arXiv:hep-th/9811090 v2 1 Mar 1999

DUALITY IN QUANTUM LIOUVILLE THEORY

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Abstract

The quantisation of the two-dimensional Liouville field theory is investigated using the path integral, on the sphere, in the large radius limit. The general form of the N-point functions of vertex operators is found and the three-point function is derived explicitly. In previous work it was inferred that the three-point function should possess a two-dimensional lattice of poles in the parameter space (as opposed to a one-dimensional lattice one would expect from the standard Liouville potential). Here we argue that the two-dimensionality of the lattice has its origin in the duality of the quantum mechanical Liouville states and we incorporate this duality into the path integral by using a two-exponential potential. Contrary to what one might expect, this does not violate conformal invariance; and has the great advantage of producing the two-dimensional lattice in a natural way.

PACS: 11.10Kk, 11.10Lm, 11.25Hf, 11.25Pm, 04.60Kz

Keywords: Liouville Theory, Path Integral, Duality

I. INTRODUCTION

The classical Liouville theory is defined by a real scalar field with an exponential potential [1]. In two dimensions it is both conformally invariant and integrable, and is of considerable importance in a variety of physical problems. The quantised theory is of similar importance, particularly in the context of string theory [2]. Accordingly, it is of interest to compute correlation functions in the quantised Liouville theory. As is well known, the scalar field itself is not a primary field with respect to the conformal group, and hence correlation functions are traditionally defined as expectation values of a set of exponential (vertex) functions which are primary fields. However, the computation of these correlation functions turns out to be rather tricky. In contrast to the Wess-Zumino-Witten theory (from which the Liouville theory can obtained by imposing a set of linear first class constraints on the Kac-Moody currents [3]), for example, there is no known closed expression for the four-point function; and although the spacetime dependence of the two-point and three-point functions is dictated, as usual, by conformal invariance, the computation of the their coefficients as functions of the parameters of the theory has turned out to be quite difficult. It has been shown that crossing symmetry relations, for a special set of four-point functions which are known in terms of hypergeometric functions, determine the coefficient of the three-point function uniquely [4]; and a form which satisfies these requirements has been proposed by Dorn and Otto and by A. and Al. Zamolodchikov [5]. We shall refer to this proposal as the DOZZ proposal. A surprising feature of the proposal is that the dependence on the parameters exhibits a two-dimensional lattice of poles, rather than the one-dimensional lattice that one would expect from a single exponential potential. The appearance of the two-dimensional lattice corresponds to an unexpected duality symmetry in the quantum theory that is not present in the classical theory. Since this symmetry disappears in the classical limit, it is the opposite of the situation normally encountered with an anomaly. As such, the

physical interpretation of the additional set of poles is unclear, and the residues of the proposed three-point function at these poles can not be immediately identified within the theory.

Motivated by the above state of affairs, we wish to consider the computation of the correlation functions in this paper, using the path-integral approach, and taking a point of view that differs slightly from the conventional one. The difference is based on the fact that, whereas the classical Liouville theory admits one primary vertex field for each conformal weight, the quantum Liouville theory admits two such fields. In particular, it admits two distinct exponential potentials which are conformally invariant. Accordingly, we take the view that, for the Action in the path integral, the natural generalisation of the classical Liouville potential is not a single exponential potential, but a linear combination of two independent exponential potentials whose parameters are arranged so as to guarantee conformal invariance. We compute the three-point function in this generalised theory and as we shall see, this hypothesis gives a straightforward explanation of the two-dimensional lattice of poles and the quantum mechanical duality mentioned above. It is shown that our expression for the three-point function reduces to the DOZZ proposal when the dimensional parameters in the theory (the coefficients of the potential terms) obey a certain duality condition. Our approach also allows us to give a simple expression for the four-point function. A spin-off of our approach is that many of our formulae are valid for any theory with two exponential potentials, like the Sinh-Gordon and Sine-Gordon theories; although it is only when the parameters are related by conformal invariance, that we can carry out explicit computations. It will be assumed that the underlying manifold on which the theory is defined is a compact Riemann surface with the topology of a two-dimensional sphere although, for computational purposes, we shall take the infinite-volume limit.

The paper is organised as follows. In Section II we present the arguments for the

two-exponential potential, write down the appropriate path integral, and specify the conditions on the parameters that make it conformally invariant.

In Section III we consider the path integral for the Liouville theory. As a consequence of the spherical topology of the base space, the Liouville field can be separated into a constant zero mode and a fluctuating component. The Liouville potential has the special property that the integration over the zero mode essentially decouples from the rest of the integral. By means of a Sommerfeld-Watson transform [6], the fluctuating part of the path integral can then be brought into a form that resembles the functional integral of a free scalar field theory with insertions of powers of vertex functions – except that the powers are not necessarily positive integers. However, the result for positive integers, obtained earlier by Dotsenko and Fateev [7], is such that the answer for the general case can be obtained by an extrapolation. In Appendix B we express the Dotsenko-Fateev result in a form which is amenable for this extrapolation. The functional integration requires the regularisation of the relevant Green's function as discussed in Appendix A, and we discuss in detail how this regularisation affects the Weyl invariance, and hence the conformal invariance of the theory. We conclude Section III by passing to the infinite volume limit in which we consider the translational invariance and scale covariance of the relevant functional integral in conformal coordinates. We also study the dependence of the N-point functions on the dimensional parameters and show how the parameters are renormalised.

In Section IV we study the covariance property of the path integral with respect to SL(2, C) transformations. We use this property to simplify the the expressions for the N-point functions.

In Section V we consider the three-point function. It is shown that the usual power law dependence of the three-point function can be obtained using the SL(2, C) covariance properties of the previous section, and turns out to have the standard form

dictated by conformal invariance. Unlike the discussions in [5] where a new set of poles is discovered in the three point function when it is analytically continued in the parameter space, our computations naturally produce the full set of poles because duality is built into our construction from the beginning.

In Section VI we show that there is no unambiguous way to define the N-point functions for $N \leq 2$. Thus the path integral should be viewed as a sort of a distribution that takes meaningful values only when it is tested against at least three vertex functions. However, we present a proposal for the two-point function.

In Section VII we show that the SL(2, C) covariance of the correlation functions is also sufficient to fix the four-point function. Apart from the usual power law dependence on the coordinate differences dictated by conformal invariance, this depends on one conformally invariant cross-ratio. The function of this cross-ratio – the so-called conformal block – is a polynomial if any one of the vertex functions has a positive integer power. In the generic case, where no power is a positive integer, one can only obtain asymptotic expansions in powers of the cross-ratio and its inverse. The coefficients of various powers in this series can be computed explicitly in terms of known three-point functions.

In Section VIII, we present our conclusions.

II. THE MODEL

The classical Euclidean Liouville Action for a real scalar field $\tilde{\phi}$, on a compact manifold, which we choose to be topologically equivalent to the two-sphere, is given by

$$S = \int d^2x \sqrt{g(x)} \big[\frac{1}{4\pi} \tilde{\phi} \Delta \tilde{\phi} + \frac{q}{2\pi} R \tilde{\phi} + V_b(\tilde{\phi}) \big] \tag{2.1}$$

where $\sqrt{g(x)}$ is the determinant of the background metric, q and b are dimensionless parameters, and R is the Ricci scalar. The Laplace-Beltrami operator Δ and the Liouville

potential V_b are given by

$$\Delta = -\frac{1}{\sqrt{g}}\partial_\mu \sqrt{g} g^{\mu\nu}\partial_\nu \quad \text{and} \quad V_b(\tilde{\phi}) = \mu_b e^{2b\tilde{\phi}} \tag{2.2}$$

respectively, μ_b being a constant parameter which has the dimensions of mass squared to compensate for the dimensions of d^2x in the Action. The classical energy momentum tensor, defined as $T_{\mu\nu} = \delta S/\delta g^{\mu\nu}$ is

$$2\pi T_{\mu\nu} = (\partial_\mu\tilde{\phi})(\partial_\nu\tilde{\phi}) - \frac{1}{2}g_{\mu\nu}(\partial\tilde{\phi})^2 + \frac{1}{2}g_{\mu\nu}V_b - q(g_{\mu\nu}\Delta - \partial_\mu\partial_\nu)\tilde{\phi} \tag{2.3}$$

The condition for Weyl invariance, which implies conformal invariance in the flat space limit, is that the trace of $T_{\mu\nu}$ be proportional to R on the mass shell. This relates the parameters q and b by the condition $qb = 1$. In conformal coordinates $g_{\mu\nu}(x) = e^{\sigma(x)}\eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the flat Euclidean metric, this implies that, on the mass shell, the only non-vanishing components of the energy-momentum tensor are

$$T_\pm = (\partial_\pm\tilde{\phi})^2 \pm q(\partial_\pm^2\tilde{\phi}) \tag{2.4}$$

As is well-known, the field $\tilde{\phi}$ is not primary with respect to the Virasoro algebra generated by $T_\pm$, but the fields $e^{2\alpha\tilde{\phi}}$ are primary fields of weight $(\alpha q, \alpha q)$. From this it follows that the only local potential that is allowed by conformal invariance *i.e.* has conformal weight (1,1), is of the exponential type shown in (2.2).

In the quantum theory, the situation is rather different because of normal ordering and the replacement of Poisson brackets by commutators. The quantum analogue of the condition that T^μ_μ be proportional to R on the mass shell, namely that $< T^\mu_\mu >$ be proportional to R, leads to a new relation between the parameters b and q, namely $qb = 1 + \hbar b^2$. The fields $e^{2\alpha\tilde{\phi}}$ remain primary, but have conformal weights $(\Delta_\alpha, \Delta_\alpha)$ where

$$\Delta_\alpha = \alpha(q - \hbar\alpha) \tag{2.5}$$

Although the latter equation differs from its classical counterpart only by a quantum correction, it nevertheless changes the structure of the theory fundamentally because it means that there are now two fields for each conformal weight Δ_α, namely

$$e^{\alpha_\pm \tilde{\phi}(x)} \quad \text{where} \quad \alpha_\pm = \frac{q \pm \sqrt{q^2 - 4\hbar\Delta_\alpha}}{2\hbar} \tag{2.6}$$

These fields are dual in the sense that

$$\alpha_+ \alpha_- = \frac{\Delta_\alpha}{\hbar} \tag{2.7}$$

Note that in the classical limit, α_- reduces to the classical value αq, while $\alpha_+ \to \infty$. The existence of dual primary fields, of definite conformal weight, raises the question as to whether these fields should be regarded as distinct or identical. From the point of view of the Virasoro algebra, there would be no problem in identifying them because the first class constraint

$$e^{2\alpha_+ \tilde{\phi}} - e^{2\alpha_- \tilde{\phi}} = 0 \tag{2.8}$$

commutes with the Virasoro generators. However, this constraint does not commute with other primary fields such as the canonical momentum $\pi_{\tilde{\phi}}$. Furthermore, from the point of view of the functional integral, it would restrict the range of the integration variable in a non-linear way. Accordingly it would seem more reasonable not to make the identification of $e^{\alpha_\pm \tilde{\phi}}$ but to treat them as distinct, dual fields. This is the point of view we shall adopt.

We now wish to consider the path integral formulation of the theory and because of the possibility of having two distinct fields of conformal weight (1, 1) in the quantum theory, we propose to start from a path integral with an Action which contains two exponential potentials

$$Z = \int d\tilde{\phi} \;\; e^{-\int d^2x \sqrt{g(x)} \left[\frac{1}{4\pi} \tilde{\phi} \Delta \tilde{\phi} + \frac{q}{2\pi} R \tilde{\phi} + V_b(\tilde{\phi}) + V_c(\tilde{\phi}) \right]} \tag{2.9}$$

leaving the parameters b and c to be determined by conformal invariance. We remark in passing that an advantage of using the general Action (2.9) is that, until we impose conformal invariance, our equations are valid for any theory with a potential which is a sum of two different exponentials. This includes, in particular, the Sinh-Gordon (and by analytic continuation, the Sine-Gordon) theory. At first sight, the proposal to use two potentials may seem rather radical but we shall see that it is perfectly compatible with conformal invariance when b and c are suitably related [8]. In fact, within the context of the path integral itself it can be shown (see Section III) that when renormalisation is taken into account, Weyl invariance, which implies conformal invariance in the flat space limit, requires that

$$1 - bq + \hbar b^2 = 0 \quad \text{and} \quad 1 - cq + \hbar c^2 = 0 \tag{2.10}$$

These equations for b and c are just the conditions that the potentials V_b and V_c have conformal weight (1,1). If we eliminate q from (2.10), we obtain a direct relationship between b and c

$$(b - c)(1 - \hbar bc) = 0 \tag{2.11}$$

This equation, which has no reference to the background metric, is actually the necessary and sufficient condition for conformal invariance in the flat space limit and distinguishes the Liouville theory from other two-exponential theories.

For $\hbar \neq 0$, there are obviously two solutions to equation (2.11). The solution $b = c$ corresponds to the case of a single potential, and can be recovered from the more general case $\hbar bc = 1$ by setting one of the dimensional parameters μ_b or μ_c equal to zero. We shall therefore consider the more general case $\hbar bc = 1$. From now on we shall normalise $\hbar = 1$ and thus we shall consider the path integral (2.9) with the condition that

$$bc = 1, \quad q = b + c \tag{2.12}$$

For simplicity of notation, however, we shall continue to use both b and c with the relationship $bc = 1$ understood. As we shall see, an important consequence of using the two dual potentials of conformal weight (1,1) in the Action is that they automatically produce the dual set of poles (in the parameter space) of the three point structure functions whose existence was inferred indirectly by other authors [5].

We conclude this section by defining the N-point function of vertex functions to be

$$\mathcal{G}_N(x_I, \alpha_I) =< \prod_{I=1}^{N} e^{2\alpha_I \tilde{\phi}(x_I)} >= \int d\tilde{\phi}\ e^{-S+2\sum_{I=1}^{N} \alpha_I \tilde{\phi}(x_I)} \tag{2.13}$$

where S is the Euclidean Action in the path integral (2.9).

III. PATH INTEGRATION, SYMMETRIES, AND RENORMALISATION

<u>*The Path Integration*</u>: It is well-known that, on a compact space which we choose to be topologically equivalent to the two-sphere, the Laplace-Beltrami operator has only one zero mode, namely the constant function ϕ_0. We therefore split the field $\tilde{\phi}$ into its zero mode and its orthogonal complement ϕ.

$$\tilde{\phi}(x) = \phi_0 + \phi(x), \quad \Delta\phi_0 = 0, \qquad \int d^2x \sqrt{g(x)}\phi(x) = 0 \tag{3.1}$$

The expression for the N-point function (2.13) then becomes

$$\begin{aligned} \mathcal{G}_N(x_I, \alpha_I) = \int d\phi_0\ e^{-2\xi\phi_0} \int d\phi\ e^{-U_b(\phi)e^{2b\phi_0}} e^{-U_c(\phi)e^{2c\phi_0}} \\ \times\ e^{-\int d^2x\sqrt{g(x)}\left[\frac{1}{4\pi}\phi\Delta\phi + \frac{q}{2\pi}R\phi\right] + 2\alpha_I\phi(x_I)} \end{aligned} \tag{3.2}$$

where

$$\xi = q - \Sigma, \qquad \Sigma = \sum_{I=1}^{N} \alpha_I \quad \text{and} \quad U_b = \int d^2x \sqrt{g(x)} V_b(\phi) \tag{3.3}$$

In arriving at the above equation we have used $\int d^2x\sqrt{g}R = 2\pi\chi$ where χ is the Euler characteristic. For a space which is topologically equivalent to the two-sphere, $\chi = 2$. Note that (2.12) implies that b and c have the same sign and because of the ϕ-integration

there is no loss of generality in assuming that both of them are positive. Then, for the zero mode integration to converge, ξ must be negative. We will assume this to be the case.

At first sight it would seem natural to make the Coulomb gas expansion

$$e^{-U_b(\phi)e^{2b\phi_0}} = \sum_m \frac{1}{m!}\Big[-U_b(\phi)e^{2b\phi_0}\Big]^m \tag{3.4}$$

and similarly for $b \to c$ in (3.2). The problem of computing correlation functions of arbitrary vertex operators in the Liouville theory then reduces to the problem of computing a double infinite series of correlation functions of powers of integrated vertex operators in a free scalar theory. This is reminiscent of the classical equivalence of the Liouville theory and the free theory by a canonical transformation [9]. However, this is a little too restrictive because the zero mode integration then forces certain combinations of the parameters to be positive integers. It is more convenient to introduce the Sommerfeld-Watson transform [6] for the exponential function

$$e^t = \frac{1}{2\pi i}\int_{-\infty}^{\infty} du \frac{(-t)^{iu}}{\Gamma(1+iu)}\frac{\pi}{\sinh \pi u} \tag{3.5}$$

The Sommerfeld-Watson transform replaces the double series by a double integral representation which has the virtue of manifestly displaying the pole-like singularities (in the parameter space) of the correlation functions.[1] The integration in equation (3.5) is along the real axis. The above equation is easily verified by closing the contour of integration in the upper half plane or the lower half plane to enclose the poles of $\sinh \pi u$ which lie along the imaginary axis at integer values and using Cauchy's residue theorem. For either choice of the contour, care should be taken to enclose the origin. Substituting

[1] This may be contrasted with the Coulomb gas method in which the exponential is expanded in powers of $U(\phi)$, in which case, the singularities in the individual terms are not apparent, although the series diverges.

(3.5) in (3.2) and continuing the parameter ξ to $i\xi$, we get

$$\begin{aligned}\mathcal{G}_N(x_I,\alpha_I) &= -\frac{1}{4}\int dudvd\phi_0\ \frac{e^{-2i(\xi-bu-cv)\phi_0}}{\sinh\pi u\sinh\pi v}\mathcal{R}_N(b,c;\alpha_I;iu,iv)\\ &= -\frac{1}{8}\int dudv\ \frac{\delta(bu+cv-\xi)}{\sinh\pi u\sinh\pi v}\mathcal{R}_N(b,c;\alpha_I;iu,iv)\end{aligned} \tag{3.6}$$

where

$$\mathcal{R}_N(b,c;\alpha_I;iu,iv) = \int d\phi\frac{U_b^{iu}}{\Gamma(1+iu)}\frac{U_c^{iv}}{\Gamma(1+iv)}e^{-\int d^2x\sqrt{g(x)}\left[\frac{1}{4\pi}\phi\Delta\phi+\frac{q}{2\pi}R\phi\right]+2\alpha_I\phi(x_I)} \tag{3.7}$$

The function $\mathcal{R}_N(b,c;\alpha_I;iu,iv)$ cannot be computed directly for arbitrary iu and iv. However, under certain conditions to be discussed later, it can be considered as an extrapolation of $\mathcal{R}_N(b,c;\alpha_I;m,n)$ where m and n are positive integers. For positive integers m and n, $\mathcal{R}_N$ can be explicitly computed because it is simply a correlation function of integrated vertex operators in a free theory of the form

$$\mathcal{R}_N(b,c;\alpha_I;m,n) = \frac{\mu_b^m\mu_c^n}{m!n!}\int\prod_{i=1}^m d^2x_i\sqrt{g(x_i)}\prod_{r=1}^n d^2y_r\sqrt{g(y_r)}\int d\phi e^{-\int d^2x\sqrt{g}(\phi\frac{\Delta}{4\pi}\phi+j\phi)} \tag{3.8}$$

where

$$j(x) = \frac{q}{2\pi}R(x) - \sum_{I=1}^N\frac{2\alpha_I}{\sqrt{g(x)}}\delta^2(x-z_I) - \sum_{i=1}^m\frac{2b}{\sqrt{g(x)}}\delta^2(x-x_i) - \sum_{r=1}^n\frac{2c}{\sqrt{g(x)}}\delta^2(x-y_r) \tag{3.9}$$

Of course, (3.8) mimics the Coulomb gas expansion. But the point is that m and n have to be extrapolated to the values iu and iv used in (3.6). Evaluating the Gaussian integral in (3.8) in the usual way we get

$$\begin{aligned}\mathcal{R}_N(b,c;\alpha_I;m,n) =& \frac{\mu_b^m\mu_c^n}{m!n!}\frac{1}{\sqrt{\det{}'\frac{\Delta}{4\pi}}}\int\prod_{i=1}^m d^2x_i\sqrt{g(x_i)}\prod_{r=1}^n d^2y_r\sqrt{g(y_r)}\\ &\times e^{\int d^2x\sqrt{g(x)}\int d^2y\sqrt{g(y)}j(x)G(x,y)j(y)}\end{aligned} \tag{3.10}$$

where the prime in $\det'$ means that the zero mode is omitted and $G(x,y)$ is the finite volume Green's function defined in Appendix A. As in the above equation, the subscripts

of x, y, and α always run from 1 to m, 1 to n, and 1 to N respectively; unless otherwise stated. With this in mind, we shall not display the ranges of these subscripts from now on for the sake of notational simplicity.

Substituting for j from (3.9) we then have

$$\mathcal{R}_N(b,c;\alpha_I;m,n) = \frac{e^{\frac{q^2}{4\pi^2}\int d^2x\sqrt{g(x)}d^2y\sqrt{g(y)}R(x)G(x,y)R(y)}}{\sqrt{\det{}'\frac{\Delta}{4\pi}}} \times \frac{\mu_b^m\mu_c^n}{m!n!}e^{\sum_{I\neq J=1}^{N}4\alpha_I\alpha_J G(z_I,z_J)}e^{-2q\sum\alpha_I p(z_I)} \times I_N \tag{3.11}$$

where

$$p(x) = \frac{1}{\pi}\int d^2y\sqrt{g(y)}\ G(x,y)R(y) \tag{3.12}$$

$$I_N(b,c;\alpha_I;m,n) = \int\prod_i d^2x_i\sqrt{g(x_i)}e^{-2bqp(x_i)}\int\prod_r d^2y_r\sqrt{g(y_r)}e^{-2cqp(y_r)} \times e^{\left[F_b(x_i,x)+F_c(y_r,y)+8G(x_i,y_r)\right]} \tag{3.13}$$

and

$$F_b(x_i,x) = 8b\sum_I\alpha_I G(x_i,z_I) + 4b^2\sum_j G(x_i,x_j) \tag{3.14}$$

and similarly for $F_c(y_r,y)$. The numerator of the first term in (3.11) will be recognised as the Polyakov term. It implies that the centre of the Virasoro algebra is $1+3q^2$, where the one comes from the Weyl anomaly for a single real scalar field. This term and the denominator $\sqrt{\left(\det{}'(\Delta/4\pi)\right)}$ of the first term in (3.11) play no further role, so in order to simplify the calculations we drop these two terms from now on. We also use conformal coordinates in which $p(x)$ reduces to $\frac{1}{2}\ln\sqrt{g(x)}$. Since the Green's function becomes singular when the arguments coincide, we have to renormalise it. This we do in the standard manner discussed in Appendix A, the end result of which is that $G(x_i,x_i)$ is taken to be $\frac{1}{4}\ln\sqrt{g(x_i)}$. Thus after renormalisation, integral (3.13) may be written in conformal coordinates as

$$I_N(b,c;\alpha_I;m,n) = \int\prod_i d^2x_i\sqrt{g(x_i)}^{W_b}\prod_r d^2y_r\sqrt{g(y_r)}^{W_c}e^{\left[F_b^R(x_i,x)+F_c^R(y_r,y)+8G(x_i,y_r)\right]} \tag{3.15}$$

where

$$W_b = 1 - qb + b^2 \quad \text{and} \quad W_c = 1 - qc + c^2 \tag{3.16}$$

and it is understood that the terms with coincident arguments $i = j$ and $r = s$ are to be omitted in F_Rs – the renormalised Fs.

<u>*Weyl Invariance*</u>: Making a Weyl transformation $\sqrt{g} \to \lambda\sqrt{g}$, we see that $I_N \to \lambda^{mW_b+nW_c} I_N$. This factor can be absorbed in the dimensional parameters μ_b and μ_c in (3.11) by the scaling $\mu_b \to \lambda^{-W_b}\mu_b$ and $\mu_c \to \lambda^{-W_c}\mu_c$. Hence on retracing the steps (3.2) to (3.16) we see that a Weyl scaling of the original integral (3.2) has the simple effect that

$$V_b \to \lambda^{-W_b} V_b \quad \text{and} \quad V_c \to \lambda^{-W_c} V_c \tag{3.17}$$

Thus, as stated in the Introduction, Weyl invariance implies $W_b = W_c = 0$, which is the same as (2.10) when $\hbar$ is restored. Having extracted the Weyl condition, we may now consider the infinite-volume limit in which the Green's function takes the standard form

$$G_0(x, y) = -\frac{1}{2}\ln\frac{\mid x - y \mid}{L} \tag{3.18}$$

L being a dimensional cut-off. Then the equation for $I_N(b, c; \alpha_I; m, n)$ in (3.15) simplifies to

$$\begin{aligned} I_N(b, c; \alpha_I; m, n) = \int \prod_i d^2x_i \sqrt{g(x_i)}^{W_b} \prod_r d^2y_r \sqrt{g(y_r)}^{W_c} \\ \times\, e^{\left[F_b^R(x_i,x) + F_c^R(y_r,y) - 4\ln\frac{|x_i - y_r|}{L}\right]} \end{aligned} \tag{3.19}$$

where

$$F_b^R(x_i, x) = -4b\sum_I \alpha_I \ln\frac{\mid x_i - z_I \mid}{L} - 2b^2 \sum_{i\neq j} \ln\frac{\mid x_i - x_j \mid}{L} \tag{3.20}$$

This equation is actually valid for the N-point functions of any theory with two exponential potentials. What distinguishes the conformally invariant Liouville theory is that

$W_b = W_c = 0$ in accordance with the Weyl condition. In particular $bc = 1$. In that case, the integral simplifies further to

$$I_N(b,c;\alpha_I;m,n) = \int dK(x_i,y_r) \prod_{i=1}^{m}\prod_{r=1}^{n}\prod_{I=1}^{N} \mid x_i - z_I \mid^{-4b\alpha_I} \mid y_r - z_I \mid^{-4c\alpha_I} \qquad (3.21a)$$

where the measure $dK(x_i,y_r)$ is given by

$$dK(x_i,y_r) = \prod_{i<j}^{m}\prod_{r<s}^{n} d^2x_i d^2y_r \ \mid x_i - x_j \mid^{-4b^2} \mid y_r - y_s \mid^{-4c^2} \mid x_i - y_r \mid^{-4} \qquad (3.21b)$$

and

$$mb + nc = \xi = q - \Sigma \qquad (3.22)$$

which is the zero mode constraint for integer values of iu and iv. It is integral (3.21) with which we shall deal in the rest of the paper. We shall now comment on its symmetries.

<u>*Translational Invariance and Scale Covariance*</u>: Integral (3.21) is obviously translationally invariant. Under a scaling, $z_I \to \lambda z_I$, it is easy to check, using (3.22), that the measure $dK(x_i,y_r)$ transforms as

$$dK(x_i,y_r) \to \mid \lambda \mid^{2(mb+nc)[q-(mb+nc)]} dK(x_i,y_r) \qquad (3.23)$$

The cross-terms between the external variables z_I and the integrated variables x_i and y_r in (3.21) produce a factor $\mid \lambda \mid^{-4(mb+nc)\Sigma}$. Putting these two results together and using (3.22) we find that the integral I is covariant in the sense that

$$I_N \to I'_N = \mid \lambda \mid^{-2(q-\Sigma)\Sigma} I_N \qquad (3.24)$$

We should emphasise that the scaling property above takes this neat form only because of the constraint due to the zero mode integration.

<u>*Renormalisation*</u>: Let us consider the dependence of the N-point function on the dimensional parameters. The μ_b and μ_c dependence is easily read off from (3.6) and

(3.8) to be of the form $\mu_b^m \mu_c^n$. The dependence on the cut-off L can be calculated in a straightforward manner from (3.10) by noting that each Green's function produces one L. Thus the total contribution is given by $e^{2(\Sigma+mb+nc)^2 \ln L}$ which, upon using (3.22) reduces to L^{2q^2}. The third dimensional parameter in the theory is the diffeomorphic invariant short-distance cut-off ds introduced in Appendix A. It is easy to see that since there is one subtraction to be made for each of the diagonal points, the total contribution is given by $e^{-2(mb^2+nc^2+\sum_I \alpha_I^2)\ln ds}$. After a little algebra it is easy to see, using (3.22) and the Weyl conditions $b^2+1=qb$ and $c^2+1=qc$, that this contribution reduces to $(ds)^{2(m+n-q^2+\sum_I \Delta_I)}$. Finally let z_0 be a fiducial point from which the distances of the positions of the external variables z_I are measured. If L_0 be the unit length in which these distances are measured, then the dependence of the N-point function on this unit length is obtained by putting together the contributions coming from the $G(z_I, z_J)$ terms in (3.11) and the scaling property of the I_N integral given in (3.24) and works out to be $\mid L_0 \mid^{-2\sum_I \Delta_I}$. Putting all the above factors together we find that the dimensionality of the N-point function is given by

$$\left(\frac{L}{ds}\right)^{2q^2} \left(\mu_b (ds)^2\right)^m \left(\mu_c (ds)^2\right)^n \left(\frac{\mid L_0 \mid}{ds}\right)^{-2\sum_I \Delta_I} \tag{3.25}$$

The first term above may be eliminated by absorbing it in the overall normalisation since it is independent of α_I. The ds in the other terms may be eliminated by renormalising the bare coupling constants μ_b and μ_c, and L_0. Thus the parametric dependence of the path integral may be written in dimensionless variables as

$$\left(\frac{\mu_b^R}{\mu}\right)^m \left(\frac{\mu_c^R}{\mu}\right)^n \tag{3.26}$$

where μ, the renormalisation scale, has dimensions of mass squared, and μ_b^R and μ_c^R are the renormalised couplings.

IV. SL(2, C) COVARIANCE OF $I_N(b, c; \alpha_I; m, n)$

Under an SL(2, C) transformation[2]

$$x_i \to \frac{ax_i + b}{cx_i + d}, \qquad ad - bc = 1 \tag{4.1}$$

we have

$$dx_i \to \frac{dx_i}{(cx_i + d)^2} \quad \text{and} \quad x_i - x_j \to \frac{x_i - x_j}{(cx_i + d)(cx_j + d)} \tag{4.2}$$

It follows that the measure $dK(x_i, y_r)$ transforms as shown below:

$$dK(x_i, y_r) \to dK(x_i, y_r) \mid cx_i + d \mid^{4b(mb+nc-q)} \mid cy_r + d \mid^{4c(mb+nc-q)} \tag{4.3}$$

The cross-terms between the external variables and the integration variables produce a product of factors of the form $\mid cx_i + d \mid^{4b\Sigma} \mid cy_r + d \mid^{4c\Sigma}$ for the integrated variables and $\mid cz_I + d \mid^{4(mb+nc)\alpha_I}$ for the external variables. Putting everything together and using (3.22) once again, we find that all the factors corresponding to the integration variables cancel and I_N transforms covariantly i.e.

$$I_N(z_I) \to I_N(z'_I) = I_N(z_I) \prod_{I=1}^{N} \Big[\frac{1}{\mid cz_I + d \mid}\Big]^{-4(q-\Sigma)\alpha_I} \tag{4.4}$$

It follows that the partition function is SL(2, C) invariant and the one-point and two-point functions are invariant with respect to two and one parameter non-compact subgroups of SL(2, C) respectively. Thus, in principle, these functions are infinite. For the three and higher point functions, on the other hand, we can use the SL(2,C) covariance to simplify the integral. To do this we let

$$z'_1 = \frac{az_1 + b}{cz_1 + d} = \zeta, \qquad z'_2 = \frac{az_2 + b}{cz_2 + d} = 0, \qquad z'_3 = \frac{az_3 + b}{cz_3 + d} = 1 \tag{4.5}$$

[2] The SL(2,C) parameters a, b, c, d should not be confused with the parameters b and c in the rest of the paper.

where ζ will be taken to infinity at the end of the calculation. Since $ad - bc = 1$, this set of equations can be solved for the parameters a, b, c, d and the solutions are given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\zeta z_{31}}{(\zeta-1)z_{23}z_{12}}} & -z_2\sqrt{\frac{\zeta z_{31}}{(\zeta-1)z_{23}z_{12}}} \\ -\frac{\zeta z_{23}+z_{12}}{\sqrt{\zeta(\zeta-1)z_{12}z_{23}z_{31}}} & \frac{\zeta z_1 z_{23}+z_3 z_{12}}{\sqrt{\zeta(\zeta-1)z_{12}z_{23}z_{31}}} \end{pmatrix} \tag{4.6}$$

Using these equations it follows that

$$cz_I + d = \frac{z_{12}z_{3I} + \zeta z_{23}z_{1I}}{\sqrt{\zeta(\zeta-1)z_{12}z_{23}z_{31}}} \tag{4.7}$$

The integral I_N then becomes

$$\begin{aligned} I_N = \prod_{I=1}^{N} \Big[\mid \sqrt{\frac{\zeta z_{23}}{(\zeta-1)z_{12}z_{31}}} z_{1I} + \sqrt{\frac{z_{12}}{\zeta(\zeta-1)z_{23}z_{31}}} z_{3I} \mid \Big]^{4(q-\Sigma)\alpha_I} \\ \times \int dK(x_i, y_r) \prod_i \prod_r \prod_{K=4}^{N} \mid x_i - r_K \mid^{-4b\alpha_K} \mid y_r - r_K \mid^{-4c\alpha_K} \\ \times \mid x_i - \zeta \mid^{-4b\alpha_1} \mid x_i \mid^{-4b\alpha_2} \mid x_i - 1 \mid^{-4b\alpha_3} \mid y_r - \zeta \mid^{-4c\alpha_1} \mid y_r \mid^{-4c\alpha_2} \mid y_r - 1 \mid^{-4c\alpha_3} \end{aligned} \tag{4.8}$$

where the r_K are given by

$$r_K = \frac{\zeta z_{K2}z_{31}}{z_{12}z_{3K} + \zeta z_{23}z_{1K}}, \qquad K \geq 4 \tag{4.9}$$

On taking the limit $\zeta \to \infty$, r_K become the conformally invariant cross-ratios

$$r_K = \frac{z_{K2}z_{31}}{z_{K1}z_{32}}, \qquad K \geq 4 \tag{4.10}$$

and

$$\begin{aligned} I_N = \mid \frac{z_{12}z_{31}}{z_{23}} \mid^{-2(q-\Sigma)\alpha_1} \prod_{I=2}^{N} \mid \frac{z_{23}}{z_{12}z_{31}} \mid^{2(q-\Sigma)\alpha_I} \mid z_{1I} \mid^{4(q-\Sigma)\alpha_I} \\ \times \int dK(x_i, y_r) \prod_i \prod_r \prod_{K=4}^{N} \mid x_i - r_K \mid^{-4b\alpha_K} \mid y_r - r_K \mid^{-4c\alpha_K} \\ \times \mid x_i \mid^{-4b\alpha_2} \mid x_i - 1 \mid^{-4b\alpha_3} \mid y_r \mid^{-4c\alpha_2} \mid y_r - 1 \mid^{-4c\alpha_3} \end{aligned} \tag{4.11}$$

where the ζ-dependent terms from the integrand cancel those coming from the prefactor of (4.8). In the above computation we have chosen to extract z_1, z_2 and z_3. But of

course, since the N-point function is completely symmetric under the permutation of the indices $I = 1 \cdots N$, we could have used any three z_I for the SL (2, C) transformations. Thus (4.11) is invariant under these permutations, although it is not manifestly so because the cross-ratios change with the permutations. This invariance is what is usually referred to as crossing symmetry. Note that it gives relations between correlation functions of the same order N, but does not connect correlation functions of different order. By using a standard relation between four and three point functions in conformal field theories, the four-point crossing relations can be used to obtain constraints on the coefficient of the three-point function. An indirect method of checking that the DOZZ proposal satisfies these constraints was used in [4]. In the next section we shall directly derive the three-point function from the results of this section.

V. THE THREE-POINT FUNCTION

The simplest case of (4.11) is the case of the three-point function which is distinguished by the fact that there are no cross-ratios. In that case, (4.11) reduces to

$$I_3(b,c;\alpha_1,\alpha_2,\alpha_3;m,n) = \mid \frac{z_{12}z_{31}}{z_{23}} \mid^{-2(q-\Sigma)\alpha_1} \mid \frac{z_{12}z_{23}}{z_{31}} \mid^{-2(q-\Sigma)\alpha_2} \mid \frac{z_{23}z_{31}}{z_{12}} \mid^{-2(q-\Sigma)\alpha_3} \mathcal{I}_3 \tag{5.1}$$

where $\mathcal{I}_3$ is defined by

$$\mathcal{I}_3(m,n) = \int dK(x_i,y_r) \prod_i \prod_r \mid x_i \mid^{-4b\alpha_2} \mid x_i - 1 \mid^{-4b\alpha_3} \mid y_r \mid^{-4c\alpha_2} \mid y_r - 1 \mid^{-4c\alpha_3} \tag{5.2}$$

Denoting the extrapolation of the above integral from integer values of m and n, by $\mathcal{I}_3(iu,iv)$ and substituting in (3.11) and (3.6), we get for the three-point function

$$\mathcal{G}_3(z_1,z_2,z_3;\alpha_1,\alpha_2,\alpha_3) = \mathcal{H}_3 \mid z_{12} \mid^{2(\Delta_3-\Delta_1-\Delta_2)} \mid z_{23} \mid^{2(\Delta_1-\Delta_3-\Delta_2)} \mid z_{31} \mid^{2(\Delta_2-\Delta_3-\Delta_1)} \tag{5.3}$$

where

$$\mathcal{H}_3(\alpha_1,\alpha_2,\alpha_3) = -\frac{1}{8} \int dudv \, \frac{\mathcal{I}_3(iu,iv)}{\Gamma(1+iu)\Gamma(1+iv)} \frac{\delta(bu+cv-\xi)}{\sinh\pi u \sinh\pi v} \Big(\frac{\mu_b^R}{\mu}\Big)^{iu} \Big(\frac{\mu_c^R}{\mu}\Big)^{iv} \tag{5.4}$$

We shall now discuss how the extrapolation is done. Note that the final result for the three-point function is got by putting together the contributions from the zero mode integral and the fluctuation mode integral. As already mentioned, the latter is known only at a discrete set of points labelled by positive integer values. The former, however, is a known function of ξ, for negative values of ξ. Therefore we continue the zero mode integral to positive values of ξ where the two contributions can be put together. Once this is done, we shall show that it is possible to continue the full result away from the positive integral values to which the fluctuation mode integral is restricted.

Using the computations of Dotsenko and Fateev [7] and the k-function introduced by the Zamolodchikovs (defined in B8), we first note, as shown in Appendix B, that for positive integer m and n the fluctuation mode contribution (5.2) can be written as

$$\mathcal{I}_3(m,n) = -m!n!\Phi_b^m\Phi_c^n \frac{k'(0)}{k'(-bm-cn)} \prod_{I=1}^{3} \frac{k(2\alpha_I)}{k(\Sigma-2\alpha_I)} \tag{5.5}$$

where

$$\Phi_b = -\pi \frac{bk(2b)}{k(b)} \tag{5.6}$$

Here the function $k(x)$ is an entire function with simple zeroes at $x = -(mb+nc)$ and $x = (m+1)b + (n+1)c$ for all $m, n \geq 0$. It has the reflection symmetry property $k(x) = k(q-x)$ and is quasi-periodic in the sense that

$$k(x+b) = k(x)\gamma(bx)b^{1-2bx}, \quad k(x+c) = k(x)\gamma(cx)c^{1-2cx}, \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \tag{5.7}$$

The terms $m!n!\Phi_b^m\Phi_c^n$ and the α_I parts of this expression have a straightforward extrapolation from m, n to u, v but, as we shall see, the extrapolation of the function $k'(-mb-nc)$ is somewhat ambiguous. So for the moment, we simply assume that it has an extrapolation to some function $\tilde{k}(-bu-cv)$. The extrapolation of (5.5) is then

$$\mathcal{I}_3(iu,iv) = -\Gamma(1+iu)\Gamma(1+iv)\Phi_b^{iu}\Phi_c^{iv} \frac{k'(0)}{\tilde{k}(-bu-cv)} \prod_{I=1}^{3} \frac{k(2\alpha_I)}{k(\Sigma-2\alpha_I)} \tag{5.8}$$

Substituting this result in the expression for $\mathcal{H}_3$, the Γ functions exactly cancel and we obtain

$$\mathcal{H}_3 = \frac{1}{8}\int dudv\ \frac{\delta(bu+cv-\xi)}{\sinh\pi u \sinh\pi v} B^{iu} C^{iv} \frac{k'(0)}{\tilde{k}(-bu-cv)} \prod_{I=1}^{3} \frac{k(2\alpha_I)}{k(\Sigma-2\alpha_I)} \tag{5.9}$$

where

$$B = \frac{\mu_b^R}{\mu}\Phi_b, \quad C = \frac{\mu_c^R}{\mu}\Phi_c \tag{5.10}$$

We may rewrite (5.9) in the form

$$\mathcal{H}_3 = \frac{1}{8} L(\xi) \frac{k'(0)}{\tilde{k}(-\xi)} \prod_{I=1}^{3} \frac{k(2\alpha_I)}{k(\Sigma-2\alpha_I)} \tag{5.11}$$

where

$$L(\xi) = \int dudv \frac{\delta(bu+cv-\xi)}{\sinh\pi u \sinh\pi v} B^{iu} C^{iv} \tag{5.12}$$

In arriving at the above equations we have used the fact that the delta function converts the functions of $bu+cv$ into functions of ξ which enabled us to remove all the k-dependent parts from the u, v integrations.

The function $L(\xi)$ is closely related to the function $\frac{k'(\xi)}{k(\xi)}$ of the Zamolodchikovs. To see this we note that the $L(\xi)$ function satisfies the following recursion relations:

$$L(\xi - ib) = -B^{-1}\Big[L(\xi) - 2\frac{C^{ib\xi}}{\sinh\pi b\xi}\Big] \tag{5.13}$$

and a similar one for $\xi \to \xi + c$. The above equation may be compared with the analogous relations we can derive for the function $\frac{k'(\xi)}{k(\xi)}$ which have, instead of $\frac{1}{\sinh\pi b\xi}$ on the right hand side $\frac{\gamma'(b\xi)}{\gamma(b\xi)}$, and similarly for $\xi \to \xi + c$. Thus, both functions have simple poles at the points $\xi = \pm(mb+nc)$, but while $L(\xi)$ has residues $B^m C^n$, $\frac{k'(\xi)}{k(\xi)}$ has residues equal to 1. Thus

$$\lim_{\xi \to mb+nc} L(\xi)k(\xi) = B^m C^n k'(mb+nc) \qquad m, n \geq 0 \tag{5.14}$$

Substituting this back in (5.11) we find that the coefficient of the three-point function may be written as

$$\mathcal{H}_3(\alpha_1,\alpha_2,\alpha_3) = \frac{1}{8}\Big[L(-\xi)\frac{k(-\xi)}{k'(-\xi)}\Big] \times \frac{1}{k(-\xi)}\prod_{I=1}^{3}\frac{k(2\alpha_I)}{k(\Sigma-2\alpha_I)} \tag{5.15}$$

Note that the factor in the parantheses on the right hand side of the above equation is finite. This factor in the two-exponential theory is more general than the corresponding factor in the one-exponential theory. If this is taken into account, an analysis similar to the one in [4] would probably reveal that the two-exponential theory is also crossing-symmetric since the rest of the expression is readily seen to agree with the conjecture of the Zamalodchikovs in [5]. It is however interesting to point out that the expression in (5.15) has a rather complicated behaviour under a reflection $\alpha \to q-\alpha$ because of the more general factor in the parantheses.

Since the dimensionless parameters $\frac{\mu_b^R}{\mu}$ and $\frac{\mu_c^R}{\mu}$ are at our disposal, we may now specialise to the case which simplifies the u, v integrations in $L(\xi)$ by letting

$$B^c = C^b \equiv A \quad \text{which implies} \quad \big(-\pi\frac{\mu_b^R}{\mu}\gamma(b^2)\big)^c = \big(-\pi\frac{\mu_c^R}{\mu}\gamma(c^2)\big)^b \tag{5.16}$$

Denoting this simplified integral by $l(\xi)$ we find

$$l(\xi) = \int dudv\ \frac{\delta(bu+cv-\xi)}{\sinh\pi u \sinh\pi v} \tag{5.17}$$

The function $l(\xi)$ has residues ± 1 at the positive and negative poles respectively while the function $\frac{k'(\xi)}{k(\xi)}$ has residues 1 at all poles. This corresponds to the fact that $l(\xi)$ is even, and $\frac{k'(\xi)}{k(\xi)}$ is odd under the reflection transformation $\xi \to q-\xi$. The important point however is that the residues coincide for positive m and n and thus

$$\lim_{\xi\to mb+nc} l(\xi)k(\xi) = k'(mb+nc) \qquad m,n \geq 0 \tag{5.18}$$

This shows that there are at least two natural choices for $\tilde{k}(\xi)$ as an extension of $k'(mb+nc)$, $\quad m,n \geq 0$ namely, $\tilde{k}(\xi) = k'(\xi)$ (odd) and $\tilde{k}(\xi) = l(\xi)k(\xi)$ (even). Any arbitrary

combination of the above two choices is also permissible. And there may be other possibilities. However, if we now require that the extension $\tilde{k}(\xi)$ does not introduce any new singularities (i.e. not already occuring in $l(\xi)$), then we see that

$$\frac{l(\xi)}{\tilde{k}(\xi)}k(\xi) = e(\xi) \tag{5.19}$$

where from (5.18), $e(\xi)$ is an entire function satisfying

$$e(mb + nc) = 1, \quad m, n \geq 0 \tag{5.20}$$

But this means that $e(\xi)$ is doubly periodic on the positive real axis, which by Jacobi's theorem [10] on doubly periodic functions means that $e = 1$ everywhere. Thus the only choice of $\tilde{k}$ that does not introduce new poles is

$$\tilde{k}(\xi) = l(\xi)k(\xi) \tag{5.21}$$

In that case

$$\mathcal{H}_3 = \frac{1}{8}\frac{k'(0)}{k(-\xi)}A^{\xi}\prod_{I=1}^{3}\frac{k(2\alpha_I)}{k(\xi + 2\alpha_I)} \tag{5.22}$$

This expression is covariant under a reflection $\alpha \to q - \alpha$ because the denominator is invariant under this transformation and the only non-trivial contributions come from the behaviour of the numerator. Substituting the above equation in (5.3) we get finally

$$\mathcal{G}_3 = \frac{1}{8}\frac{k'(0)}{k(-\xi)}A^{\xi}\prod_{I=1}^{3}\frac{k(2\alpha_I)}{k(\xi + 2\alpha_I)} \mid z_{12} \mid^{2(\Delta_3-\Delta_1-\Delta_2)} \mid z_{23} \mid^{2(\Delta_1-\Delta_3-\Delta_2)} \mid z_{31} \mid^{2(\Delta_2-\Delta_3-\Delta_1)} \tag{5.23}$$

Note that this expression is exactly the one that was conjectured by the Zamolodchikovs on the basis of the one-potential theory. Thus we have shown that this result can be derived in the two-potential theory in a natural way. Furthermore, whereas in the one-potential theory, only one set of poles could be physically identified; in the two-potential theory, the full lattice of poles can be identified.

VI. THE TWO-POINT FUNCTION

The two-point function can be evaluated directly along the lines of the three-point function. It is easy to see that the translational invariance and scale covariance properties of the path integral imply that the fluctuating part has the following structure

$$I_2(b,c;\alpha_1,\alpha_2;m,n) = \mid z_{12} \mid^{2(q-\Sigma)\Sigma} \mathcal{I}_2 \tag{6.1}$$

where

$$\mathcal{I}_2 = \int dK(x_i,y_r) \prod_i \prod_r \mid x_i - 1 \mid^{-4b\alpha_1} \mid y_r - 1 \mid^{-4c\alpha_1} \mid x_i \mid^{-4b\alpha_2} \mid y_r \mid^{-4c\alpha_2} \tag{6.2}$$

Substituting for I_2 from the above equations in (3.11) and (3.6), we get

$$\mathcal{G}_2(z_1,z_2,\alpha_1,\alpha_2) = \mathcal{I}_2(b,c;\alpha_1,\alpha_2;m,n) \mid z_{12} \mid^{-2(\Delta_1+\Delta_2)} \tag{6.3}$$

The integral in (6.2) is essentially the same as the one encountered in the computation of the three-point function. But, in contrast to the latter, it is infinite or zero according as $D = \Delta_1 - \Delta_2 = 0$ or not. This is due to the SL(2, C) covariance and can be seen as follows: Under any SL(2, C) transformation with parameters $\{a,b,c,d\}$, the integral and the quantity $\mid z_1 - z_2 \mid^{-1}$ pick up the usual factors

$$\mid cz_1 + d \mid^{-2\Delta_1} \mid cz_2 + d \mid^{-2\Delta_2} \quad \text{and} \quad \mid cz_1 + d \mid\mid cz_2 + d \mid \tag{6.4}$$

respectively. Consider now the stability subgroup S of z_1 and z_2. This is a one-parameter subgroup whose standard parameters $\{a_s,b_s,c_s,d_s\}$ say, are functions of one free parameter. Since the transformations belonging to the subgroup leave both the integral and the quantity $\mid z_1 - z_2 \mid$ invariant, we have from (6.4)

$$I_2 = \mid c_s z_1 + d_s \mid^{-2\Delta_1} \mid c_s z_2 + d_s \mid^{-2\Delta_2} I_2 \quad \text{and} \quad \mid c_s z_1 + d_s \mid\mid c_s z_2 + d_s \mid = 1 \tag{6.5}$$

and thus

$$\mathcal{I}_2(\xi,\eta) = (c_s z_1 + d_s)^D \mathcal{I}_2(\xi,\eta) \quad \text{where} \quad \xi = q - \alpha_1 - \alpha_2,\ \eta = \alpha_1 - \alpha_2, \ \text{and} \ D = \xi\eta \tag{6.6}$$

which shows that either $I_2 = 0$ or $D = 0$. In the case that $D = 0$, the integral is invariant under a change of variable corresponding to the subgroup and is therefore infinite. To obtain a finite result, the integration corresponding to the one-parameter subgroup must be factored out.

Since $D = (q - \alpha_1 - \alpha_2)(\alpha_1 - \alpha_2)$ we see that $D = 0$ corresponds to either $\alpha_1 = \alpha_2$ or $\alpha_1 = q - \alpha_2$ *i.e.* the parameters α_1 and α_2 are either equal or reflection conjugate. The fact that the two-point function is zero except for these values is not surprising when we recall that the two-point function may be regarded as the inner product of two formal states of the kind $\mid \alpha, z >= e^{\alpha\phi(z)} \mid 0 >$ and these would be expected to be orthogonal unless the parameters were conjugate in some sense. However, as already mentioned, the integral is infinite and has to be regulated by factoring out the one-parameter subgroup S. A natural way to regulate it is to note from (5.2) and (6.2) that the two-point function is the $\alpha_3 \to 0$ limit of the three-point function with coefficient

$$l(\xi - \alpha_3) I_3(\alpha_1, \alpha_2, \alpha_3) = \frac{k'(0)k(2\alpha_1)k(2\alpha_2)k(2\alpha_3)}{k(\alpha_3 - \xi)k(\alpha_3 + \xi)k(\alpha_3 - \eta)k(\alpha_3 + \eta)} \tag{6.7}$$

Since by definition $D = \xi\eta$, we see that $D \neq 0$ implies $\xi \neq 0$ and $\eta \neq 0$ and from (6.7) we see that in this case the function does indeed vanish as $\alpha_3 \to 0$. On the other hand $D = 0$ implies $\xi = 0$ or $\eta = 0$, in which cases (6.7) becomes

$$\frac{k'(0)k(2\alpha_1)k(2\alpha_2)k(2\alpha_3)}{k(\alpha_3)k(\alpha_3)k(\alpha_3 - \eta)k(\alpha_3 + \eta)} \quad \text{or} \quad \frac{k'(0)k(2\alpha_1)k(2\alpha_2)k(2\alpha_3)}{k(\alpha_3 - \xi)k(\alpha_3 + \xi)k(\alpha_3)k(\alpha_3)} \tag{6.8}$$

respectively. A short computation shows that in the limit $\alpha_3 \to 0$ these become

$$\frac{2}{\alpha_3} \quad \text{and} \quad \frac{2}{\alpha_3}\frac{k(\xi)}{k(-\xi)} \tag{6.9}$$

Interpreting the universal constant $2/\alpha_3$ as the integral over the stability subgroup that has to be factored out, we obtain finally

$$G_2(q-\alpha,\alpha;z_1,z_2) = G_2(\alpha,q-\alpha;z_1,z_2) =\mid z_{12}\mid^{-4\Delta} \tag{6.10}$$

$$G_2(\alpha,\alpha;z_1,z_2) = G_2(q-\alpha,q-\alpha;z_1,z_2) = N(\alpha)\mid z_{12}\mid^{-4\Delta} \tag{6.11}$$

where

$$N(\alpha) = \Big[\frac{k(\xi)}{k(-\xi)}\Big]_{\xi=q-2\alpha} = \frac{k(2\alpha)}{k(2\alpha-q)} \tag{6.12}$$

The ambiguity in the two-point function may indicate that the extrapolated two-point function (and hence the one-point and partition functions) do not really exist *i.e.* that the functional integral is a kind of distribution which takes meaningful values only when tested against products of at least three external fields.

VII. THE FOUR-POINT FUNCTION

The four-point function may be calculated along the same lines as the three-point function. It is straightforward to see that it takes the form

$$\begin{aligned}\mathcal{G}_4(z_1,z_2,z_3,z_4;\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= P(z_{IJ})\times Q(r,\bar{r})\\ &\times\int dudv\,\frac{\delta(bu+cv-\xi)}{\sinh u\sinh v}I(r,\bar{r},u,v)\end{aligned} \tag{7.1}$$

where

$$P(z_{IJ}) =\mid z_{12}\mid^{-2(\Delta_1+\Delta_2-\Delta_3-\Delta_4)}\mid z_{23}\mid^{2(-\Delta_1+\Delta_2+\Delta_3+\Delta_4)}\mid z_{13}\mid^{2(\Delta_1-\Delta_2+\Delta_3-\Delta_4)}\mid z_{14}\mid^{4\Delta_4} \tag{7.2}$$

$$Q(r,\bar{r}) =\mid r\mid^{4\alpha_2\alpha_4}\mid r-1\mid^{4\alpha_3\alpha_4} \tag{7.3}$$

and $I(r,\bar{r},m,n)$ is given by

$$\begin{aligned}I(r,\bar{r},m,n) &= \int dK(x_i,y_r)\prod_i\prod_r \mid x_i-r\mid^{-4b\alpha_4}\mid y_r-r\mid^{-4c\alpha_4}\\ &\times\mid x_i\mid^{-4b\alpha_2}\mid y_r\mid^{-4c\alpha_2}\mid x_i-1\mid^{-4b\alpha_3}\mid y_r-1\mid^{-4c\alpha_3}\end{aligned} \tag{7.4}$$

In the special case when both $-4b\alpha_4$ and $-4c\alpha_4$ are positive integers (which implies that b is rational), the integral in (7.4) can be expanded as a polynomial in r with coefficients which are the three-point structure constants. Although α_4 is singled out here, it is clear that as long as one of the exponents is a positive integer, by choosing an appropriate SL(2, C) transformation, the four-point function is a polynomial in a suitable cross-ratio.

In the generic case, where none of the αs is a positive integer, if one divides the integral (7.4) into sections for which x_i and y_r are greater or less than r, one can expand each of these sections in an infinite power series in r or r^{-1}. The simplest example is for $m = 1,\ n = 0$ when we have an integral of the form

$$\int d^2x \ \mid x \mid^{-4b\alpha_2} \mid x-1 \mid^{-4b\alpha_3} \mid x-r \mid^{-4b\alpha_4} \tag{7.5}$$

These integrals can be split into parts $I_>$ and $I_<$ for which $\mid x \mid > \mid r \mid$ and $\mid x \mid < \mid r \mid$ respectively such that

$$I(r,\bar{r}) = I_> + I_< \tag{7.6}$$

$I_>$ and $I_<$ have the expansions

$$\begin{aligned} I_>(r,\bar{r}) &= \sum_{p=0}^{\infty} \int d^2x \ \mid x \mid^{-4b\alpha_2} \mid x-1 \mid^{-4b\alpha_3} \binom{-4b\alpha_4}{p} \mid r \mid^p \mid x \mid^{-p} \mid x \mid^{-4b\alpha_4} \\ &= \sum_{p=0}^{\infty} \binom{-4b\alpha_4}{p} \mid r \mid^p \int d^2x \mid x \mid^{-4b(\alpha_2+\alpha_4+\frac{p}{4b})} \mid x-1 \mid^{-4b\alpha_3} \\ &\sum_{p=0}^{\infty} \binom{-4b\alpha_4}{p} \mathcal{H}_3(-\alpha_2-\alpha_4-\alpha_3-\frac{p}{4b}+c, \alpha_2+\alpha_4+\frac{p}{4b}, \alpha_3) \mid r \mid^p \end{aligned} \tag{7.7a}$$

and

$$\begin{aligned} I_<(r,\bar{r}) &= \sum_{p=0}^{\infty} \int d^2x \ \mid x \mid^{-4b\alpha_2+p} \mid x-1 \mid^{-4b\alpha_3} \binom{-4b\alpha_4}{p} \mid r \mid^{-p} \\ &= \sum_{p=0}^{\infty} \binom{-4b\alpha_4}{p} \mid r \mid^{-p} \int d^2x \mid x \mid^{-4b(\alpha_2-\frac{p}{4b})} \mid x-1 \mid^{-4b\alpha_3} \\ &\sum_{p=0}^{\infty} \binom{-4b\alpha_4}{p} \mathcal{H}_3(-\alpha_2-\alpha_3+\frac{p}{4b}+c, \alpha_2-\frac{p}{4b}, \alpha_3) \mid r \mid^{-p} \end{aligned} \tag{7.7b}$$

respectively where the coefficients $\mathcal{H}_3$ are just the three-point coefficients already discussed in Section V. Note that each of these expansions is only an asymptotic expansion of the full integral.

VIII. CONCLUSIONS

In this paper we have considered the quantisation of the two-dimensional Liouville field theory by computing the N-point functions of vertex operators using path integral methods. It is argued that the standard one exponential Liouville potential admits a two-exponential generalisation because, in the quantum theory, there are two fields, rather than one, with a given conformal weight. It is shown that the two-exponential theory is not only conformally invariant but also has a built-in duality symmetry which was inferred earlier from the form of the N-point functions of the standard Liouville theory. We have derived expressions for the N-point functions and explicitly computed the three-point function. We have shown that the coefficient of the three-point function exhibits a two-dimensional lattice of poles in the parameter space. Unlike previous work where the existence of the two-dimensional lattice (as opposed to a one-dimensional lattice) was inferred in an indirect manner, the existence of the two-dimensional lattice is shown to be a natural and direct consequence of the quantum mechanical duality symmetry of the Liouville theory.

Unlike the three-point function, the lower point functions are invariant under non-compact sub-groups of SL(2, C) and are therefore infinite. In principle they can be made finite by factoring out the integral over the relevant subgroups. We propose a method of doing this for the two-point function and show that it vanishes unless the two vertex parameters are either equal or reflection conjugate. The four-point function can also be studied using our methods. We have briefly discussed its most important properties. In particular we have shown that when one of the vertex functions has an integer power,

the conformal block of the four-point function is given by a known polynomial of the cross-ratio. In general, however, the conformal block can only be obtained as a sum of asymptotic expansions in the cross-ratio and its inverse.

We conclude by suggesting some generalisations. It is probable that the main aspects of the formalism we have presented are equally valid for Toda theories and for their supersymmetric generalisations. It is also clear from our analysis that much of the calculation is valid for any two-exponential potential theory. It would be therefore interesting to see how far we can extend these results to interesting integrable field theories like the Sine-Gordon and Sinh-Gordon theories. We hope to generalise our analysis to these theories in the future.

Appendix A: The Finite-Volume Green's Function

The Green's function $G(x,y)$ on a compact two-dimensional space S^2 of volume Ω is the inverse of $\sqrt{g}\Delta_x$ where Δ_x is the two-dimensional Laplace-Beltrami operator defined in (2.2). Since on a two-dimensional compact space the only zero mode of Δ_x is the constant function it follows that $G(x,y)$ is the (unique) solution of the equation

$$\Delta_x G(x,y) = \frac{\pi}{\sqrt{g}}\delta^2(x-y) \quad \text{on} \quad L_2(S) \ominus P_0 \tag{A1}$$

which is orthogonal to the zero modes

$$\int d^2z\sqrt{g(z)}G(z,y) = \int d^2y\sqrt{g(y)}G(z,y) = 0 \tag{A2}$$

the expression $L_2(S) \ominus P_0$ denoting the usual Hilbert space for S minus the projection P_0 on the zero modes. It is not difficult to verify that in conformal coordinates the solution of (A1) is

$$G(x,y) = G_0(x,y) - \frac{1}{\Omega}[\rho(x)+\rho(y)] + \frac{1}{\Omega^2}\int d^2z\sqrt{g(z)}\rho(z) \tag{A3}$$

where

$$G_0(x,y) = -\frac{1}{2}\ln\frac{\mid x-y \mid}{L} \tag{A4}$$

L is the renormalization scale, and

$$\rho(x) = \int d^2y\sqrt{g(y)}G_0(x,y) \tag{A5}$$

Clearly it is the ρ terms in (A3) that make $G(x,y)$ orthogonal to the zero modes.

It is of special interest to study the short-distance behaviour of $G_0(x,y)$ i.e. the limit $x \to y$, because we shall have to define $G(x,y)$ for coincident points. In conformal coordinates the line-element is given by

$$ds^2 = \sqrt{g(y)}dyd\bar{y} \tag{A6}$$

and hence from (A4)

$$\Big(G_0(x,y)\Big)_{x\to y} \to -\frac{1}{2}\ln\frac{\mid dy \mid}{L} = -\frac{1}{2}\ln(\frac{ds}{L}) + \frac{1}{4}\ln(\sqrt{g}) \qquad (A7)$$

where ds is the geodesic distance between x and y. It is clear that as $x \to y$ the right-hand side of (A7) diverges, but the point is that the divergent part contains only ds, which is a diffeomorphic *invariant*. This means that we can absorb the divergent part of G in the renormalisation scale L, in a diffeomorphic invariant manner, leaving only the $\ln\sqrt{g}$ term. We then interpret the renormalised version $G_0^R(x,x)$ of $G_0(x,x)$ as

$$G_0^R(y,y) = \frac{1}{4}\ln(\sqrt{g(y)}) \qquad (A8)$$

An important point to note is that, although $G_0(x,y)$ is invariant with respect to the Weyl transformations $\sqrt{g(x)} \to \lambda(x)\sqrt{g(x)}$ the renormalised quantity $G_0^R(x,x)$ is not. In fact it has the Weyl transformation

$$G_0^R(x,x) \quad \to \quad G_0^R(x,x) + \frac{1}{4}\ln(\lambda(x)) \qquad (A9)$$

It is also worth noting that, whereas $G_0(x,y)$ is invariant with respect to all (rigid and local) Weyl transformations, the functions $\rho(x)$ in (A5) are invariant only with respect to rigid Weyl transformations. Hence the full Green's function $G(x,y)$ is invariant with respect to rigid Weyl transformations for all Ω but is invariant with respect to all Weyl transformations only in the infinite-volume limit.

Appendix B: The Dotsenko-Fateev Integral

The integral

$$I = \prod_{i<j}^{m}\prod_{r<s}^{n}\int d^2x_i d^2y_r \mid x_i - x_j \mid^{4\rho} \mid y_r - y_s \mid^{4\rho'} \mid x_i - y_r \mid^{-4} \\ \times \mid x_i \mid^{2\alpha} \mid x_i - 1 \mid^{2\beta} \mid y_r \mid^{2\alpha'} \mid y_r - 1 \mid^{2\beta'} \tag{B1}$$

which depends, apart from ρ and ρ', on the six parameters $m, n, \alpha, \beta, \alpha', \beta'$ has been computed by Dotsenko and Fateev [7]. By using the dictionary

$$\rho = -b^2, \quad \rho' = -c^2, \quad \alpha = -2b\alpha_1, \quad \beta = -2b\alpha_2, \quad \alpha' = -2c\alpha_1, \quad \beta' = -2c\alpha_2 \tag{B2}$$

and the definition

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \tag{B3}$$

their result for (5.2) may be written in the form

$$I = \Big[m!n!\pi^{m+n}b^{-8mn}\big(\gamma(-b^2)\big)^{-m}\big(\gamma(-c^2)\big)^{-n}\Big](XYZ)^{-1} \tag{B4}$$

where

$$X = \prod_{I=1}^{3}\prod_{l=0}^{m-1}\gamma(2b\alpha_I + lb^2), \quad Y = \prod_{I=1}^{3}\prod_{l=0}^{n-1}\gamma(2c\alpha_I + m + lc^2) \tag{B5}$$

$$Z = \prod_{l=1}^{m}\gamma(1+lb^2)\prod_{l=1}^{n}\gamma(1+m+lc^2) \tag{B6}$$

and α_3 is an auxiliary variable defined by

$$\alpha_3 \equiv q - (\alpha_1 + \alpha_2) - mb - nc \tag{B7}$$

The problem with this expression, from our point of view, is that it is not in a form that readily admits an extrapolation to non-integer values of m and n. This situation can be remedied by using the function $k(x)$ defined by the Zamolodchikovs as

$$\ln(k(x)) = \int_0^\infty \frac{dt}{t}\Big((\frac{q}{2} - x)^2 e^{-2t} - \frac{\sinh^2(\frac{q}{2} - x)t}{\sinh(bt)\sinh(ct)}\Big) \tag{B8}$$

in the range $0 < x < q$, and elsewhere by analytic continuation. As already mentioned in the Section V, the relevant properties of $k(x)$ are that it has the symmetry property $k(x) = k(q-x)$, it is an entire function with zeros at $x = -mb - nc$ and $x = (m+1)b + (n+1)c$ for any non-negative integers m and n and that it is related to $\gamma(x)$ by the recursion relation

$$\gamma(bx) = \frac{k(x+b)}{k(x)} b^{(2bx-1)} \tag{B9}$$

¿From the recursion relation it follows that

$$\gamma(b\chi + lb^2) = \frac{k\big(\chi + (l+1)b\big)}{k(\chi + lb)} b^{(2b\chi + 2lb^2 - 1)} \tag{B10}$$

and thus

$$\prod_{l=m_1}^{m_2} \gamma(b\chi + lb^2) = \frac{k\big(\chi + (m_2+1)b\big)}{k(\chi + m_1 b)} b^{m[2b\chi - 1 + Mb^2]} \tag{B11}$$

where $m = m_2 - m_1 + 1$ and $M = m_1 + m_2$. In particular

$$\prod_{l=0}^{m-1} \gamma(b\chi + lb^2) = \frac{k(\chi + mb)}{k(\chi)} b^{m[(2b\chi - 1) + (m-1)b^2]} \tag{B12}$$

and

$$\prod_{l=1}^{m} \gamma(b\chi + lb^2) = \frac{k\big(\chi + (m+1)b\big)}{k(\chi + b)} b^{m[(2b\chi - 1) + (m+1)b^2]} \tag{B13}$$

Similar results are valid for $b \to c, m \to n$. These relations permit us to write the m and n products occurring in X, Y and Z as ratios of single functions. Thus using (B12) for X and Y we obtain

$$X = \big(b\big)^{m[4b\Sigma - 3 + 3(m-1)b^2]} \prod_{I=1}^{3} \frac{k(2\alpha_I + mb)}{k(2\alpha_I)} \tag{B14}$$

and

$$Y = \Big(c\Big)^{n[4c\Sigma - 3 + 6m + 3(n-1)c^2]} \prod_{I=1}^{3} \frac{k(2\alpha_I + mb + nc)}{k(2\alpha_I + mb)} \tag{B15}$$

where $\Sigma = \alpha_1 + \alpha_2 + \alpha_3$. A crucial point is that when we combine these expressions to form XY the factors $k(2\alpha_I + mb)$ in X and Y cancel to give

$$XY = \big(b\big)^{(mb - nc)\Sigma - 6mn} \prod_{I=1}^{3} \frac{k(2\alpha_I + mb + nc)}{k(2\alpha_I)} \tag{B16}$$

which, using $k(q-x)=k(x)$ and the definition of α_3, may be written as

$$XY = \big(b\big)^{(mb-nc)\Sigma-6mn}\prod_{I=1}^{3}\frac{k(\Sigma-2\alpha_I)}{k(2\alpha_I)} \quad \text{where} \quad \Sigma=\alpha_1+\alpha_2+\alpha_3 \tag{B17}$$

It is this cancellation that is responsible for the equality of the two-exponential and one-exponential computations. In order to apply the same procedure to Z we have to be a little careful as the k-functions have zeros at relevant points. To allow for this we use the formula (B13) with $\epsilon\neq 0$ for Z, and take the limit $\epsilon\to 0$. We then obtain

$$Z = \big(b\big)^{\delta}\Big\{\Big[\frac{k\big(c+(m+1)b+\epsilon\big)}{k(c+b+\epsilon)}\Big]\Big[\frac{k\big((1+m)b+(n+1)c+\epsilon\big)}{k\big((1+m)b+c+\epsilon\big)}\Big]\Big\}_{\epsilon=0} \tag{B18}$$

where

$$\delta = m[1+(m+1)b^2]-n[(1+2m)+(n+1)c^2] \tag{B19}$$

Here again the k factors that contain m but not n cancel, and we obtain

$$Z = \big(b\big)^{(mb-nc)(2q-\Sigma)-2nm}\Big[\frac{k\big((1+m)b+(n+1)c+\epsilon\big)}{k(c+b+\epsilon)}\Big]_{\epsilon=0} \tag{B20}$$

Note that the terms proportional to q in the exponent δ add rather than cancel as they did for XY. This is because the summation runs from 1 to m rather than 0 to $m-1$ as it did for X and Y. Using $k(x)=k(q-x)$ and the definition of α_3, this may be written as

$$\begin{aligned} Z &= \big(b\big)^{(mb-nc)(2q-\Sigma)-2nm}\times\Big[\frac{k\big((-mb-nc)-\epsilon\big)}{k(q+\epsilon)}\Big]_{\epsilon=0} \\ &= -\big(b\big)^{(mb-nc)(2q-\Sigma)-2nm}\times\frac{k'(-mb-nc)}{k'(q)} \end{aligned} \tag{B21}$$

where in the last step we have used L'Hospital's rule. Combining the factors XYZ we then have

$$XYZ = -\big(b\big)^{[2(mb-nc)q-8mn]}\times\frac{k'(-mb-nc)}{k'(0)}\prod_{I=1}^{3}\frac{k(\Sigma-2\alpha_I)}{k(2\alpha_I)} \tag{B22}$$

Hence if we define

$$\Phi_b = -\pi(b^2)^{2-qb}\gamma(b^2) = -\pi\frac{bk(2b)}{k(b)} \tag{B23}$$

and use (B4) we have finally

$$I_{mn} = -m!n!\Phi_b^m \Phi_c^n \frac{k'(0)}{k'(-mb-nc)} \prod_{I=1}^{3} \frac{k(2\alpha_I)}{k(\Sigma - 2\alpha_I)} \qquad (B24)$$

Acknowledgements: This work was begun in collaboration with I. Sachs and C. Wiesendanger. The discussion of Weyl invariance in this paper is inspired to a large extent by some unpublished work of the latter. It is a pleasure to thank J. Teschner for several useful discussions.

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