arXiv:math/0209254v1 [math.AG] 19 Sep 2002

SOME REDUCTIONS ON JACOBIAN PROBLEM IN TWO VARIABLES

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Abstract. Let $f = (f_1, f_2)$ be a regular sequence of affine curves in $\mathbb{C}^2$. Under some reduction conditions achieved by composing with some polynomial automorphisms of $\mathbb{C}^2$, we show that the intersection number of curves (f_i) in $\mathbb{C}^2$ equals to the coefficient of the leading term x^{n-1} in g_2, where $n = \deg f_i$ $(i = 1, 2)$ and (g_1, g_2) is the unique solution of the equation $y\mathcal{J}(f) = g_1 f_1 + g_2 f_2$ with $\deg g_i \leq n-1$. So the well-known Jacobian problem is reduced to solving the equation above. Furthermore, by using the result above, we show that the Jacobian problem can also be reduced to a special family of polynomial maps.

1. Introduction

Let $f = (f_1, f_2)$ be a pair of polynomials in two variables (x, y). Let $\mathcal{J}(f) = \frac{\partial(f_1,f_2)}{\partial(x,y)}$ be its Jacobian. The well known Jacobian Conjecture in two variables says that: if $\mathcal{J}(f) \equiv 1$, then the map $f : \mathbb{C}^2 \to \mathbb{C}^2$ is invertible and the inverse map f^{-1} is also a polynomial map. For the history and well known results about this conjecture, see [2] and [14]. For the two-variable case, there are numerous partial results. Here we just mention a few of them. Abhyankar [1] shows that the conjecture is equivalent to any two affine curves (f_i) $(i = 1, 2)$ with the Jacobian condition having exactly one intersection point at infinity and gives a proof that any two such curves have at most two intersection points at infinity. He also proves the conjecture under the condition that $k(x, y)$ is a Galois field extension over $k(f_1, f_2)$. Note that this is also proved by Markar-Limanov [8]. Nakai and Baba [11] generalize a theorem of Magnus [9] and prove the conjecture if one of $d_i = \deg f_i$ $(i = 1, 2)$ is a prime, or 4, or if $d_1 = 2p \geq d_2$ for some prime number p. Wright [13] proves that f is invertible if and only if the Jacobian matrix $J(f)$ can be written as a product of elementary and diagonal matrices in $GL_2(k[X])$. Finally, with some help from computers, Moh [10] shows the conjecture is true if $d_i = \deg f_i \leq 100$ $(i = 1, 2)$.

Key words and phrases. Residue, intersection numbers and Jacobian conjecture.
2000 Mathematics Subject Classification: 14R15, 14C17.

It is well known that the Jacobian Conjecture is equivalent to saying that the polynomial map f is injective when $\mathcal{J}(f) \equiv 1$ (See, for example, [2], [12]). Equivalently, if the Jacobian Conjecture is true, then any two affine curves (f_i) $(i = 1, 2)$ in $\mathbb{C}^2$ with $\mathcal{J}(f) \equiv 1$ have (at most) one intersection point in $\mathbb{C}^2$. One natural question we may ask here is whether or not there is some "nice" relationship between $\mathcal{J}(f)$ and the total number (counting multiplicity) of intersection points of the affine curves (f_1) and (f_2) in the affine space $\mathbb{C}^2$. Our first main result will be that, under some reduction conditions achieved by composing with some polynomial automorphisms of $\mathbb{C}^2$, the answer to the question above is "Yes". To be more precise, we first show that, by composing with some polynomial automorphisms of $\mathbb{C}^2$ to the polynomial map $f = (f_1, f_2)$, we may assume that:

(**RC1**) the leading homogeneous parts of f_1 and f_2 are both x^n for some $n \in \mathbb{N}$;

(**RC2**) all intersection points of (f_1) and (f_2) in the affine space $\mathbb{C}^2$ lie on the line $\{y = 0\}$.

From now on, we will assume $f = (f_1, f_2)$ satisfies the reduction conditions above for the rest of this section. Our first main result is

Theorem 1.1. (1) *there is a unique polynomial solution* $g = (g_1, g_2)$ *for the equation*

$$y\mathcal{J}(f) = f_1 g_1 + f_2 g_2 \tag{1.1}$$

with $\deg g_i = n - 1$ $(i = 1, 2)$.

(2) *the total intersection number* (*counting multiplicity*) *of the affine curves* (f_1) *and* (f_2) *in* $\mathbb{C}^2$ *equals to the coefficient of* x^{n-1} *of* g_2.

Since the Jacobian condition $\mathcal{J}(f) \equiv 1$ and the total intersection number of (f_1) and (f_2) in $\mathbb{C}^2$ can be preserved in our reduction procedure, the Jacobian conjecture in two variables is reduced to the problem solving the polynomial equation (1.1) for the polynomial maps $f = (f_1, f_2)$ with the reduction conditions above. The partial solution $g_i(x, 0)$ $(i = 1, 2)$ to the equation (1.1) is given in Proposition 5.4 in the case that all the intersection points of (f_1) and (f_2) in $\mathbb{C}^2$ are normal crossing. Note that the solution of the equations of the form (1.1) is the so called the membership problems, which is one of the most important problem in computational algebra. It has been studied from many different ways, see [4] by using Bezout Identities and residues and [3], [5] by using Gröbner bases. We hope that some results from the membership problem can provide some new insights to the Jacobian problem via Theorem 1.1.

Our second main result is

Theorem 1.2. *Suppose that all intersection points of* (f_1) *and* (f_2) *in* $\mathbb{C}^2$ *are normal crossing, then the polynomial map* $f = (f_1, f_2)$ *can always be written as*

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} h_1 & -k_2 \\ h_2 & k_1 \end{pmatrix} \begin{pmatrix} r(x) \\ y\lambda(x) \end{pmatrix} \tag{1.2}$$

where $\det \begin{pmatrix} h_1 & -k_2 \\ h_2 & k_1 \end{pmatrix} = \mathcal{J}(f)$ *and* $r(x)$ *and* $\lambda(x)$ *are polynomials in one variable related by*

$$r(x)\mu(x) + r'(x)\lambda(x) = 1 \tag{1.3}$$

$$\deg \lambda(x) \leq \deg r(x) \tag{1.4}$$

for some polynomial $\mu(x)$.

From Theorem 1.2, one immediately sees that the Jacobian problem is reduced to the following

Conjecture 1.3. *Let* $f = (f_1, f_2)$ *be of the form* (1.2) *with the matrix* $\begin{pmatrix} h_1 & -k_2 \\ h_2 & k_1 \end{pmatrix}$ *being invertible. Then, for any* $(r(x), \lambda(x))$ *related by* (1.3), (1.4) *and* $\deg(r(x)) \geq 2$, *the Jacobian* $\mathcal{J}(f) \neq c$ *for any* $c \in \mathbb{C}^*$.

The arrangement of the paper is as follows. In Section 2, for the convenience of the readers, we fix some notation and recall some results in the theory of residues and intersection numbers which will play the key roles in our later arguments. In Section 3, we first recall Noether's $AF+BG$ theorem, then derive some consequences which later will give the degree upper bound of the solutions of equation (1.1). In Section 4, we show that, by composing certain polynomial automorphisms of $\mathbb{C}^2$, the reduction conditions $(RC1)$ and $(RC2)$ can be achieved. In Section 5, we give the proofs for our main results Theorem 1.1 and Theorem 1.2.

This paper is a revised version of one of the topics in the author's Ph.D Thesis in The University of Chicago. The author is very grateful to his adviser, Professor Spencer Bloch for encouragement and discussions. The author also thanks Professor Mohan Kumar, Madhav Nori and David Wright for personal communications on this subject.

2. Residues and Intersection Numbers

Notation:

1) Let $[X_0, X_1, \cdots, X_n]$ be the homogeneous coordinates for $\mathbb{C}P^n$. Set $U_i = \{X_i \neq 0\}$ $(i = 0, 1, 2, \cdots, n)$. We use $x_1, x_2, \cdots, x_n$ to denote

the Euclidean coordinate systems for U_0. We usually use small letters f, g, so on, to denote the polynomials $f(x_1, \cdots, x_n)$, $g(x_1, \cdots, x_n)$ in n variables and use the corresponding capital letters to denote their homogenized polynomials in $X_0, X_1, \cdots, X_n$, i.e. $F(X_0, X_1, \cdots, X_n) = X_0^d F(\frac{X_1}{X_0}, \frac{X_2}{X_0} \cdots, \frac{X_n}{X_0})$, where d is the total degree of the polynomial f in $x_1, x_2, \cdots, x_n$.

2) Let H, F, G be three homogeneous polynomials in X_i, $(i = 0, 1, 2)$. Suppose F and G intersect discretely at $p \in \mathbb{C}P^2$. We say that *the restriction of H at p lies in the ideal generated by F and G*, denoted by $H|_p \in < F, G >_p$, if the following condition is hold:

Let U_i be an affine open subset defined above for $\mathbb{C}P^2$ such that $p \in U_i$ for some $0 \leq i \leq 2$. Here let us assume that $i = 2$. Set $f(X_0, X_1) = F(X_0, X_1, 1)$, $g(X_0, X_1) = G(X_0, X_1, 1)$ and $h(X_0, X_1) = H(X_0, X_1, 1)$. Then, as holomorphic functions near $p \in U_2$, we have $h|_p \in < f, g >_p$, i.e. h lies in the ideal generated by f and g.

It is easy to see that the condition above does not depend on the choices of the affine open subset U_i.

A sequence $(f_1, f_2, \cdots, f_n)$, where $f_i \in \mathcal{O}_0$, the germs of holomorphic functions at $0 \in \mathbb{C}^n$, is said to be *regular* at $0 \in \mathbb{C}^n$ if there is an open neighborhood U of $0 \in \mathbb{C}^n$ such that 0 is the only common zeros of f_i $(i = 1, 2, \cdots, n)$. This is equivalent to saying that the Jacobian $\mathcal{J}(f)$ is not identically 0. A sequence $(f_1, f_2, \cdots, f_n)$, where $f_i \in \mathcal{O}$ is said to be *regular* on $\mathbb{C}^n$ if they intersect only at discrete points, or equivalently, f is regular at any point of $\mathbb{C}^n$.

For a regular sequence $f = (f_1, f_2, \cdots, f_n)$ at $0 \in \mathbb{C}^n$ and a holomorphic function $h \in \mathcal{O}_0$. Set

$$\omega(z) = \frac{h(z)}{f_1 f_2 \cdots f_n} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \tag{2.1}$$

We define the residue of the meromorphic form at $0 \in \mathbb{C}^n$ to be to be

$$\mathrm{Res}_{\{0\}}\omega(z) = \int_\Gamma \frac{h(z)}{f_1 f_2 \cdots f_n} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \tag{2.2}$$

where $\Gamma = \{z \in \mathbb{C}^n : |f_1(z)| = \epsilon, |f_2(z)| = \epsilon, \cdots, |f_n(z)| = \epsilon\}$ for some small $\epsilon > 0$.

Proposition 2.1. 1) *The residue $Res_{\{0\}}\omega$ is alternating with respect to the permutations of f_i $(i = 1, 2, \cdots, n)$.*

2) *Let I be the ideal generated by f_i $(i = 1, 2, \cdots, n)$. For any $g, h \in \mathcal{O}$, set*

$$Res_{\{0\}}(g, h) = \int_\Gamma \frac{g(z)h(z)}{f_1 f_2 \cdots f_n} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \tag{2.3}$$

Then, $g \in I$ if and only if $Res_{\{0\}}(g,h) = 0$ for any $h \in \mathcal{O}$. In other words, the bilinear pairing

$$Res_{\{0\}} : \mathcal{O}/I \times \mathcal{O}/I \to \mathbb{C} \tag{2.4}$$

induced by (2.3) *is non-singular.*

3) **(Transition Formula)** *Suppose that $g = (g_1, g_2, \cdots, g_n)$ be another regular sequence at $0 \in \mathbb{C}^n$ with $g^{-1}(0) = \{0\}$. If $\{g_1, g_2, \cdots, g_n\} \subset I$, say $g_i(z) = \sum_{j=1}^n a_{i,j}(z) f_j(z)$ $(i = 1, 2, \cdots, n)$, for some $a_{i,j}(z) \in \mathcal{O}$. Then for any $h \in \mathcal{O}$, we have*

$$\begin{aligned} & Res_{\{0\}}\left(\frac{h(z)}{f_1 f_2 \cdots f_n} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n\right) \\ &= Res_{\{0\}}\left(\frac{h(z)\det(A)}{g_1 g_2 \cdots g_n} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n\right) \end{aligned} \tag{2.5}$$

The intersection number $D_1 \cdot D_2 \cdots D_n$ of the divisors $D_i = (f_i)$ at point $0 \in \mathbb{C}^n$ is defined to be

$$\begin{aligned} (D_1 \cdot D_2 \cdots D_n)(0) &= \int_\Gamma \frac{1}{f_1 f_2 \cdots f_n} df_1 \wedge df_2 \wedge \cdots \wedge df_n \\ &= \int_\Gamma \frac{\mathcal{J}(f)(z)}{f_1 f_2 \cdots f_n} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \end{aligned} \tag{2.6}$$

Proposition 2.2. 1) *The intersection number $(D_1 \cdot D_2 \cdots D_n)(0)$ is always a positive integer and equals to the degree d of the holomorphic map f at point $0 \in \mathbb{C}^n$. It also equals to the complex dimension of the vector space $\mathcal{O}/I$. In particular, for any $h \in \mathcal{O}$ with $h(0) = 0$, we have $h^d \in I$.*

2) *For any $h \in \mathcal{O}$, we have*

$$Res_{\{0\}}(h, \mathcal{J}(f)) = \deg(f) h(0) \tag{2.7}$$

Proposition 2.3. *For any $h \in \mathcal{O}$ with $h(0) = 0$, we have $\mathcal{J}(f)h \in I$.*

Proof: For any $g \in \mathcal{O}$, consider

$$\begin{aligned} \mathrm{R}es_{\{0\}}(g, \mathcal{J}(f)h) &= \mathrm{R}es_{\{0\}}(gh, \mathcal{J}(f)) \\ &= \deg(f)(gh)(0) \\ &= 0 \end{aligned} \tag{2.8}$$

Then, by Proposition 2.1, 2), we have $\mathcal{J}(f)h \in I$. □

Since the residue is defined locally, we can generalize it to complex manifolds.

Theorem 2.4. (Residue Theorem) *Let M be a compact complex manifold of dimensional n, ω a meromorphic $(n,0)$ form on M which has only simple pole over effective divisors D_i $(i=1,2,\cdots,n)$. Suppose D_i's intersect only at discrete points v_k $(k=1,2,\cdots,m)$, then*

$$\sum_{j=1}^{m} Res_{\{v_k\}}\omega = 0 \tag{2.9}$$

3. Noether's AF+BG Theorem and Some Consequences

First let us recall Noether's $AF+BG$ theorem, for the proof of this theorem, see [6]. Let H,F,G be three homogeneous polynomials in X_i, $(i=0,1,2)$. Let $deg(H)=m$, $deg(F)=k$ and $deg(G)=\ell$. Suppose that $a=m-k\geq 0$, $b=m-\ell\geq 0$ and F and G have only discrete intersection points. If $H|_p \in < F,G >_p$ for any $p\in (F)\cdot(G)$, i.e. the restriction of H at any intersection point of F and G lies in the ideal generated by the restriction of F and G. (See the notations fixed at the beginning of the Section 2). Then there are homogeneous polynomials A and B of degree a and b, respectively, such that $H=AF+BG$. Furthermore, the pair (A,B) is unique up to the following sense: if $(\tilde{A},\tilde{B})$ is another such a pair, then there exist a homogeneous polynomial C such that $\tilde{A}=A+CG$ and $\tilde{B}=B-CF$. Clearly if $a<\ell$ or $b<k$, the pair (A,B) is uniquely determined by H,F,G.

Next, we derive some consequences of Noether's $AF+BG$ theorem, which will play crucial roles in our later argument.

Proposition 3.1. *Let h,f,g be polynomials in x,y. Suppose that the affine curves (f) and (g) intersect only at discrete points in $\mathbb{C}^2$ and $h_p\in < f,g >_p$ for any intersection point $p\in(f)\cdot(g)$. Then there exists a pair of polynomials $(a(x,y),b(x,y))$ such that*

$$h(x,y)=a(x,y)f(x,y)+b(x,y)g(x,y)$$

Furthermore, the pair (a,b) is unique up to the similar sense as above.

Note that, the proposition above is stronger than Hilbert's Nullstellensatz (See [7]), which claims only that $h(x,y)$ is in the radical of the ideal generated by $f(x,y)$ and $g(x,y)$ in general.

Proof: We first embed the curves $\mathcal{C}_i$ for $(i=1,2)$ into the projective space $\mathbb{C}P^2$ by considering the homogenized polynomials H,F,G of h,f,g respectively. Observe that F,G still intersect discretely in $\mathbb{C}P^2$. We can choose $m\in\mathbb{N}$ large enough such that $(X_0^m H)_p \in < F,G >_p$ for any $p\in(F)\cdot(G)\cdot\{X_0=0\}$ in $\mathbb{C}P^2$. Then apply Noether's $AF+BG$ theorem to the homogeneous polynomial $X_0^m H$, we have

$X_0^n H = AF + BG$ for some homogeneous polynomials A and B, then restrict to the open set $U_0 \simeq \mathbb{C}^2$, we get $h = af + bg$. $\square$

Unlike Noether's Theorem, Proposition 3.1 does not tell us much about the degrees of the polynomials $a(x, y)$ and $b(x, y)$. But when $h(x, y)$ has the form $\mathcal{J}(f)h(x, y)$, we have

Proposition 3.2. *Let f_i $(i = 1, 2)$ as above, then, for any polynomial $h(x, y)$ which vanishes at all intersection points of (f_1) and (f_2) in $\mathbb{C}^2$, there exist a pair (g_1, g_2) of polynomials in x, y such that*

$$\mathcal{J}(f)h(x, y) = g_1 f_1 + g_2 f_2 \tag{3.1}$$

and

$$\deg(g_1) \leq \deg(f_2) + \deg(h) - 2 \tag{3.2}$$

$$\deg(g_2) \leq \deg(f_1) + \deg(h) - 2 \tag{3.3}$$

Furthermore, if $\deg(\mathcal{J}(f)h(x, y)) < 2\, min(\deg(f_1), \deg(f_2))$, the solution $g = (g_1, g_2)$ of (3.1) is unique.

Before we give the proof of the proposition above, we need the following

Lemma 3.3. *Let f_i $(i = 1, 2)$ as above and F_i their homogenized polynomials. Then $\frac{\partial(F_1,F_2)}{\partial(X_1,X_2)}\big|_p \in < F_1, F_2 >_p$ for any $p \in (F_1)\cdot(F_2)\cdot\{X_0 = 0\}$.*

Proof: Let p be any intersection point of (F_1) and (F_2) in $\{X_0 = 0\} \subset \mathbb{C}P^2$. Without losing any generality, we may assume $p \in U_2$. Observe that for any homogeneous polynomial $F(X_0, X_1, X_2)$ of degree n, we always have

$$\begin{aligned} nF(X_0, X_1, X_2) &= X_0 \frac{\partial}{\partial X_0} F(X_0, X_1, X_2) + X_1 \frac{\partial}{\partial X_1} F(X_0, X_1, X_2) \\ &\quad + X_2 \frac{\partial}{\partial X_2} F(X_0, X_1, X_2) \end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial X_0} F(X_0, X_1, X_2) &= \frac{1}{X_0} \Big(nF(X_0, X_1, X_2) - X_1 \frac{\partial}{\partial X_1} F(X_0, X_1, X_2) \\ &\quad - X_2 \frac{\partial}{\partial X_2} F(X_0, X_1, X_2) \Big) \end{aligned} \tag{3.5}$$

We calculate the Jacobian $\frac{\partial(F_1,F_2)}{\partial(X_0,X_1)}$ as following

$$\begin{aligned}
\frac{\partial(F_1,F_2)}{\partial(X_0,X_1)} &= \begin{pmatrix} \frac{\partial F_1}{\partial X_0} & \frac{\partial F_1}{\partial X_1} \\ \frac{\partial F_2}{\partial X_0} & \frac{\partial F_2}{\partial X_1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{X_0}\left(d_1F_1 - X_1\frac{\partial}{\partial X_1}F_1 - X_2\frac{\partial}{\partial X_2}F_1\right) & \frac{\partial F_1}{\partial X_1} \\ \frac{1}{X_0}\left(d_2F_2 - X_1\frac{\partial}{\partial X_1}F_2 - X_2\frac{\partial}{\partial X_2}F_2\right) & \frac{\partial F_2}{\partial X_1} \end{pmatrix} \\
&= \frac{1}{X_0}\left(d_1F_1\frac{\partial F_2}{\partial X_1} - d_2F_2\frac{\partial F_1}{\partial X_1}\right) + \frac{X_2}{X_0}\frac{\partial(F_1,F_2)}{\partial(X_1,X_2)} \qquad (3.6)
\end{aligned}$$

By Proposition 2.3, $X_0\frac{\partial(F_1,F_2)}{\partial(X_0,X_1)}\big|_p \in < F_1, F_2 >_p$. Thus

$$\begin{aligned}
X_2\frac{\partial(F_1,F_2)}{\partial(X_1,X_2)} &= X_0\frac{\partial(F_1,F_2)}{\partial(X_0,X_1)} - \left(d_1F_1\frac{\partial F_2}{\partial X_1} - d_2F_2\frac{\partial F_1}{\partial X_1}\right) \\
&\equiv X_0\frac{\partial(F_1,F_2)}{\partial(X_0,X_1)} \mod \mathrm{I}_p \qquad (3.7)
\end{aligned}$$

Hence $\frac{\partial(F_1,F_2)}{\partial(X_1,X_2)}\big|_p \in < F_1, F_2 >_p$. □

Remark 3.4. *If* $\deg(\mathcal{J}(f)) = m$, *let* $J(X_0,X_1,X_2) = X_0^m\mathcal{J}(f)(\frac{X_1}{X_0},\frac{X_2}{X_0})$ *be the homogenized polynomial of the Jacobian* $\mathcal{J}(f)$. *Then it is straightforward to check that*

$$\frac{\partial(F_1,F_2)}{\partial(X_1,X_2)} = X_0^{d_1+d_2-2-m}J(X_0,X_1,X_2) \qquad (3.8)$$

where $d_i = \deg(f_i)$ *for* $i = 1, 2$.

Proof of Proposition 3.2: Consider the homogeneous polynomial $\frac{\partial(F_1,F_2)}{\partial(X_1,X_2)}H$ whose restriction on U_0 is $\mathcal{J}(f_1,f_2)h(x,y)$.

Claim: For any intersection point p of the divisors (F_1) and (F_2) in $\mathbb{C}P^2$, we have $\frac{\partial(F_1,F_2)}{\partial(X_1,X_2)}H \in < F_1, F_2 >_p$.

When $p \in U_0$, the claim follows from Proposition 2.3 and Remark 3.4. When $p \in \{X_0 = 0\}$, it follows from Lemma 3.3.

Now apply Noether's $AF + BG$ Theorem to $\frac{\partial(F_1,F_2)}{\partial(X_1,X_2)}H$, we have

$$\frac{\partial(F_1,F_2)}{\partial(X_1,X_2)}H = G_1F_1 + G_2F_2 \qquad (3.9)$$

for some homogeneous polynomials G_i $(i = 1, 2)$ with

$$\begin{aligned}\deg(G_i) &= \deg \frac{\partial(F_1, F_2)}{\partial(X_1, X_2)} H - \deg F_i \\ &= \deg(F_1) + \deg(F_2) + \deg(H) - 2 - \deg(F_i)\end{aligned}$$

Restrict to $U_0 \simeq \mathbb{C}^2 \subset \mathbb{C}P^2$, we get (3.1).

When $\deg(\mathcal{J}(f)h(x, y)) < 2\min(\deg(f_1), \deg(f_2))$, the uniqueness of $g = (g_1, g_2)$ follows from the uniqueness of $G = (G_1, G_2)$ in Noether's $AF + BG$ theorem. $\square$

4. **Reductions on polynomial maps**

To consider the total intersection number of a regular sequence $f = (f_1, f_2)$ in two variables in the affine space $\mathbb{C}^2$, we first perform the following reductions by applying some polynomial automorphisms of $\mathbb{C}^2$ with $f = (f_1, f_2)$.

Suppose the affine curves (f_1) and (f_2) in $\mathbb{C}^2$ intersect at discrete points $v_0, v_1, \cdots, v_N$ (without counting multiplicities). First we can choose two generic lines l_1 and l_2 to form a linear basis for $\mathbb{C}^2$, such that $v_0 = l_1 \cap l_2$ and $v_i - v_j \notin l_2$ for any $i \neq j$. Let (a_i, b_i) be the coordinate of v_i $(i = 1, 2, \cdots, N)$ with respect to the basis (l_1, l_2), then $v_0 = (0, 0)$ and $a_i \neq a_j$ for any $i \neq j$.

Now, for any $m > 0$, there exist a polynomial $p(x)$ such that

1) $p(a_i) = b_i$ for any $i = 0, 1, 2, \cdots N$,

2) $\deg(p(x)) \geq m$.

To see such a polynomial $p(x)$ always exists, we first choose and write $p(x) = c_{N-1}x^{m+N-1} + c_{N-2}x^{m+N-2} + \cdots + c_0 x^m$, then the equations $p(a_i) = b_i$ for $1 \leq i \leq N$ give the family of the linear equations

$$\begin{pmatrix} a_1^{m+N-1} & a_1^{m+N-2} & \cdots & a_1^m \\ a_2^{m+N-1} & a_2^{m+N-2} & \cdots & a_2^m \\ \vdots & \vdots & & \vdots \\ a_N^{m+N-1} & a_N^{m+N-2} & \cdots & a_N^m \end{pmatrix} \begin{pmatrix} c_{N-1} \\ c_{N-2} \\ \vdots \\ c_0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} \tag{4.1}$$

Hence (4.1) has a unique solution. Also note that $p(0) = 0$ since $m > 0$, therefore $p(a_i) = b_i$ for any $i = 0, 1, \cdots, N$.

Observe that the polynomial map $u = (x, y + p(x))$ has Jacobian $\mathcal{J}(u) \equiv 1$ and is invertible with the polynomial inverse $u^{-1} = (x, y - p(x))$. Therefore, the affine curves $(f_1 \circ u)$ and $(f_2 \circ u)$ have same total intersection number in $\mathbb{C}^2$ as the affine curves (f_1) and (f_2). Actually, $f_1 \circ u$ and $f_2 \circ u$ intersect in $\mathbb{C}^2$ only at points $(a_i, 0)$ ($i = 0, 1, 2, \cdots, N$) which all are on the line $\{y = 0\}$. Another observation is that if we

choose m large enough, the leading terms of $f_1 \circ u$ and $f_2 \circ u$ will depends only on x. Replacing f_1 or f_2 by $f_1 + f_2$ if it is necessary, we may assume that $f_1 \circ u$ and $f_2 \circ u$ have same degree. By composing certain linear automorphisms to $f = (f_1, f_2)$ from the left or right, which do not change the Jacobians, we may assume that the leading terms of $f_i \circ u$ are both x^n.

From the reductions above, we see that, without losing any generality, for any regular sequence $f = (f_1, f_2)$ of polynomials, composing some polynomial automorphism to f if it is necessary, we may assume f satisfies the following conditions:

Reduction Conditions:

(RC1): $f_i = x^n +$ lower degree terms ($i = 1, 2$),

(RC2): all intersection points of (f_1) and (f_2) in $\mathbb{C}^2$ lie on the line $\{y = 0\}$.

Remark 4.1. *From the reduction procedure above, it is easy to see that the Jacobian $\mathcal{J}(f)$ will not be changed if we choose properly the polynomial automorphisms composed to $f = (f_1, f_2)$. In particular, the Jacobian condition $\mathcal{J}(f) \equiv 1$ can be preserved during our reduction procedures.*

5. Main Results

From now on and for the rest of this paper, we will always assume $f = (f_1, f_2)$ satisfies the reduction conditions $(RC1)$ and $(RC2)$.

Let F_i $(i = 1, 2)$ be the homogenized polynomial of f_i. Note that, by the reduction condition $(RC1)$, the only intersection point the curves of (F_1) and (F_2) in $\{X_0 = 0\} \subset \mathbb{C}P^2$ is the point $[0, 0, 1]$. We denote it by v_∞. Let $(a_i, 0)$ $(i = 0, 1, 2, \cdots, N)$ be the intersection points of (F_1) and (F_2) in $\mathbb{C}^2 \subset \mathbb{C}P^2$ with $a_0 = 0$. Set $r(x) = \prod_{i=0}^{N}(x - a_i)$.

Proposition 5.1. *There exist a pair of polynomials (k_1, k_2) and a unique pair of polynomials (g_1, g_2) such that*

$$\mathcal{J}(f)y = g_1 f_1 + g_2 f_2 \tag{5.1}$$

$$\mathcal{J}(f)r(x) = k_1 f_1 + k_2 f_2 \tag{5.2}$$

with $deg(g_i) \leq n - 1$ and $deg(k_i) \leq n + N - 1$ $(i = 1, 2)$.

Proof: This is a direct consequence of Proposition 3.2 for the polynomials $h(x, y) = y$ and $h(x, y) = r(x)$, respectively. □

Theorem 5.2. *With the same notation as above, the coefficient of the term x^{n-1} of g_2 equals to the intersection number of f_1 and f_2 in $\mathbb{C}^2$.*

Note that Proposition 5.1 and Theorem 5.2 imply Theorem 1.1, the first main result stated in Section 1.

Proof: Let $m = \deg(\mathcal{J}(f))$ and $J = X_0^m \mathcal{J}(f)(\frac{X_1}{X_0}, \frac{X_2}{X_0})$. Note that under the reduction conditions $(RC1)$ and $(RC2)$, we have $m < 2n-2$.

Consider the meromorphic $(2,0)$ form ω on $\mathbb{C}P^2$ which is defined as following:

$$\omega(x,y) = \frac{\mathcal{J}(f)}{f_1 f_2} dx \wedge dy \quad \text{on} \quad U_0 \tag{5.3}$$

$$\omega(x_0,x) = \frac{x_0^{2n-3-m} J(x_0,x)}{F_1(x_0,x)F_2(x_0,x)} dx_0 \wedge dx \quad \text{on} \quad U_2$$

$$\omega(x_0,y) = -\frac{x_0^{2n-3-m} J(x_0,y)}{F_1(x_0,y)F_2(x_0,y)} dx_0 \wedge dy \quad \text{on} \quad U_1$$

where (x,y), (x_0,x) and (x_0,y) are the Euclidean coordinates for U_0, U_2 and U_1, respectively.

It is easy to check that ω is well defined $(2,0)$ form on $\mathbb{C}P^2$ and has pole only at the effective divisors (F_1) and (F_2). Then by the Residue Theorem 2.4, we have

$$Res_{v_\infty}\omega = -\sum_{i=0}^{i=N} \mathrm{Res}_{v_i}\omega \tag{5.4}$$

which, by (5.3), is the negative of the intersection number of (f_1) and (f_2) in $\mathbb{C}^2$.

We can calculate the residue of ω at v_∞ as following: Note that from Remark 3.4 and (5.1), we have

$$X_0^{2n-2-m} JY = G_1F_1 + G_2F_2 = (G_1+G_2)F_1 + G_2(F_2 - F_1)$$

Note that

$$F_1 = X_1^n + X_0 B_1 \tag{5.5}$$

$$F_2 = X_1^n + X_0 B_2 \tag{5.6}$$

for some homogeneous polynomials B_i. So $F_2 - F_1 = X_0(B_2 - B_1)$ is divisible by X_0, hence so is $G_1 + G_2$. Let $B = X_0^{-1}(F_2 - F_1)$, then

$$X_0^{2n-3-m} JY = X_0^{-1}(G_1+G_2)F_1 + G_2 B \tag{5.7}$$

Restrict to U_2, we have

$$x_0^{2n-3-m}J(x_0,x) = x_0^{-1}(G_1+G_2)(x_0,x)F_1(x_0,x) + G_2(x_0,x)B(x_0,x)$$

By Proposition 2.1, the Transition Formula (2.5), we have

$$\begin{aligned}
\mathrm{Res}_{v_\infty}\omega &= \int_{v_\infty} \frac{x_0^{2n-3-m}J(x_0,x)}{F_1(x_0,x)F_2(x_0,x)}dx_0\wedge dx \\
&= \int_{v_\infty} \frac{G_2B}{F_1(x_0,x)F_2(x_0,x)}dx_0\wedge dx \\
&= -\int_{v_\infty} \frac{G_2}{x^n x_0}dx_0\wedge dx \\
&= -\text{the coefficient of } x^{n-1} \text{ of } G_2.
\end{aligned}$$

Then by (5.4), we are done. □

It is interesting and probably a little surprising to see that the total intersection number of (f_1) and (f_2) in $\mathbb{C}^2$ and the Jacobian $\mathcal{J}(f)$ are related in the algebraic way provided by Proposition 5.1 and Theorem 5.2. Considering the Jacobian problem, we immediately have

Corollary 5.3. *The Jacobian Conjecture for two variables is equivalent to the following statement: Suppose that $f=(f_1,f_2)$ satisfies the Reduction Conditions $(RC1)$, $(RC2)$ and $\mathcal{J}(f)\equiv 1$. Let $g=(g_1,g_2)$ be the unique solution of the equation*

$$y = g_1f_1 + g_2f_2 \tag{5.8}$$

with $\deg(g_i)\le n-1$. *Then the coefficient of x^{n-1} of g_2 equals* 1.

Unfortunately, the equation (5.1) is not quite easy to solve in general, even though it is a linear equation and has a unique solution $g=(g_1,g_2)$ with the degree condition $\deg g_i \le n-1$ $(i=1,2)$. One question, which we think, might be interesting is to look more closely at the algorithm using Gröbner bases and to see if we can get more insight to solution of the equation (5.1) or (5.8).

In the next proposition, we give the partial solution $g_i(x,0)$ of g_i in terms of $f_i(x,0)$ $(i=1,2)$ under the condition that (f_1) and (f_2) have only transversal intersection points in $\mathbb{C}^2$.

Write $f_1=\sum_{i=0}^{n-1}a_i(x)y^i$, $f_2=\sum_{i=0}^{n-1}b_i(x)y^i$, $g_1=\sum_{i=0}^{n-1}c_i(x)y^i$ and $g_2=\sum_{i=0}^{n-1}d_i(x)y^i$, then we have

Proposition 5.4. *Suppose $\mathcal{J}(f)(v_i) \neq 0$ for any $i = 0, 1, 2, \cdots, N$. Then,*

$$c_0(x) = -r'(x)\frac{b_0(x)}{r(x)} \tag{5.9}$$

$$d_0(x) = r'(x)\frac{a_0(x)}{r(x)} \tag{5.10}$$

where $r(x) = \Pi_{i=0}^{N}(x - a_i)$ as before and $r'(x) = \frac{dr}{dx}(x)$.

Proof: From the equations (5.1) and (5.2), let $y = 0$, we get

$$0 = c_0(x)a_0(x) + d_0(x)b_0(x) \tag{5.11}$$

$$\mathcal{J}(f)(x, 0)r(x) = k_1(x, 0)a_0(x) + k_2(x, 0)b_0(x) \tag{5.12}$$

Note that $r(x)$ is the greatest common divisor of $a_0(x)$ and $b_0(x)$, so $a_0(x)/r(x)$ and $b_0(x)/r(x)$ are coprime to each other. Dividing $r(x)$ from the both sides of (5.11), we get $0 = c_0(x)(a_0(x)/r(x)) + d_0(x)(b_0(x)/r(x))$. Therefore there exists a polynomial $\eta(x)$ such that $c_0(x) = -\eta(x)(b_0(x)/r(x))$, $d_0(x) = \eta(x)(a_0(x)/r(x))$ and $\deg(\eta(x)) \leq N$. It is easy to check that

$$k_1(x, 0)d_0(x) - k_2(x, 0)a_0(x) = \mathcal{J}(f)(x, 0)\eta(x) \tag{5.13}$$

Now apply the Transition Formula (2.5) to the equations (5.1) and (5.2), we get

$$\begin{aligned}
1 &= \mathrm{Res}_{\{v_i\}}\left(\frac{\mathcal{J}(f)}{f_1 f_2}dx \wedge dy\right) \\
&= \mathrm{Res}_{\{v_i\}}\left(\frac{\mathcal{J}(f)(g_1k_2 - g_2k_1)}{(\mathcal{J}(f)y)(\mathcal{J}(f)r(x))}dx \wedge dy\right) \\
&= \mathrm{Res}_{\{v_i\}}\left(\frac{(g_1k_2 - g_2k_1)\mathcal{J}(f)^{-1}}{yr(x)}dx \wedge dy\right) \\
&= \mathrm{Res}_{\{v_i\}}\left(\frac{(g_1k_2 - g_2k_1)(x, 0)\mathcal{J}(f)^{-1}(x, 0)}{yr(x)}dx \wedge dy\right) \\
&= -\mathrm{Res}_{\{v_i\}}\left(\frac{(k_1(x, 0)d_0(x) - k_2(x, 0)a_0(x))\mathcal{J}(f)^{-1}(x, 0)}{yr(x)}dx \wedge dy\right) \\
&= -\mathrm{Res}_{\{v_i\}}\left(\frac{\eta(x)}{yr(x)}dx \wedge dy\right) \\
&= \frac{\eta(a_i)}{r'(a_i)}
\end{aligned}$$

Hence, we have $\eta(a_i) = r'(a_i)$ for any $i = 0, 1, 2, \cdots, N$. Since $\deg(\eta(x)) \leq \deg(r'(x))$. we have $\eta(x) = r'(x)$. $\square$

Under the conditions above, we can choose the following special solution $k = (k_1, k_2)$ for the equation (5.1) as follows:

Set

$$k_1(x,y) = \frac{1}{y}(r(x)g_1(x,y) + r'(x)f_2(x,y)) \tag{5.14}$$

$$k_2(x,y) = \frac{1}{y}(r(x)g_2(x,y) - r'(x)f_1(x,y)) \tag{5.15}$$

Note that from equations (5.9) and (5.10), it is easy to check that k_i $(i = 1, 2)$ defined above are polynomials.

From (5.14) $\times f_1+$ (5.15)$\times f_2$, we get

$$\mathcal{J}(f)r(x) = k_1(x,y)f_1(x,y) + k_2(x,y)f_2(x,y) \tag{5.16}$$

From (5.14) $\times g_2-$ (5.15) $\times g_1$, we get

$$\mathcal{J}(f)r'(x) = k_1(x,y)g_2(x,y) - k_2(x,y)g_1(x,y) \tag{5.17}$$

Since that $r(x)$ and $r'(x)$ are coprime to each other, there exist polynomials $\lambda(x)$ and $\mu(x)$ such that

1)

$$r(x)\mu(x) + r'(x)\lambda(x) = 1 \tag{5.18}$$

2) $\deg \lambda(x) \leq N$ and $\deg \mu(x) \leq N - 1$.

Set

$$h_1(x,y) = \mu(x)f_1(x,y) + \lambda(x)g_2(x,y) \tag{5.19}$$

$$h_2(x,y) = \mu(x)f_2(x,y) - \lambda(x)g_1(x,y) \tag{5.20}$$

Then, from the equations (5.14)-(5.20), we have

$$\det \begin{pmatrix} h_1 & -k_2 \\ h_2 & k_1 \end{pmatrix} = \mathcal{J}(f) \tag{5.21}$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} h_1 & -k_2 \\ h_2 & k_1 \end{pmatrix} \begin{pmatrix} r(x) \\ y\lambda(x) \end{pmatrix} \tag{5.22}$$

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -h_2 & k_1 \\ h_1 & k_2 \end{pmatrix} \begin{pmatrix} r'(x) \\ y\mu(x) \end{pmatrix} \tag{5.23}$$

In particular, we have proved Theorem 1.2, the second main result stated in Section 1.

Example 5.5. *Consider $f = (f_1, f_2)$, where*

$$f_1(x,y) = x + y + x^n$$
$$f_2(x,y) = y + x^n$$

Note that $\mathcal{J}(f) \equiv 1$ *and* f *is a polynomial automorphism of* $\mathbb{C}^2$ *with the inverse* $f^{-1} = (f_1^{-1}, f_2^{-1})$, *where*

$$\begin{aligned} f_1^{-1}(x,y) &= x - y \\ f_2^{-1}(x,y) &= y - (x-y)^n \end{aligned}$$

Hence, (f_1) *and* (f_2) *intersect only at* $0 \in \mathbb{C}^2$ *with multiplicity* 1 *and in this case, we have* $r(x) = x$, $\mu(x) = 0$ *and* $\lambda(x) = 1$.

It is easy to check that the unique solution $g = (g_1, g_2)$ *of equation* (5.8) *is given by*

$$\begin{aligned} g_1(x,y) &= -x^{n-1} \\ g_2(x,y) &= x^{n-1} + 1 \end{aligned}$$

Hence the coefficient of x^{n-1} *of* g_2 *is same as the total intersection number of* (f_1) *and* (f_2) *in* $\mathbb{C}^2$ *which is* 1.

We choose the matrix

$$\begin{pmatrix} h_1 & -k_2 \\ h_2 & k_1 \end{pmatrix} = \begin{pmatrix} 1 + x^{n-1} & 1 \\ x^{n-1} & 1 \end{pmatrix}$$

Then we have $\det \begin{pmatrix} h_1 & -k_2 \\ h_2 & k_1 \end{pmatrix} = \mathcal{J}(f) \equiv 1$ *and*

$$\begin{aligned} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} 1 + x^{n-1} & 1 \\ x^{n-1} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} &= \begin{pmatrix} -x^{n-1} & 1 \\ 1 + x^{n-1} & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

which are the equations (5.22) *and* (5.23), *respectively, in this case.*

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