+\frac{U'}{\Lambda^2}}{\frac{U'}{\Lambda^2}}\right) = \lambda_\Lambda\left(\frac{\rho}{\Lambda^2} - \delta\kappa_\Lambda\right) \tag{2.52}$$

for $U' \geq 0$. For theories with characteristic mass scale much smaller than the ultraviolet cutoff (such as in the symmetric phase in the vicinity of the phase transition) we have $U'/\Lambda^2 \to 0$. This gives for the renormalized quartic coupling

$$U'' = \frac{\lambda_\Lambda}{1 - \frac{NC}{2}\lambda_\Lambda \ln\left(\frac{U'}{\Lambda^2/e}\right)} \to 0. \tag{2.53}$$

The same happens near the minimum of the potential in the phase with spontaneous symmetry breaking, where no renormalized mass can be generated for the radial mode. All this is in agreement with the arguments for the "triviality" of the scalar theory in the limit that the ultraviolet cutoff is removed (see for example ref. [24] and refs. therein).

c) Two dimensions

We finally turn to two dimensions, where the most general solution of eq. (**2**.9) is given by

$$\frac{2}{NC}\rho + \ln\left(\frac{|u'|}{1+u'}\right) = F\left(u'e^{2t}\right), \tag{2.54}$$

with

$$C = 2v_2 = \frac{1}{4\pi} \tag{2.55}$$

and ρ dimensionless. We consider a quartic classical potential according to eq. (**2**.19), with

$$\kappa_\Lambda = \rho_{0\Lambda}, \qquad \lambda_\Lambda = \frac{\bar{\lambda}_\Lambda}{\Lambda^2}. \tag{2.56}$$

This leads to

$$u'e^{2t} = \lambda_\Lambda\left(\rho - \kappa_\Lambda\right) + \frac{NC}{2}\lambda_\Lambda \ln\left(\frac{1+u'e^{2t}}{(1+u')e^{2t}}\right). \tag{2.57}$$

For any choice of classical parameters the minimum of the effective average potential runs to zero at a scale

$$t_s = -\frac{\kappa_\Lambda}{NC}. \tag{2.58}$$

At $t=0$ the renormalized potential has a minimum at zero and is given by the expression

$$\frac{U'}{\Lambda^2} = \lambda_\Lambda\left(\rho-\kappa_\Lambda\right) - \frac{NC}{2}\lambda_\Lambda \ln\left(\frac{\frac{U'}{\Lambda^2}}{1+\frac{U'}{\Lambda^2}}\right). \tag{2.59}$$

The renormalized theory cannot be defined in the phase with spontaneous symmetry breaking and there is no phase transition, in agreement with the Mermin-Wagner theorem [29].

3. The Abelian Higgs model in four dimensions:

We now turn to gauge theories, for which exact renormalization equations have also been obtained [9]. As an example we discuss the Abelian Higgs model with N real scalars, in four dimensions. The evolution equation can be written in the form (for the details see ref. [19])

$$\frac{\partial u'}{\partial t} = -2u' + 2\tilde{\rho}u'' + (N-1)v_4 u'' L_1^4(u') + v_4(3u''+2\tilde{\rho}u''')L_1^4(u'+2\tilde{\rho}u'') + 6v_4 e^2 L_1^4(2e^2\tilde{\rho}), \tag{3.1}$$

with v_4 given by eq. (**2.**2). We have again neglected the small wavefunction renormalization effects for the scalar field. We recognize the contributions of the Goldstone and radial modes. The last term in eq. (**3.**1) is the contribution of the gauge field. It involves the gauge coupling e^2, whose evolution can be computed independently [9, 19]. Since the resulting running for e^2 is only logarithmic in $d=4$, it is a good approximation to neglect it in the following and use a constant e^2. The "threshold" functions L_1^4 are given by eq. (**2.**8). The contribution of the radial mode introduces higher derivatives in the evolution equation, making an explicit solution impossible. We shall again neglect this contribution, as in the first part of the paper. We should point out, however, that the resulting approximate evolution equation does not become exact any more in the large N limit. The purpose of this section is simply to demonstrate that the correct physical behaviour is incorporated in the full evolution equation, even though the approximate equation that we are using does not permit quantitative accuracy. After the omission of the contribution of the radial mode, the resulting partial differential equation is first order and can be solved with the method of characteristics. We have not managed to obtain an analytical solution in closed form, even though a numerical solution is possible. For this reason we make an additional approximation which is not crucial for the physical behaviour that we are interested in (see below). We set $L_1^4(u') = L_1^4(0) = -2$ in the contributions of the Goldstone modes, while maintaining the full "threshold" function in the contribution of the gauge field. As a result we cannot observe the decoupling of the scalar modes or the approach to convexity for the effective potential. However, we preserve the full effect of the gauge field on the form of the potential. We thus finally arrive to the following evolution equation

$$\frac{\partial u'}{\partial t} - 2\tilde{\rho}\frac{\partial u'}{\partial \tilde{\rho}} + B\frac{\partial u'}{\partial \tilde{\rho}} + \frac{De^2}{1+2e^2\tilde{\rho}} + 2u' = 0, \tag{3.2}$$

with

$$B = 2(N-1)v_4 = \frac{N-1}{16\pi^2}$$
$$D = 12 v_4 = \frac{3}{8\pi^2}. \qquad (\mathbf{3}.3)$$

The most general solution of eq. (**3**.2) is given by

$$\frac{u'}{2\tilde{\rho}-B} + \frac{De^2}{2(Be^2+1)}\frac{1}{2\tilde{\rho}-B} - \frac{De^4}{2(Be^2+1)^2}\ln\left(\frac{2e^2\tilde{\rho}+1}{|2\tilde{\rho}-B|}\right) = F\left((2\tilde{\rho}-B)e^{2t}\right). \qquad (\mathbf{3}.4)$$

The function F is determined through the initial condition for the potential. Assuming a quartic classical potential given by eqs. (**2**.19), (**2**.48) we find

$$F(x) = \lambda_\Lambda \frac{x+B}{2x} - \lambda_\Lambda \kappa_\Lambda \frac{1}{x} + \frac{De^2}{2(Be^2+1)}\frac{1}{x} - \frac{De^4}{2(Be^2+1)^2}\ln\left(\frac{e^2(x+B)+1}{|x|}\right). \qquad (\mathbf{3}.5)$$

In fig. 5 we plot the potential which results from eqs. (**3**.4), (**3**.5) for a certain choice of the parameters of the theory. Initially the effective average potential has only one minimum at a non-zero value of ρ. As k is lowered a second minimum appears around zero, which eventually becomes the absolute minimum of the potential. The discontinuity in the expectation value signals the presence of a first order phase transition. The development of the minimum around zero is caused by the logarithmic terms in eqs. (**3**.4), (**3**.5). The situation is typical of a Coleman-Weinberg phase transition triggered by radiative corrections [30]. The effective potential $U = U_0$ can be calculated from eqs. (**3**.4), (**3**.5) in the limit $t \to -\infty$, with $u', \tilde{\rho} \to \infty$, so that $u'e^{2t} \sim 1$, $\tilde{\rho}e^{2t} \sim 1$. We find

$$\frac{U'(\rho)}{\Lambda^2} = \lambda_\Lambda\left[\frac{\rho}{\Lambda^2} - \left(\kappa_\Lambda - \frac{B}{2} - \frac{De^2}{2\lambda_\Lambda(Be^2+1)}\right)\right] + \frac{De^4}{(Be^2+1)^2}\frac{\rho}{\Lambda^2}\ln\left(\frac{2e^2\frac{\rho}{\Lambda^2}}{e^2\left(2\frac{\rho}{\Lambda^2}+B\right)+1}\right). \qquad (\mathbf{3}.6)$$

Without the logarithmic term the phase transition in dependence to κ_Λ would have been second order. The presence of the last term in eq. (**3**.6) results in the development of a barrier near a secondary minimum at the origin. This leads to a weakly first order transition, with a discontinuity for the expectation value much smaller than the minimum of the classical potential. (For a detailed discussion of the Coleman-Weinberg transition using the full evolution equation see ref. [19].) The effective potential of eq. (**3**.6) is not convex. As we have mentioned already, the reason for this is the approximation of the "threshold" function by a constant in the evolution equation. Starting from eq. (**3**.6) we can derive a relation between the different mass scales in this model. We denote the scalar field mass at the minimum ρ_0 of the potential by $m^2 = 2U''(\rho_0)\rho_0$, the gauge field mass by $M^2 = 2e^2\rho_0$, and the scalar mass around the origin by $m_s^2 = U'(0)$. We find

$$\frac{m^2}{M^2} + 2\frac{m_s^2}{M^2} = \frac{3e^2}{8\pi^2}\frac{1}{(Be^2+1)(Be^2+1+M^2/\Lambda^2)}. \qquad (\mathbf{3}.7)$$

The well-known result of Coleman and Weinberg [30] is obtained for $M^2/\Lambda \ll 1$, if we neglect the scalar contributions by setting $B = 0$, and consider a potential with zero curvature at the origin by setting $m_s^2 = 0$.

4. Conclusions

We presented analytical solutions of the exact renormalization group equation for the effective average action which are not restricted in the vicinity of a possible infrared attractive fixed point. This allowed us to investigate universal and non-universal aspects of phase transitions for a variety of models. We neglected the effects of wavefunction renormalization and approximated the action by the potential and a standard kinetic term. We solved the evolution equation for the potential as a function of the field and the running scale k. We presented analytical solutions for the $O(N)$-symmetric scalar theory in the large N limit in three, four and two dimensions. The omission of the effects of wavefunction renormalization is justified in four and three dimensions by the vanishing of the anomalous dimension to leading order in $1/N$. Parts of our results have been obtained in the past [14, 7]. However, the emphasis of our work lies in the detailed study of the influence of the classical parameters of the theory on the possible phase transitions. We also gave a crude treatment of the Abelian Higgs model in four dimensions. The enhanced complexity of the evolution equation for this last model made necessary the use of additional approximations, which, however, do not affect the qualitative conclusions.

a) For the $O(N)$-symmetric scalar theory in the large N limit in three dimensions we distinguish two types of behaviour: I) For a classical ϕ^4 potential given by eq. (**2.**19) the renormalized theory has a second order phase transition in dependence on κ_Λ. The universal behaviour near the transition is governed by the Wilson-Fisher fixed point and can be parametrized by critical exponents. II) For a classical ϕ^6 potential given by eq. (**2.**38) there is a parameter range for which the renormalized theory has a second order phase transition in dependence on κ_Λ, with universal critical behaviour. For another parameter range the theory has a first order phase transition. The two regions are separated by a tricritical line (at $\lambda_\Lambda = 0$) which displays tricritical behaviour with mean field exponents.
b) For the same theory in four dimensions the solution reproduces the "triviality" of the critical theory and the vanishing of the renormalized quartic coupling.
c) In two dimensions the solution demonstrates that the renormalized theory cannot be defined in the phase with spontaneous symmetry breaking and there is no phase transition, in agreement with the Mermin-Wagner theorem.
d) For the Abelian Higgs model in four dimensions we reproduce the Coleman-Weinberg first order phase transition which is triggered by radiative corrections.

Our results on the universal behaviour of the three-dimensional scalar theory and on the four-dimensional Abelian Higgs model are in agreement with refs. [16, 17, 20, 19], where the evolution equations for the effective average action were studied with other methods. Thus they provide an additional argument for the validity and applicability of the method of the effective average action in a wide range of problems. The most important aspect of our solutions, however, is that they are fully analytical and not restricted in the vicinity of a possible infrared fixed point. They provide a concise, transparent picture of universal and non-universal behaviour, at all values of the field and the effective infrared cutoff, for a variety of physical systems.

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Figure captions

Fig. 1 The effective average potential $U_k(\rho)$ at different scales for a scalar model in the large N limit. The classical potential is given by eq. (**2.**19) with $\kappa_\Lambda = 1.2NC$ and $\lambda_\Lambda = 0.3/NC$ $(d = 3)$.

Fig. 2 The derivative $u'(\tilde{\rho})$ of the rescaled potential at different scales for a scalar model in the large N limit. The classical potential is given by eq. (**2.**19) with κ_Λ slightly smaller than NC and $\lambda_\Lambda = 0.3/NC$ $(d = 3)$.

Fig. 3 The derivative $u'(\tilde{\rho})$ of the rescaled potential at different scales for a scalar model in the large N limit. The classical potential is given by eq. (**2.**38) with $\kappa_\Lambda = NC$, $\lambda_\Lambda = 0.3/NC$ and $\sigma_\Lambda = 0.84/(NC)^2$ $(d = 3)$.

Fig. 4 The effective average potential $U_k(\rho)$ at different scales for a scalar model in the large N limit. The classical potential is given by eq. (**2.**38) with $\kappa_\Lambda = 1.2NC$, $\lambda_\Lambda = 0.3/NC$ and $\sigma_\Lambda = 1/(NC)^2$ $(d = 3)$.

Fig. 5 The approximate effective average potential $U_k(\rho)$ at different scales for an Abelian Higgs model with $e^2 = 2$ and $N = 11$ real scalars. The classical potential is given by eq. (**2.**19) with $\kappa_\Lambda = 1$ and $\lambda_\Lambda = 0.01$ $(d = 4)$.

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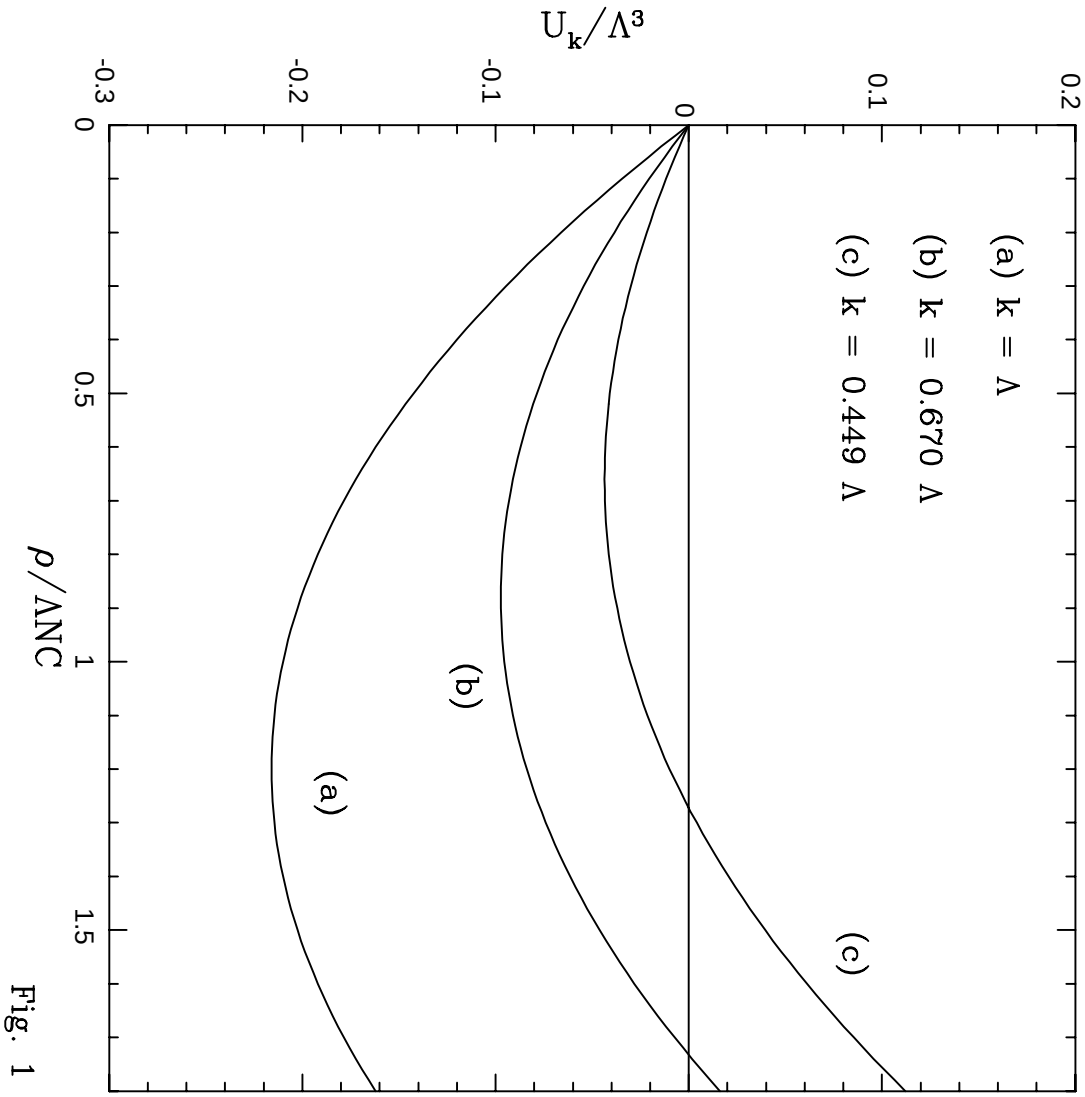


Fig. 1

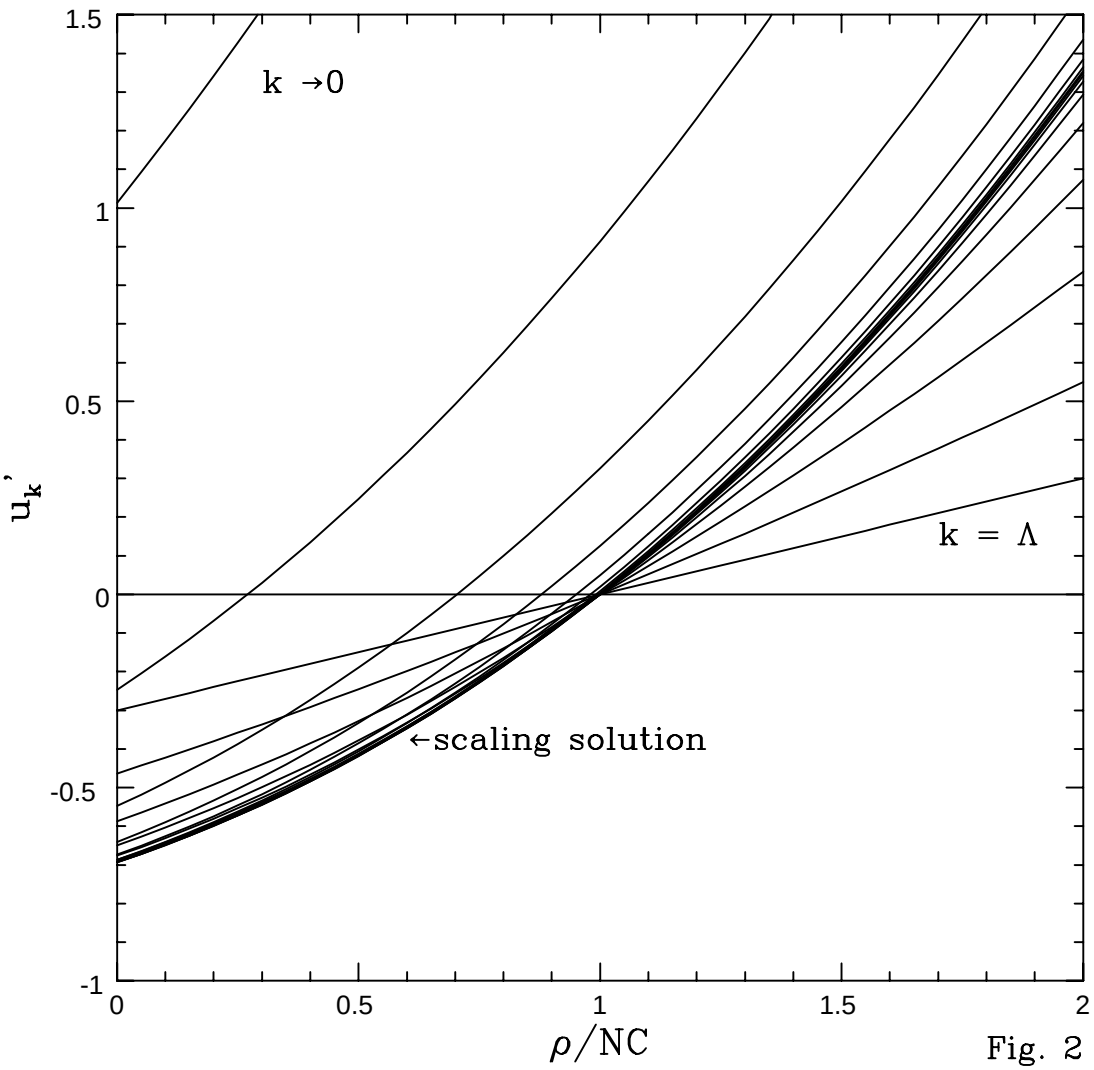
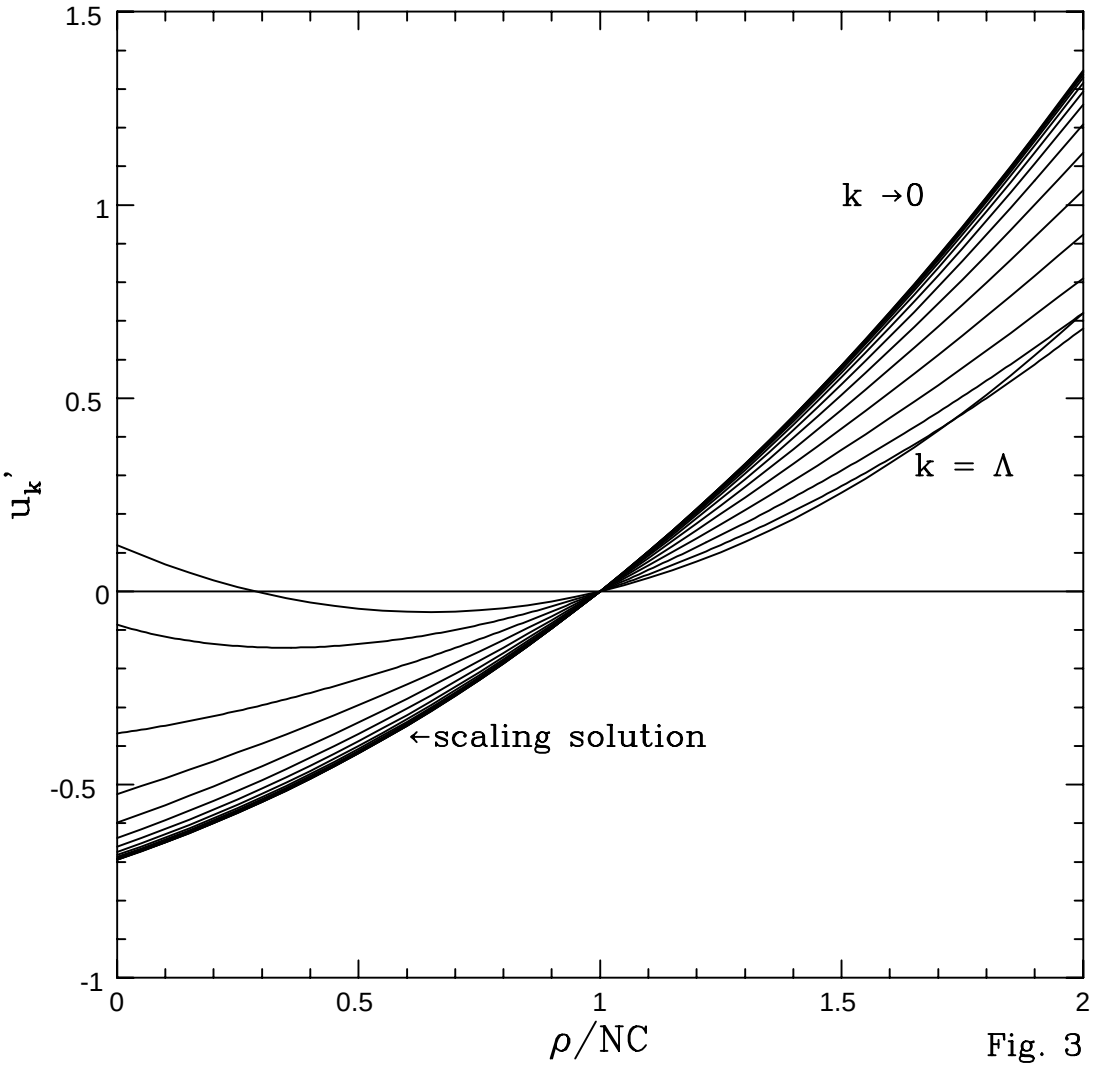


Fig. 2



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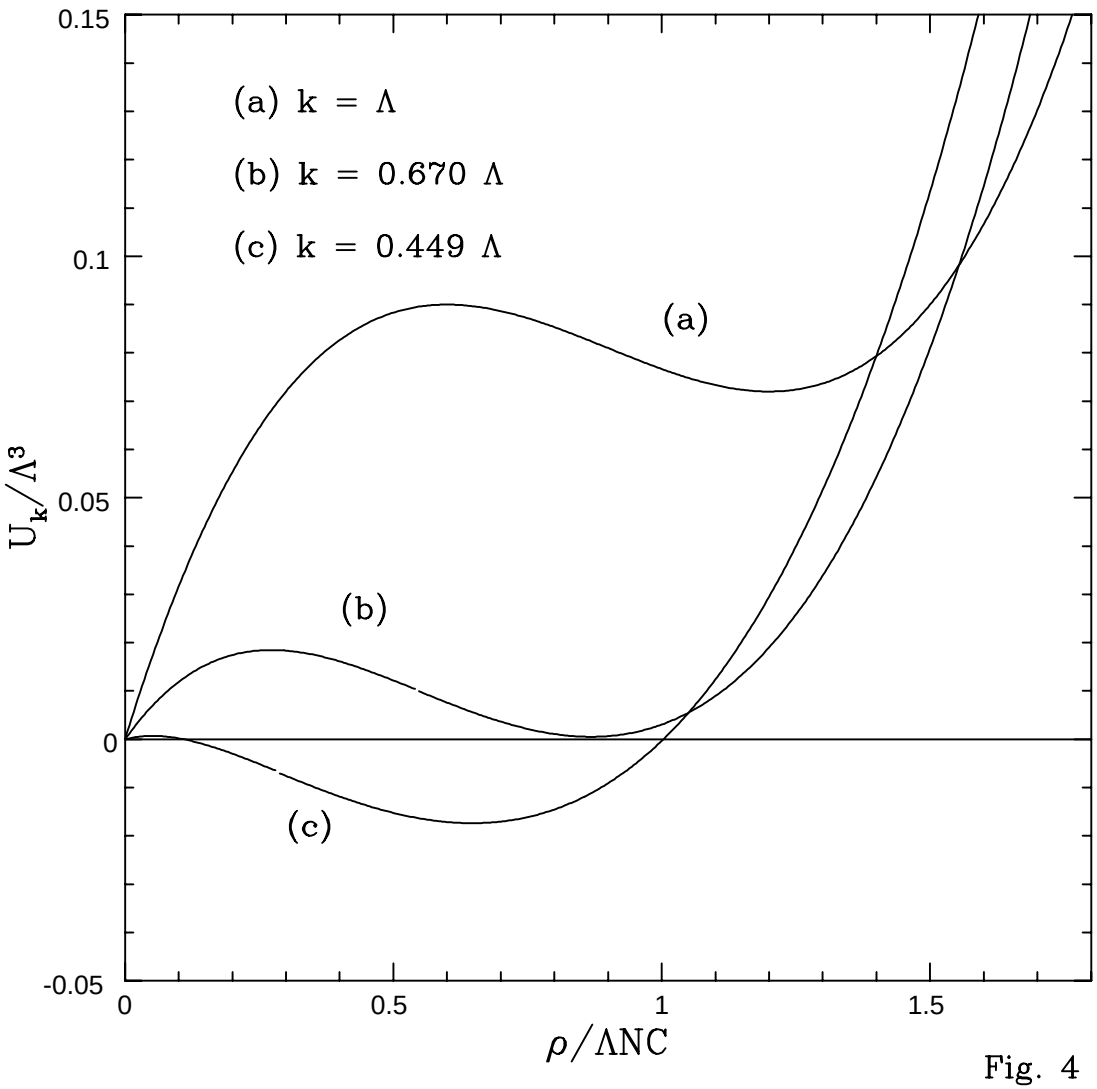
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0.05
0
-0.05
0
0.5
1
1.5
(a) k = Λ
(b) k = 0.670 Λ
(c) k = 0.449 Λ
(a)
(b)
(c)
U_k/Λ^3
$\rho/\Lambda NC$

Fig. 4

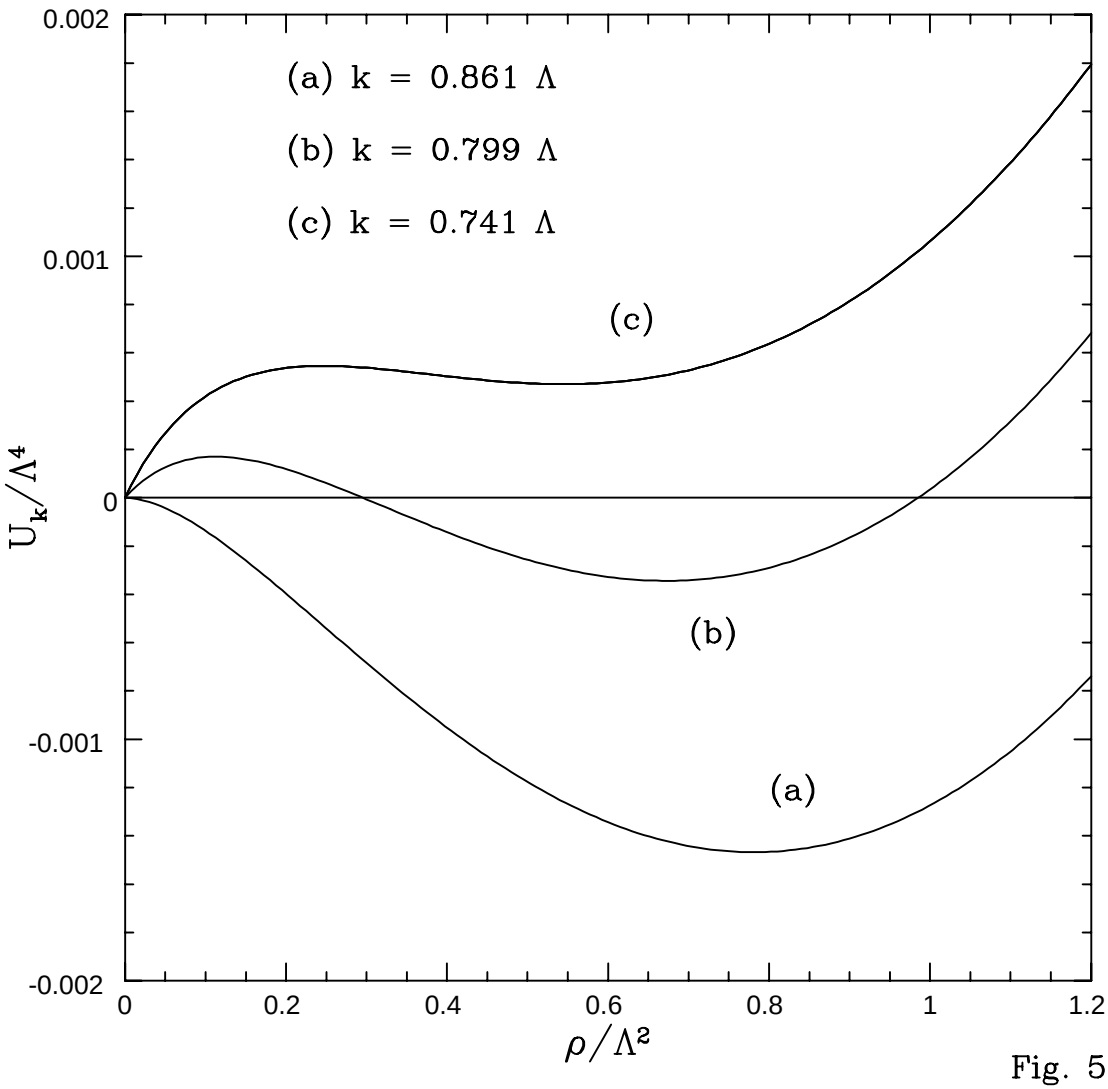
(a) k = 0.861 Λ
(b) k = 0.799 Λ
(c) k = 0.741 Λ
(c)
(b)
(a)
0.002
0.001
0
-0.001
-0.002
0
0.2
0.4
0.6
0.8
1
1.2
U_k/Λ^4
ρ/Λ^2

Fig. 5