arXiv:hep-th/9810161 v1 21 Oct 1998

LAPTH–703/98
HD-THEP-98/52

World Line Path Integrals as a Calculational Tool in Quantum Field Theory *

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Abstract: We report on the status of the string-inspired world line path integral formalism, a recently developed powerful tool for the reorganisation of standard perturbative amplitudes in quantum field theory. The method is outlined and the present range of its applicability surveyed. The emphasis is on QED and QCD photon/gluon amplitudes, with a short discussion of axial couplings.

*Talk given by C. S. at 6th International Conference on Path Integrals from peV to TeV: 50 years from Feynman's Paper, Florence, Italy, 25-29 Aug 1998.

1 Historical Remarks

50 years ago Feynman developed, in the paper which we are celebrating at this conference [1], the formulation of quantum mechanics in terms of path integrals. In the following years he invented the manifestly relativistic framework for perturbative calculations in second-quantized quantum field theory which we are still using today. What is perhaps not so well-known is that, at the time, Feynman was also experimenting with an alternative approach to quantum field theory based on his previous work on the quantum mechanical path integral. While he did not publish much about this first-quantized approach, in the appendix A of [2] he shortly discusses it for the case of scalar quantum electrodynamics, "for its own interest as an alternative to the formulation of second quantization". What he states here is that the amplitude for a charged scalar particle to move, under the influence of the external potential A_μ, from point x to x' in Minkowski space is given by

$$\int_0^\infty dT \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau) \mathrm{e}^{-\frac{1}{2}im^2T} \exp\Big[-\frac{i}{2}\int_0^T d\tau (\frac{dx_\mu}{d\tau})^2 - i\int_0^T d\tau \frac{dx_\mu}{d\tau} A_\mu(x(\tau)) \\ -\frac{i}{2}e^2 \int_0^T d\tau \int_0^T d\tau' \frac{dx_\mu}{d\tau}\frac{dx_\nu}{d\tau'} \delta_+(x(\tau)-x(\tau'))\Big] \qquad (1.1)$$

That is, for a fixed value of the proper-time T one can construct the amplitude as a certain quantum mechanical path integral. This path integral is to be performed over the set of all open trajectories running from x to x' in the given proper-time. The action consists of the familiar kinetic term, and two interaction terms. Of those the first one represents the interaction with the external field, to all orders in the field, while the second one describes an arbitrary number of virtual photons emitted and re-absorbed along the trajectory of the particle (δ_+ denotes the photon propagator). This simple path integral formula thus corresponds to an infinite number of Feynman diagrams. Feynman then shows how to extend it to a path integral formula for the complete scalar QED S-matrix. Extensions to spinor QED were also constructed by Feynman and others [3].

While Feynman himself did not make further use of those formulas, other authors over the years applied them to a variety of problems in quantum field theory, ranging from QED [4] to nonabelian gauge theory [5], anomalies [6], and meson-nucleon theory [7]. Still it is fair to say that this approach

never became really popular, nor did any standard way for calculating this type of path integral emerge.

Renewed interest in the first-quantized approach was generated in recent years following the work of Bern and Kosower, who succeeded in deriving new rules for the construction of one-loop QCD amplitudes by analyzing the infinite string tension limits of the corresponding amplitudes in an appropriate string model [8]. Those string amplitudes are represented in terms of the Polyakov path integral, a first-quantized path integral of the same type as Feynman's "world line" path integral above. It is thus not too surprising that it turned out to be possible [9] to rederive those "Bern-Kosower Rules" by representing the QED/QCD one-loop effective action in terms of such path integrals, and evaluating them in a way analogous to string theory [10].

2 Photon Scattering in Quantum Electrodynamics

In the case of spinor QED, the generalization of eq.(1.1) appropriate in the "stringy" context is the following,

$$\Gamma[A] = -2\int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x\mathcal{D}\psi \exp\left[-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi\dot{\psi} + ieA_\mu \dot{x}^\mu - ie\psi^\mu F_{\mu\nu}\psi^\nu\right)\right] \tag{2.1}$$

Here $\Gamma[A]$ denotes the one-loop (Euclidean) effective action for the Maxwell field due to an electron loop. Now we have, in addition to a path integral over the closed loops in spacetime $x^\mu(\tau), x^\mu(T) = x^\mu(0)$, an additional path integral over the space of antiperiodic Grassmann functions $\psi^\mu(\tau)$ representing the electron spin. In the "string–inspired" approach, the path integrals over x and ψ are evaluated by one-dimensional perturbation theory, using the Green's functions

$$\begin{aligned} \langle x^\mu(\tau_1) x^\nu(\tau_2)\rangle &= -g^{\mu\nu} G_B(\tau_1,\tau_2) = -g^{\mu\nu}\left[|\tau_1 - \tau_2| - \frac{(\tau_1-\tau_2)^2}{T}\right] \\ \langle \psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle &= \frac{1}{2}\, g^{\mu\nu} G_F(\tau_1,\tau_2) = \frac{1}{2}\, g^{\mu\nu}\mathrm{sign}(\tau_1-\tau_2) \end{aligned} \tag{2.2}$$

(The bosonic Wick contraction is actually carried out in the relative coordinate of the loop, while the integration over its average position yields energy–momentum conservation.)

One–loop scattering amplitudes are obtained by specializing the background to a finite sum of plane waves of definite polarization. In the case of scalar QED this leads to the following extremely compact "Bern-Kosower master formula" for the one-loop (off-shell, dimensionally regularized) N-photon amplitude,

$$\Gamma[k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N] = (-ie)^N \int_0^\infty \frac{dT}{T}[4\pi T]^{-\frac{D}{2}} e^{-m^2T} \prod_{i=1}^N \int_0^T d\tau_i$$
$$\times \exp\Big\{ \sum_{i,j=1}^N [\frac{1}{2}G_{Bij}k_i\cdot k_j - i\dot{G}_{Bij}\varepsilon_i\cdot k_j + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j]\Big\} \,|_{\mathrm{multi-linear}} \quad (2.3)$$

($G_B(\tau_1,\tau_2)\equiv G_{B12}$ etc.). Here it is understood that only the terms linear in all the $\varepsilon_1,\ldots,\varepsilon_N$ have to be taken. D denotes the spacetime dimension. Besides the Green's function G_B also its first and second derivatives appear, $\dot{G}_B(\tau_1,\tau_2) = \mathrm{sign}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T}, \ddot{G}_B(\tau_1,\tau_2) = 2\delta(\tau_1-\tau_2) - \frac{2}{T}$. Dots generally denote a derivative acting on the first variable. For the fermion QED case an analogous formula can be written using a superfield formalism [11, 12]. Alternatively the additional terms from the Grassmann path integration can also be generated by performing a certain partial integration algorithm on the above expression, and then applying a simple "substitution rule" on the result [8]. For example, the following representation is obtained after partial integration for the one-loop 4-photon amplitude in scalar QED [13],

$$\begin{aligned}
\Gamma[\{k_i,\varepsilon_i\}] &= e^4\int_0^\infty \frac{dT}{T}[4\pi T]^{-\frac{D}{2}} e^{-m^2T}\prod_{i=1}^4\int_0^T d\tau_i\,(Q_4^4+Q_4^3+Q_4^2-Q_4^{22})\mathrm{e}^{\frac{1}{2}G_{Bij}k_i\cdot k_j}\\
Q_4^4 &= \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41}Z_4(1234) + 2 \text{ permutations}\\
Q_4^3 &= \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\dot{G}_{B4i}\varepsilon_4\cdot k_i + 3 \text{ perm.}\\
Q_4^2 &= \dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Big\{\dot{G}_{B3i}\varepsilon_3\cdot k_i\dot{G}_{B4j}\varepsilon_4\cdot k_j\\
&\quad +\frac{1}{2}\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4\Big[\dot{G}_{B3i}k_3\cdot k_i - \dot{G}_{B4i}k_4\cdot k_i\Big]\Big\} + 5 \text{ perm.}\\
Q_4^{22} &= \dot{G}_{B12}\dot{G}_{B21}Z_2(12)\dot{G}_{B34}\dot{G}_{B43}Z_2(34) + 2 \text{ perm.}
\end{aligned} \quad (2.4)$$

Here summation over dummy indices from $1,\ldots,4$ is understood, and

$$
\begin{aligned}
Z_2(ij) &\equiv \varepsilon_i \cdot k_j \varepsilon_j \cdot k_i - \varepsilon_i \cdot \varepsilon_j k_i \cdot k_j \\
Z_n(i_1 i_2 \ldots i_n) &\equiv \mathrm{tr} \prod_{j=1}^{n} \Big[k_{i_j} \otimes \varepsilon_{i_j} - \varepsilon_{i_j} \otimes k_{i_j} \Big] \quad (n \geq 3)
\end{aligned}
$$

This formula has all properties which one could possibly demand from an integral representation for this amplitude, namely i) it provides a maximal gauge invariant decomposition, ii) manifest term-by-term UV finiteness, iii) permutation symmetry, iv) it represents the complete amplitude rather than a single Feynman diagram. The full set of Bern-Kosower rules allows one to use it to construct, by mere pattern matching, not only its spinor QED equivalent, but also the corresponding gluonic amplitudes.

3 QED in a Constant Field

The presence of an additional constant external field $F_{\mu\nu}$ can be shown to change the master formula eq.(2.3) to [12, 14]

$$
\begin{aligned}
\Gamma[k_1, \varepsilon_1; \ldots; k_N, \varepsilon_N] &= (-ie)^N \int_0^\infty \frac{dT}{T} [4\pi T]^{-\frac{D}{2}} e^{-m^2 T} \det{}^{-\frac{1}{2}} \left[\frac{\sin(eFT)}{eFT} \right] \\
&\times \prod_{i=1}^{N} \int_0^T d\tau_i \exp\Big\{ \sum_{i,j=1}^{N} [\frac{1}{2} k_i \cdot \mathcal{G}_{Bij} \cdot k_j - i\varepsilon_i \cdot \dot{\mathcal{G}}_{Bij} \cdot k_j + \frac{1}{2} \varepsilon_i \cdot \ddot{\mathcal{G}}_{Bij} \cdot \varepsilon_j] \Big\} \,|_{\mathrm{multi-linear}}
\end{aligned}
\tag{3.1}
$$

where $z = eFT$, and

$$
\begin{aligned}
\mathcal{G}_B(\tau_1, \tau_2) &= \frac{T}{2z^2} \Big(\frac{z}{\sin(z)} \, \mathrm{e}^{-iz\dot{G}_{B12}} + iz\dot{G}_{B12} - 1 \Big) \\
\mathcal{G}_F(\tau_1, \tau_2) &= G_{F12} \frac{\mathrm{e}^{-iz\dot{G}_{B12}}}{\cos(z)}
\end{aligned}
\tag{3.2}
$$

For example, using this formula with $N = 2$ it takes only a few lines to calculate the QED vacuum polarization tensor in a general constant field, a calculation which in field theory is already substantial. The $N = 3$ case was used in [15] for a recalculation of the photon splitting amplitude in a magnetic field, and also showed a significant gain in efficiency.

Much less explored are presently the QED amplitudes involving external scalars [16] or fermions [17], for which no equally elegant formulation has been found so far as in the photon case.

4 Other Field Theories

Eq.(2.1) generalizes to the case of an external non-abelian gauge field simply by the addition of a colour trace, and the path-ordering operator. The accomodation of internal gluons is also possible but more involved; it requires the introduction of auxiliary degrees of freedom in the loop whose contributions have to be projected out [9, 12].

While worldline representations for gauge couplings have been discussed for decades, only following the development of the "string-inspired" formalism systematic searches for generalizations to other field theories were undertaken. Appropriate path integrals are now available for the fermion loop coupled to external (pseudo-)scalars [18, 19], axial-vectors [20, 21], and antisymmetric tensors [21]. Those constructions were based on a dimensional reduction procedure from six-dimensional gauge theory. A particularly simple and direct construction exists for the case where only a vector field A and axial-vector field A_5 are present [22]. It expresses the effective action $\Gamma[A, A_5]$ in terms of the same path integral formula as in eq.(2.1), with the worldline Lagrangian replaced by

$$L(\tau) \quad \longrightarrow \quad L(\tau) - 2i\hat{\gamma}_5 \dot{x}^\mu \psi_\mu \psi_\nu A_5^\nu + i\hat{\gamma}_5 \partial_\mu A_5^\mu + (D-2)A_5^2 \qquad (4.1)$$

Here the operator $\hat{\gamma}_5$ has the effect of flipping the boundary conditions for the Grassmann path integral from antiperiodic to periodic. In the presence of pseudo-scalars or axial-vectors the evaluation of fermion loops is expected to give rise to ε – tensors. In the worldline formalism those are produced by the Grassmann zero mode integral which one has for periodic boundary conditions. The above worldline Lagrangian turned out to be very suitable to the calculation of the vector – axial vector amplitude in a constant external field [23].
For an application of the pseudoscalar path integral to axion decay see [24]. Steps towards an extension to curved backgrounds were undertaken in [25].

5 Multiloop Extension

Since the one-loop parameter integral representations obtained in the worldline formalism are generally valid off-shell, they can be used to construct higher – loop amplitudes by sewing. A more elegant route to multi-loop extension is provided by the introduction of worldline multiloop Green's functions [26]. Those are effective worldline propagators taking the effect of propagator insertions into the one-loop graph into account. As shown in [27] they are the leading-order coefficients of the corresponding string-theoretic worldsheet Green's functions in the $\frac{1}{\alpha'}$ – expansion.

In either case one arrives at integral representations for multiloop amplitudes in ϕ^3 – theory [26, 28], QED [11, 16] or QCD [29] that are of a similar form as our one-loop formulas above. However here we find the additional interesting feature that a single worldline parameter integral may contain contributions from many Feynman diagrams of different topologies. In [11] the usefulness of this property was demonstrated for the case of the 2-loop scalar and spinor QED β – functions. The concept of multiloop worldline propagators can be generalized to the constant external field case as in the one-loop case. This was used for recalculations of the 2-loop scalar and spinor QED Euler-Heisenberg Lagrangians [12, 30].

6 Conclusions

To summarize, by now there is sufficient practical experience indicating that the worldline path integral formalism is an excellent tool for the calculation of the QED photon S-matrix in general, and definitely superior to standard field theory for problems involving constant external fields. While here we have concentrated on amplitude calculations, similar improvements on field theory were found also in calculations of the effective action itself in the inverse mass expansion [31]. By now a variety of extensions to other types of amplitudes exist, though those have not been sufficiently tested yet to allow for general statements on their efficiency as a calculational tool.

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