arXiv:math/0409534v2 [math.CV] 2 Nov 2004

HESSIAN NILPOTENT POLYNOMIALS AND THE JACOBIAN CONJECTURE

WENHUA ZHAO

Abstract. Let $z = (z_1, \cdots, z_n)$ and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2}$ the Laplace operator. The main goal of the paper is to show that the well-known Jacobian conjecture without any additional conditions is equivalent to the following what we call *vanishing conjecture*: for any homogeneous polynomial $P(z)$ of degree $d = 4$, if $\Delta^m P^m(z) = 0$ for all $m \geq 1$, then $\Delta^m P^{m+1}(z) = 0$ when $m >> 0$, or equivalently, $\Delta^m P^{m+1}(z) = 0$ when $m > \frac{3}{2}(3^{n-2} - 1)$. It is also shown in this paper that the condition $\Delta^m P^m(z) = 0$ $(m \geq 1)$ above is equivalent to the condition that $P(z)$ is Hessian nilpotent, i.e. the Hessian matrix $\operatorname{Hes} P(z) = (\frac{\partial^2 P}{\partial z_i \partial z_j})$ is nilpotent. The goal is achieved by using the recent breakthrough work of M. de Bondt, A. van den Essen [BE1] and various results obtained in this paper on Hessian nilpotent polynomials. Some further results on Hessian nilpotent polynomials and the vanishing conjecture above are also derived.

Contents

1. **Introduction** 2
2. **Deformed Inversion Pairs of Formal Power Series** 5
2.1. Notation and Conventions 5
2.2. Deformed Inversion Pairs of Formal Power Series 7
3. **Deformed Inversion Pairs of HN Power Series** 9
4. **A Criterion for Hessian nilpotency** 15
5. **Hessian Nilpotent Polynomials** 19
5.1. A Criterion for Hessian Nilpotency of Homogeneous Harmonic Polynomials 19
5.2. Some Examples of HNP's and HNS 23
6. **More Properties of HN Polynomials** 24
6.1. An Identity of HN Formal Power Series 24
6.2. A Vanishing Property of HN Formal Power Series 25
6.3. Isotropic Properties of Homogeneous HN Polynomials 25

2000 *Mathematics Subject Classification.* 33C55, 39B32, 14R15, 31B05.
Key words and phrases. Hessian nilpotent polynomials, Deformed inversion pairs, the Heat equation, harmonic polynomials, the Jacobian conjecture.

7. **Vanishing Conjectures of HN Polynomials** 30
References 32

1. Introduction

Let $z = (z_1, z_2, \cdots, z_n)$ and $F(z) = z - H(z)$ be a formal map from $\mathbb{C}^n$ to $\mathbb{C}^n$ with $o(H(z)) \geq 2$ and $G(z)$ the formal inverse map of $F(z)$. The well-known Jacobian conjecture first proposed by Keller [Ke] in 1939 claims that, *if $F(z)$ is a polynomial map with the Jacobian $j(F)(z) = 1$, the inverse map $G(z)$ must also be a polynomial map.* Despite intense study from mathematicians in more than half a century, the conjecture is still wide open even for the case $n = 2$. In 1998, S. Smale [S] included the Jacobian conjecture in his list of 18 important mathematical problems for 21st century. For more history and known results on the Jacobian conjecture, see [BCW], [E] and references there. Recently, M. de Bondt and A. van den Essen [BE1] (Also see G. Meng [M]) have made a breakthrough on the Jacobian conjecture. They reduced the Jacobian conjecture to polynomial maps $F(z) = z - H(z)$ with $H(z) = \nabla P(z) = (\frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2}, \cdots, \frac{\partial P}{\partial z_n})$ for some polynomials $P(z) \in \mathbb{C}[z]$. In this paper, we will refer to this reduction as the *gradient reduction* and the condition $H(z) = \nabla P(z)$ for some $P(z) \in \mathbb{C}[[z]]$ as the *gradient condition*. Note that, by Poincaré lemma, a formal map $F(z) = z - H(z)$ with $o(H(z)) \geq 2$ satisfies the gradient condition if and only if its Jacobian matrix $JF(z)$ is symmetric. Following the terminology in [BE1], we also call the formal maps satisfying the gradient condition *symmetric* formal maps.

For further discussion, let us fix the following notions. A power series $P(z) \in \mathbb{C}[[z]]$ is said to be HN (*Hessian nilpotent*) if its Hessian matrix $\operatorname{Hes} P(z) = (\frac{\partial^2 P}{\partial z_i \partial z_j})$ is nilpotent. Let t be a formal parameter which commutes with z. The *deformed inversion pair* $Q_t(z)$ of any $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$ is the unique power series $Q_t(z) \in \mathbb{C}[[z, t]]$ with $o(Q_t(z)) \geq 2$ such that the formal map $G(z) = z + t\nabla Q_t(z)$ is the inverse map of $F(z) = z - t\nabla P(z)$. Recently, G. Meng [M] and D. Wright [Wr1] have derived a tree expansion formula for the inverse map $G(z)$ of $F(z) = F_{t=1}(z)$. In [Z2], by studying the PDE satisfied by the deformed inversion pairs $Q_t(z)$, the author has derived a recurrent formula and a binary rooted tree expansion formula for the deformed inversion pairs $Q_t(z)$. Furthermore, some close relationships among the deformed inversion pairs, the Legendre transform, the inviscid Burgers' equations and the Jacobian conjecture are also clarified in [Z2]. For

some other recent results on symmetric polynomial or formal maps, see [BE1]–[BE5], [EW], [M], [Wr1], [Wr2] and [Z2].

In this paper, we will use some general results in [Z2] to study HNS (Hessian nilpotent power series) $P(z)$ and their deformed inversion pairs $Q_t(z)$. Furthermore, by using the *gradient reduction* in [BE1] and various results derived in this paper on HN polynomials, we will show that the Jacobian conjecture is equivalent to what we call *vanishing conjectures* of homogeneous HN polynomials. (See the discussion below.) We first derive the PDE's satisfied by $Q_t(z)$, $\Delta^k Q_t^m$ $(k, m \geq 1)$ and $\exp(sQ_t(z))$ $(s \in \mathbb{C}^\times)$, where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial z_i^2}$ is the Laplace operator. In particular, we show in Theorem 3.2 that $\exp(sQ_t(z))$ $(s \in \mathbb{C}^\times)$ is the unique power series solution of the Cauchy problem of the Heat equation with the initial condition $\exp(sQ_t(z))|_{t=0} = \exp(sP(z))$. We then derive a uniform formula (See Theorem 3.4.) for the powers $Q_t^k(z)$ $(k \geq 1)$ of the deformed inversion pairs $Q_t(z)$ of HNS $P(z)$. We also prove a general theorem, Theorem 4.1, on a relationship between $\{\operatorname{Tr}\operatorname{Hes}^m(P(z)) | m \geq 1\}$ and $\{\Delta^m P^m(z) | m \geq 1\}$ for any power series $P(z)$. From this theorem, we show in Theorem 4.3 that, for any formal power series $P(z)$, it is HN if and only if $\Delta^m P^m(z) = 0$ for any $m \geq 1$, or equivalently, $\Delta^m P^m(z) = 0$ for any $1 \leq m \leq n$. Finally, we prove some identities, vanishing properties and isotropic properties of $\{\Delta^k P^m(z) | m, k \geq 0\}$ for HNS or HNP's (Hessian nilpotent polynomials) $P(z)$. Some close relationships of the deformed pairs $Q_t(z)$ of HNS or HNP's $P(z)$ with the Heat equation and the Jacobian conjecture are also clarified. In particular, we show that the Jacobian conjecture without any additional conditions is equivalent to the following *vanishing conjectures*: for any HNP $P(z)$ of degree $d = 4$, $\Delta^m P^{m+1}(z) = 0$ for $m >> 0$, or more precisely, for all $m > \frac{3}{2}(3^{n-2} - 1)$.

One remark is that, due to the identity $\operatorname{Tr}\operatorname{Hes}(P) = \Delta P$, any HNS $P(z)$ is automatically harmonic, i.e. $\Delta P(z) = 0$. Note that harmonic polynomials (See [ABR], [H] and [T].) are among the most classical objects in mathematics and have been very well studied. The classical study on harmonic polynomials started from Legendre, Laplace, Jacobi in the late eighteen century. The modern generalizations of harmonic polynomials, namely, spherical functions, were first studied by Cartan and Weyl in the 1930's and later by Gelfand, Harish-Chandra, etc. It is quite surprising to see that, first, HNP's as a family of very special harmonic polynomials are closely related with the notorious Jacobian conjecture. Secondly, it seems that HNP's have been overlooked and have not been studied until the recent work of M. de Bondt, A. van den Essen [BE1] and G. Meng [M]. Besides the connections with

the Jacobian conjecture discussed above, another interesting aspect of HNP's is their connection with the classical inviscid Burgers' equation in Diffusion theory and also the Heat equation. Actually, the vanishing conjecture above is also equivalent to saying that the power series solutions of certain Cauchy problems of the inviscid Burgers' equation and the Heat equation must be polynomials and the exponentials of polynomials, respectively. (See discussion in Section 4 in [Z2] and Conjecture 3.3 in this paper.) Considering the connections of HNP's with the classical objects described above, we believe that HNP's deserve much more attentions from mathematicians.

Considering the length of this paper, we give the following detailed arrangement description. In Section 2, we first fix some notation and definitions that are needed throughout the rest of this paper. We then briefly recall certain results obtained in [Z2] and prove some preliminary results including the PDE (See Corollary 2.7) satisfied by $\Delta^k Q_t^m$ $(k, m \geq 1)$ for the deformed inversion pairs $Q_t(z)$ of any power series $P(z)$. In Section 3, for any HNS $P(z)$ and its deformed inversion pair $Q_t(z)$, we derive the PDE's satisfied by $Q_t(z)$ and $\exp(sQ_t(z))$ $(s \in \mathbb{C}^\times)$, from which we derive with two different proofs a uniform formula Eq. (3.8) for $Q_t^k(z)$ $(k \geq 1)$. In Section 4, we prove a general theorem, Theorem 4.1, on a relationship between $\{\operatorname{Tr}\operatorname{Hes}^m(P(z)) | m \geq 1\}$ and $\{\Delta^m P^m(z) | m \geq 1\}$ for the universal formal power series $P(z)$ with $o(P(z)) \geq 2$. From this theorem, we deduce a criterion in Theorem 4.3 for the Hessian nilpotency of a formal power series $P(z)$ in terms of certain vanishing properties of $\{\Delta^m P^m(z) | m \geq 1\}$. In Section 5.1, by using a fundamental theorem of harmonic polynomials (See Theorem 5.2), we derive a criterion in Proposition 5.3 for Hessian nilpotency of homogeneous harmonic polynomials. In Section 5.2, we give constructions for some HNP's and HNS. In Section 6, by using some of the main results in the previous sections, we prove more properties of HNS or HNP's $P(z)$. We prove in Proposition 6.1 an identity and in Theorem 6.2 an equivalence of certain vanishing properties of $\{\Delta^k P^m(z) | k, m \geq 1\}$ for HNS $P(z) \in \mathbb{C}[[z]]$. In Theorem 6.3, we show some isotropic properties for homogeneous HNP's. In Section 7, we discuss some applications to the Jacobian conjecture. We formulate the *vanishing conjecture*, Conjecture 7.1, for (not necessary homogeneous) HNP's and the *homogeneous vanishing conjecture*, Conjecture 7.3, for homogeneous HNP's. We show in Proposition 7.4 that both conjectures above are equivalent to the Jacobian conjecture.

Finally, some remarks on this paper are as follows. First, for convenience, we will fix $\mathbb{C}$ as our base field. But, all results, formulas as well as their proofs (except the 1st proof of Theorem 3.4) obtained in

this paper hold or work equally well if one replace $\mathbb{C}$ by any $\mathbb{Q}$-algebra. Secondly, we will not restrict our study just on HNP's. Instead, we will formulate and prove results for HN formal power series whenever they hold in this general setting. Thirdly, for any HNP's or locally convergent HNS $P(z)$, all formal power series involved in this paper are locally convergent. This can be seen either from the fact that any local analytic map with non-zero Jacobian at the origin has a locally convergent inverse, or from the well-known Cauchy-Kowaleskaya theorem (See [R], for example.)

Acknowledgment: The author is very grateful to Professor Arno van den Essen who has carefully read through the first preprint of this paper and pointed out many misprints and several mistakes. Great thanks also go to Professor David Wright for personal communications, especially for informing the author some of his own recent results. The author also would like to thank Professor Mohan Kumar for personal communications.

2. **Deformed Inversion Pairs of Formal Power Series**

In this section, we first fix some notation and definitions that are needed in this paper. We then briefly recall certain results obtained in [Z2] and prove some preliminary results.

2.1. **Notation and Conventions.** Once and for all, we fix the following notation and conventions.

(1) We fix $n \geq 1$ and set $z = (z_1, z_2, \cdots, z_n)$. For any $\mathbb{Q}$-algebra k, we denote by $k[z]$ (resp. $k[[z]]$) the polynomial algebra (resp. formal power series algebra) over k in z_i $(1 \leq i \leq n)$.

(2) For any $\mathbb{Q}$-algebra k, by a formal map $F(z)$ from k^n to k^n, we simply mean $F(z) = (F_1(z), F_2(z), \cdots, F_n(z))$ with $F_i(z) \in k[[z]]$ $(1 \leq i \leq n)$. We denote by $J(F)$ and $j(F)$ the Jacobian matrix and the Jacobian of $F(z)$, respectively.

(3) We denote by Δ the Laplace operator $\sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2}$. Note that, a polynomial or formal power series $P(z)$ is said to be *harmonic* if $\Delta P = 0$.

(4) For any $k \geq 1$ and $U(z) = (U_1(z), U_2(z), \cdots, U_k(z)) \in \mathbb{C}[[z]]^{\times k}$, we set

$$o(U(z)) = \min_{1 \leq i \leq k} o(U_i(z))$$

and, when $U(z) \in \mathbb{C}[z]^{\times k}$,

$$\deg U(z) = \max_{1 \leq i \leq k} \deg U_i(z).$$

For any $U_t(z) \in \mathbb{C}[t][[z]]^{\times k}$ or $\mathbb{C}[[z,t]]^{\times k}$ $(k \geq 1)$ for some formal parameter t, the notation $o(U_t(z))$ and $\deg U_t(z)$ always stand for the order and the degree of $U_t(z)$ with respect to z, respectively.

(5) For any $P(z) \in \mathbb{C}[[z]]$, we denote by $\nabla P(z)$ the gradient of $P(z)$, i.e. $\nabla P = (\frac{\partial P}{\partial z_1}, \frac{\partial P}{\partial z_2}, \cdots, \frac{\partial P}{\partial z_n})$. We denote by $\mathrm{Hes}\,(P)(z)$ the Hessian matrix of $P(z)$, i.e. $\mathrm{Hes}\,(P)(z) = (\frac{\partial^2 P(z)}{\partial z_i \partial z_j})$.

(6) All n-vectors in this paper are supposed to be column vectors unless stated otherwise. For any vector or matrix U, we denote by U^t its transpose. The standard $\mathbb{C}$-bilinear form of n-vectors is denoted by $< \cdot, \cdot >$.

The following lemma will be very useful in our later arguments.

Lemma 2.1. *For any $P(z) \in \mathbb{C}[[z]]$ and $m \geq 1$, we have*

$$\Delta P^{m+1}(z) = (m+1)P^m \Delta P + m(m+1)P^{m-1} < \nabla P, \nabla P > \tag{2.1}$$

or

$$P^{m-1} < \nabla P, \nabla P >= \frac{1}{m(m+1)} \left(\Delta P^{m+1} - (m+1)P^m \Delta P\right). \tag{2.2}$$

Furthermore, when $P(z)$ is harmonic, we have

$$P^{m-1}\Delta P^2 = \frac{2}{m(m+1)} \Delta P^{m+1}. \tag{2.3}$$

Proof: Consider

$$\begin{aligned}
\Delta P^{m+1} &= \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_i} P^{m+1} \\
&= (m+1) \sum_{i=1}^{n} \frac{\partial}{\partial z_i} (P^m \frac{\partial P}{\partial z_i}) \\
&= (m+1) \sum_{i=1}^{n} P^m \frac{\partial^2 P}{\partial z_i^2} + m(m+1) \sum_{i=1}^{n} P^{m-1} \frac{\partial P}{\partial z_i} \frac{\partial P}{\partial z_i} \\
&= (m+1)P^m \Delta P + m(m+1) < \nabla P, \nabla P > P^{m-1}.
\end{aligned}$$

Hence, we get Eq. (2.1) and (2.2).

Now suppose that $P(z)$ is harmonic, i.e. $\Delta P = 0$. By Eq. (2.1) with $m = 1$, we have $\Delta P^2 = 2 < \nabla P, \nabla P >$ or $< \nabla P, \nabla P >= \frac{1}{2}\Delta P^2$. It is easy to see that, in this case, Eq. (2.3) follows directly from Eq. (2.2). □

2.2. **Deformed Inversion Pairs of Formal Power Series.** For any $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$, we set $F(z) = z - \nabla P(z)$. It is shown in §1.1 in [M] (Also see Lemma 3.1 in [Z2].) that there is a unique $Q(z) \in \mathbb{C}[[z]]$ with $o(Q(z)) \geq 2$ such that the formal inverse of $F(z)$ is given by $G(z) = z + \nabla Q(z)$. We call $Q(z)$ the *inversion pair* of $P(z)$. Furthermore, following the arguments in [Z1] and [Z2], we also consider the deformation $F_t(z) = z - t\nabla P(z)$ of $F(z)$, where t is a formal parameter which commutes with variables z_i $(1 \leq i \leq n)$. By Lemma 3.1 in [Z2], we know that there exists a unique $Q_t(z) \in \mathbb{C}[[z,t]]$ with $o(Q_t(z)) \geq 2$ such that the formal inverse $G_t(z)$ of $F_t(z)$ is given by $G_t(z) = z + t\nabla Q_t(z)$. Note that, when $o(P(z)) \geq 3$, we actually have $Q_t(z) \in \mathbb{C}[t][[z]]$ and $Q_{t=1}(z) = Q(z)$. In general, $tQ_t(z)$ is nothing but the inversion pair of $tP(z)$ over the $\mathbb{Q}$-algebra $\mathbb{C}[[t]]$. Another way to look at the inversion pair is as follows. Set $U(z) = \frac{1}{2}\sum_{i=1}^n z_i^2 - P(z)$ and $V(z) = \frac{1}{2}\sum_{i=1}^n z_i^2 + Q(z)$, then $V(z)$ is exactly the Legendre transform (See [Ar], [M] and [Z2].) of the formal power series $U(z)$.

Definition 2.2. *For any $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$, $Q_t(z) \in \mathbb{C}[[z,t]]$ defined above is called the deformed inversion pair of $P(z)$.*

Another important definition is the following.

Definition 2.3. *For any $P(z) \in \mathbb{C}[[z]]$, we say $P(z)$ is HN (Hessian nilpotent) if its Hessian matrix $Hes(P) = \left(\frac{\partial^2 P}{\partial z_i \partial z_j}\right)$ is nilpotent.*

Remark 2.4. *Note that, $Tr\,Hes(P) = \Delta P$ for any $P(z) \in \mathbb{C}[[z]]$. Hence any HN formal power series is harmonic. But the converse is not true. For some examples of HNP's and HNS, see Subsection 5.2.*

Throughout the rest of this paper, for any formal power series $P(z) \in \mathbb{C}[[z]]$, we will fix the notation $F(z)$, $F_t(z)$, $G(z)$, $G_t(z)$, $Q(z)$ and $Q_t(z)$ defined above unless stated otherwise. We will use the short words HNS and HNP for "HN power series" and "HN polynomial", respectively. Furthermore, we also define a sequence of formal power series $\{Q_{[m]}(z) | m \geq 1\}$ by writing

$$Q_t(z) = \sum_{m=1}^{\infty} t^{m-1} Q_{[m]}(z). \tag{2.4}$$

Lemma 2.5. *For any formal power series $P(z)$, we have*

(a)

$$(\Delta Q_t)(F_t) = \sum_{k=1}^{\infty} t^{k-1}\, Tr\, Hes^k(P). \tag{2.5}$$

(b) *$P(z)$ is HN if and only if $Q_t(z)$ is harmonic as a formal power series in z, and if and only if $Q_t(z)$ is HN as a formal power series in z.*

Proof: (a) Set $N_t(z) = \nabla Q_t(z)$. It is easy to check that

$$JN_t(z) = \operatorname{Hes} Q_t(z), \tag{2.6}$$

$$\operatorname{Tr} \operatorname{Hes}(Q_t) = \Delta Q_t. \tag{2.7}$$

Then Eq. (2.5) follows directly from the equations above and Eq. (2.4) in [Z2].

(b) follows directly by applying Lemma 2.2 in [Z2] to the formal map $F(z) = z - H(z)$ with $H(z) = \nabla P(z)$. $\square$

The following theorem which was first proved in the unpublished preprint [Z1] and later [Z2] (See Theorem 3.6 and Proposition 3.7 in [Z2].) will play a fundamental role in this paper.

Theorem 2.6. [Z2] *For any $Q_t(z) \in \mathbb{C}[[z,t]]$ with $o(Q_t(z)) \geq 2$ and $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$, the following statements are equivalent.*

(1) *$Q_t(z)$ is the deformed inversion pair of $P(z)$.*
(2) *$Q_t(z)$ is the unique power series solution of the following Cauchy problem of PDE's.*

$$\begin{cases} \frac{\partial Q_t(z)}{\partial t} = \frac{1}{2} < \nabla Q_t, \nabla Q_t >, \\ Q_{t=0}(z) = P(z). \end{cases} \tag{2.8}$$

Furthermore, we have the following recurrent formula.

$$Q_{[1]}(z) = P(z), \tag{2.9}$$

$$Q_{[m]}(z) = \frac{1}{2(m-1)} \sum_{\substack{k,l \geq 1 \\ k+l=m}} < \nabla Q_{[k]}(z), \nabla Q_{[l]}(z) > \tag{2.10}$$

for any $m \geq 2$.

Corollary 2.7. *For any $k \geq 0$ and $m \geq 1$, we have*

$$\frac{\partial}{\partial t} \Delta^k Q_t^m(z) = \frac{1}{2(m+1)} \Delta^{k+1} Q_t^{m+1} - \frac{1}{2} \Delta^k (Q_t^m \Delta Q_t). \tag{2.11}$$

Proof: Since $\frac{\partial}{\partial t}$ and Δ^k $(k \geq 0)$ commute, by applying Δ^k $(k \geq 1)$ to Eq. (2.11) with $k = 0$, we get Eq. (2.11) for any $k \geq 1$. Therefore we may assume $k = 0$.

Consider

$$\frac{\partial}{\partial t} Q_t^m(z) = m Q_t^{m-1} \frac{\partial Q_t}{\partial t}$$

Applying Eq. (2.8) in Theorem 2.6:

$$= \frac{m}{2} Q_t^{m-1} < \nabla Q_t, \nabla Q_t >$$

Applying Eq. (2.2) in Lemma 2.1 to $Q_t(z)$:

$$\begin{aligned} &= \frac{m}{2} \frac{1}{m(m+1)} (\Delta Q_t^{m+1} - (m+1) Q_t^m \Delta Q_t) \\ &= \frac{1}{2(m+1)} \Delta Q_t^{m+1} - \frac{1}{2} Q_t^m \Delta Q_t. \end{aligned}$$

□

3. **Deformed Inversion Pairs of HN Power Series**

In this section, we study deformed inversion pairs $Q_t(z)$ of HNS (Hessian Nilpotent Formal Power Series) $P(z) \in \mathbb{C}[[z]]$. We first derive the PDE's satisfied by $Q_t(z)$ and $\exp(sQ_t(z))$ ($s \in \mathbb{C}^\times$). We then discuss some relationships among deformed inversion pairs, the Heat equation and the Jacobian conjecture. Note that similar relationships among deformed inversion pairs of formal power series (not necessarily HN), the inviscid Burgers' equations and the Jacobian conjecture have been discussed in [Z2]. Finally, we derive with two different proofs a uniform non-recurrent formula (See Eq. (3.8).) for $Q_t^k(z)$ ($k \geq 1$).

Theorem 3.1. *For any $Q_t(z) \in \mathbb{C}[[z,t]]$ with $o(Q_t(z)) \geq 2$ and HNS $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$, the following statements are equivalent.*

(1) *$Q_t(z)$ is the deformed inversion pair of $P(z)$.*
(2) *$Q_t(z)$ is the unique power series solution of the following Cauchy problem of PDE.*

$$\begin{cases} \frac{\partial Q_t(z)}{\partial t} & = \frac{1}{4} \Delta Q_t^2, \\ Q_{t=0}(z) & = P(z). \end{cases} \tag{3.1}$$

Furthermore, we have the following recurrent formula.

$$Q_{[1]}(z) = P(z), \tag{3.2}$$

$$Q_{[m]}(z) = \frac{1}{4(m-1)} \Delta \sum_{\substack{k,l \geq 1 \\ k+l=m}} Q_{[k]}(z) Q_{[l]}(z) \tag{3.3}$$

for any $m \geq 2$.

Proof: First, by Lemma 2.5, (b), we have $\Delta Q_t(z) = 0$. Therefore, for any $m \geq 1$, $Q_{[m]}(z)$ is harmonic. Secondly, for any harmonic formal power series $U(z)$,V(z)$\in \mathbb{C}[[z]]$, it is easy to check that we have

$$\Delta(UV) = 2 < \nabla U, \nabla V > . \tag{3.4}$$

By using the facts above, it is easy to see that the implication (1) $\Rightarrow$ (2) and also the recurrent formulas Eq. (3.2), (3.3) follow directly from Theorem 2.6.

To see (2) $\Rightarrow$ (1), we denote by $\widetilde{Q}_t(z)$ the deformed inversion pair of $P(z)$. By the fact proved above, we know that $\widetilde{Q}_t(z)$ also satisfies Eq. (3.1). Since the power series solution of the Cauchy problem Eq. (3.1) is unique, which is given recursively by Eq. (3.2) and (3.3), we have $\widetilde{Q}_t(z) = Q_t(z)$. Therefore (2) $\Rightarrow$ (1) also holds. $\square$

A relation of deformed inversion pairs $Q_t(z)$ of HNS $P(z)$ with the Heat equation is given by the following theorem.

Theorem 3.2. *Let $P(z) \in \mathbb{C}[[z]]$ be HN with $o(P(z)) \geq 2$ and $Q_t(z)$ its deformed inversion pair. For any non-zero $s \in \mathbb{C}$, set*

$$U_{t,s}(z) = \exp(sQ_t(z)) = \sum_{k=0}^{\infty} \frac{s^k Q_t^k(z)}{k!}.$$

Then, $U_{t,s}(z)$ is the unique formal power series solution of the following Cauchy problem of the Heat equation.

$$\begin{cases} \frac{\partial U_{t,s}}{\partial t}(z) & = \frac{1}{2s}\Delta U_{t,s}(z), \\ U_{t=0,s}(z) & = \exp(sP(z)). \end{cases} \tag{3.5}$$

Proof: The uniqueness can be proved by viewing $U_{t,s}(z)$ as a power series in t with coefficients in $\mathbb{C}[[z]]$ and showing that the coefficients of t^k $(k \geq 1)$ are recurrently determined by the coefficient of t^0 which is $U_{t=0,s}(z) = \exp(sP(z))$. We skip the details here. For a similar argument, see the proof of Proposition 2.5 in [Z2]. Note that, when $P(z)$ is locally convergent, $Q_t(z)$ and $\exp(sQ_t(z))$ $(s \in \mathbb{C}^\times)$ are locally convergent. Then the uniqueness in this case also follows from the Cauchy-Kowaleskaya theorem (See [R]) in PDE.

Now we show that $U_{t,s}(z)$ satisfies Eq. (3.5). First note that, the initial condition in Eq. (3.5) follows immediately from the one in Eq. (2.8). Secondly, by Lemma 2.5, (b), we have $\Delta Q_t(z) = 0$.

Consider

$$\frac{\partial U_{t,s}}{\partial t} = \frac{\partial \exp(sQ_t)}{\partial t} = s\exp(sQ_t)\frac{\partial Q_t}{\partial t} \tag{3.6}$$

Applying Eq. (2.8):

$$= \frac{s}{2}U_{t,s} < \nabla Q_t, \nabla Q_t > .$$

On the other hand, we have

$$\Delta U_{t,s} = \sum_{i=1}^{n} \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_i}e^{sQ_t} = s\sum_{i=1}^{n}\frac{\partial}{\partial z_i}(\frac{\partial Q_t}{\partial z_i}e^{sQ_t}) \tag{3.7}$$

Using the fact that $\Delta Q_t = 0$:

$$= s^2\sum_{i=1}^{n}\frac{\partial Q_t}{\partial z_i}\frac{\partial Q_t}{\partial z_i}e^{sQ_t} = s^2 U_{s,t} < \nabla Q_t, \nabla Q_t > .$$

By combining Eq. (3.6) and (3.7), we see that $U_{t,s}(z)$ does satisfy the PDE in Eq. (3.5). □

By combining the gradient reduction in [BE1], [M] and the homogeneous reduction in [BCW], [Y] on the Jacobian conjecture, we see that the Jacobian conjecture can be reduced to polynomial maps $F(z) = z - \nabla P(z)$ with $P(z) \in \mathbb{C}[z]$ homogeneous of degree $d = 4$. By Theorem 3.1 and Theorem 3.2, it is easy to see that the Jacobian conjecture is equivalent to the following conjecture.

Conjecture 3.3. *For any homogeneous HNP $P(z)$ of degree $d \geq 2$, the unique solutions of the Cauchy problems Eq. (3.5) and Eq. (2.8) must be a polynomial in (z, t) and the exponential of a polynomial (z, t), respectively.*

Since it has been proved by Wang [Wa] that the Jacobian conjecture holds for polynomial maps $F(z)$ of $\deg F(z) \leq 2$, hence Conjecture 3.3 is true for $d \leq 3$. For more discussion on relationships of HNP's and the Jacobian conjecture, see Section 7.

Next we give two different proofs for the following uniform formula for the powers $Q_t^k(z)$ $(k \geq 1)$ of the deformed inversion pairs $Q_t(z)$ of HNS $P(z)$.

Theorem 3.4. *Suppose $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$ is HN. Then, for any $k \geq 1$, we have*

$$Q_t^k(z) = k! \sum_{m=0}^{\infty} \frac{t^m}{2^m m!(m+k)!} \Delta^m P^{m+k}(z). \tag{3.8}$$

In particular, for any $m \geq 1$,

$$Q_{[m]}(z) = \frac{1}{2^{m-1} m!(m-1)!} \Delta^{m-1} P^m(z). \tag{3.9}$$

First Proof: First, note that Eq. (3.9) follows directly from Eq. (3.8) with $k = 1$ and the definition Eq. (2.4) of $Q_{[m]}(z)$ $(m \geq 1)$. To prove Eq. (3.8), we consider the formal power series

$$\exp(\frac{t}{2s}\Delta) \exp(sP) = \sum_{k=0}^{\infty} \frac{t^k}{(2s)^k k!} \Delta^k \exp(sP). \tag{3.10}$$

It is easy to check that the series above is also a formal power series solution of the Cauchy problem Eq. (3.5). Hence, by Theorem 3.2 and the uniqueness of the power series solution of Eq. (3.5), we have

$$\exp(sQ_t) = \exp(\frac{t}{2s}\Delta) \exp(sP). \tag{3.11}$$

By comparing the coefficients of s^k $(k \geq 1)$ of the both sides of the equation above, we get Eq. (3.8). □

The proof above for Theorem 3.4 is shorter but less intriguing than the second proof below, which begins with the following lemma.

Lemma 3.5. *Let $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$ be HN and $Q_t(z)$ the deformed inversion pair of $P(z)$. Then, for any $k, l \geq 1$, we have*

$$\frac{\partial^l Q_t^k}{\partial t^l} = \frac{\Delta^l Q_t^{k+l}}{2^l (k+1)(k+2) \cdots (k+l)}. \tag{3.12}$$

Proof: We fix $k \geq 1$ and use the mathematical induction on $l \geq 1$. First, by Lemma 2.5, (b), we have $\Delta Q_t(z) = 0$. Then Eq. (3.12) for $l = 1$ follows directly from Eq. (2.11).

Now we assume that Eq. (3.12) holds for $l = l_0 \geq 1$ and consider the case $l = l_0 + 1$.

$$\begin{aligned}\frac{\partial^l Q_t^k}{\partial t^l} &= \frac{\partial}{\partial t}\left(\frac{\partial^{l_0} Q_t^k}{\partial t^{l_0}}\right)\\ &= \frac{1}{2^{l_0}(k+1)(k+2)\cdots(k+l_0)}\frac{\partial}{\partial t}\Delta^{l_0} Q_t^{k+l_0}\\ &= \frac{k+l_0}{2^{l_0}(k+1)(k+2)\cdots(k+l_0)}\Delta^{l_0}(Q_t^{k+l_0-1}\frac{\partial Q_t}{\partial t})\end{aligned}$$

Applying the PDE in Eq. (3.1):

$$= \frac{1}{2^{l_0}(k+1)(k+2)\cdots(k+l_0-1)}\frac{1}{4}\Delta^{l_0}(Q_t^{k+l_0-1}\Delta Q_t^2)$$

Applying Eq. (2.3) to $Q_t^{k+l_0-1}\Delta Q_t^2$:

$$\begin{aligned}&= \frac{1}{2^{l_0+2}(k+1)\cdots(k+l_0-1)}\frac{2}{(k+l_0)(k+l_0+1)}\Delta^{l_0+1}Q_t^{k+l_0+1}\\ &= \frac{1}{2^{l_0+1}(k+1)(k+2)\cdots(k+l_0+1)}\Delta^{l_0+1}Q_t^{k+l_0+1}\\ &= \frac{1}{2^{l}(k+1)(k+2)\cdots(k+l)}\Delta^{l}Q_t^{k+l}.\end{aligned}$$

Hence, Eq. (3.12) holds for $l = l_0 + 1$. □

2nd Proof of Theorem 3.4: First, by the initial condition in Eq. (2.8), we have

$$\Delta^l Q_t^{k+l}(z)\big|_{t=0} = \Delta^l P^{k+l}(z) \tag{3.13}$$

for any $k \geq 1$ and $l \geq 0$.

Secondly, by setting $t = 0$ in Eq. (3.12) and applying the equation above, we see that the coefficient of t^l of $Q_t^k(z)$ is equal to

$$\frac{1}{l!}\frac{\Delta^l P^{k+l}}{2^l(k+1)(k+2)\cdots(k+l)} = \frac{k!}{2^l l!(k+l)!}\Delta^l P^{k+l}.$$

Hence Eq. (3.8) holds. □

By comparing the coefficients of s^k ($k \leq 0$) of the both sides of Eq. (3.10), we see that $\Delta^k P^m = 0$ for any $k \geq m$, which is equivalent to saying that $\Delta^m P^m = 0$ for any $m \geq 1$. Note that the later statement also follows from Eq. (3.8) with $k = 1$ and the fact that $Q_t(z)$ is harmonic.

Corollary 3.6. *For any HNS $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$, we have $\Delta^m P^m = 0$ for any $m \geq 1$.*

Later, we will show in Theorem 4.3 that the converse of the corollary above is also true.

Note that, by setting $s = 1$ in Eq. (3.11), we have the following formula.

$$\exp(Q_t) = \exp(\frac{t}{2}\Delta)\exp(P). \tag{3.14}$$

Actually, a more delicate formula (See Eq. (3.20) below.) can be derived as follows.

Set

$$\widetilde{Q}_t(z) = Q_t(z) - P(z), \tag{3.15}$$

$$\widetilde{U}_t(z) = \exp(\widetilde{Q}_t(z)), \tag{3.16}$$

$$\Lambda_P(z) = \sum_{i=1}^{n} \frac{\partial P(z)}{\partial z_i}\frac{\partial}{\partial z_i}. \tag{3.17}$$

Lemma 3.7. *For any HNS $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$, let $\widetilde{Q}_t(z)$, $\widetilde{U}_t(z)$ and Λ_P be as above. Then we have*

(a) $\widetilde{Q}_t(z)$ is the unique power series solution of the following Cauchy problem of PDE's.

$$\begin{cases} \frac{\partial \widetilde{Q}_t(z)}{\partial t} & = \frac{1}{4}\Delta\widetilde{Q}_t^2(z) + \Lambda_P\widetilde{Q}_t(z) + \frac{1}{4}\Delta P^2, \\ \widetilde{Q}_{t=0}(z) & = 0. \end{cases} \tag{3.18}$$

(b) $\widetilde{U}_t(z)$ is the unique power series solution of the following Cauchy problem of PDE's.

$$\begin{cases} \frac{\partial \widetilde{U}_t(z)}{\partial t} & = (\frac{1}{2}\Delta + \Lambda_P + \frac{1}{4}\Delta P^2)\ \widetilde{U}_t(z), \\ \widetilde{U}_{t=0}(z) & = 1. \end{cases} \tag{3.19}$$

The proof of this lemma is straightforward and similar as the proof of Theorem 3.2, so we omit it here. From Eq. (3.19), it is also easy to derive the following formula.

Proposition 3.8. *For any HNS $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$, we have*

$$\exp(\widetilde{Q}_t) = \exp(t(\frac{1}{2}\Delta + \Lambda_P + \frac{1}{4}\Delta P^2)) \cdot 1. \tag{3.20}$$

Proof: We first set

$$V_t(z) = \exp(t(\frac{1}{2}\Delta + \Lambda_P + \frac{1}{4}\Delta P^2)) \cdot 1. \tag{3.21}$$

Note that $V_{t=0}(z) = 1$. Now we consider

$$\begin{aligned}\frac{\partial V_t}{\partial t} &= (\frac{1}{2}\Delta + \Lambda_P + \frac{1}{4}\Delta P^2)\exp(t(\frac{1}{2}\Delta + \Lambda_P + \frac{1}{4}\Delta P^2)) \cdot 1 \\ &= (\frac{1}{2}\Delta + \Lambda_P + \frac{1}{4}\Delta P^2)\ V_t(z).\end{aligned}$$

Therefore $V_t(z)$ satisfies the PDE in Eq. (3.19). On the other hand, by Lemma 3.7, (b) above, $\exp(\widetilde{Q}_t)$ also satisfies Eq. (3.19). Hence, Eq. (3.20) follows from the uniqueness of power series solutions of the Cauchy problem Eq. (3.19). □

One interesting aspect of the formula above is as follows. It shows the differential operator Λ_P and the operator of the multiplication by ΔP^2 also play important roles for deformed inversion pairs $Q_t(z)$. For example, from Eq. (3.20), it is easy to see that we have the following corollary.

Corollary 3.9. *For any $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$ such that $\Delta^2 P(z) = 0$, we have $\widetilde{Q}_t(z) = 0$, or in other words, $Q_t(z) = P(z)$.*

4. A Criterion for Hessian nilpotency

Let $a = \{a_I | I \in \mathbb{N}^n,\ |I| \geq 2\}$ be a set of variables which commute with each other. Let $P(z) = \sum_{I\in\mathbb{N}^n} a_I z^I$ be the universal formal power series in z with $o(P(z)) \geq 2$. We will also view $P(z)$ as a formal power series in z with coefficients in $\mathbb{C}[a]$, i.e. $P(z) \in \mathbb{C}[a][[z]]$.

For any $m \geq 1$, we set

$$u_m(P) = \operatorname{Tr}\operatorname{Hes}^m(P), \tag{4.1}$$

$$v_m(P) = \Delta^m P^m. \tag{4.2}$$

In this section, we prove a general theorem, Theorem 4.1, about a relation between $\{u_m(P)|m \geq 1\}$ and $\{v_m(P)|m \geq 1\}$. Consequently, we get a criterion for Hessian nilpotency of formal power series $P(z) \in \mathbb{C}[[z]]$ in terms of certain vanishing properties of $\{\Delta^m P^m(z)|m \geq 1\}$.

Let $P(z) \in \mathbb{C}[a][[z]]$, $\{u_m(P)|m \geq 1\}$ and $\{v_m(P)|m \geq 1\}$ be defined as above.

For any $k \geq 1$, we define $\mathcal{U}_k(P)$ (resp. $\mathcal{V}_k(P)$) to be the ideal in $\mathbb{C}[a][[z]]$ generated by $\{u_m(P)|1 \leq m \leq k\}$ (resp. $\{v_m(P)|1 \leq m \leq k\}$) and all their partial derivatives of any order. For convenience, we also set $\mathcal{U}_0(P) = \mathcal{V}_0(P) = 0$.

The first main result of this subsection is the following theorem.

Theorem 4.1. *For any $k \geq 1$, $\mathcal{U}_k(P) = \mathcal{V}_k(P)$ as ideals in $\mathbb{C}[a][[z]]$.*

One immediate consequence of Theorem 4.1 is the following corollary.

For any $k \geq 1$, we define $\widetilde{\mathcal{U}}_k(P)$ (resp. $\widetilde{\mathcal{V}}_k(P)$) to be the ideal in $\mathbb{C}[a]$ generated by all coefficients of $\{u_m(P)|1 \leq m \leq k\}$ (resp. $\{v_m(P)|1 \leq m \leq k\}$).

Corollary 4.2. [1] *For any $k \geq 1$, $\widetilde{\mathcal{U}}_k(P) = \widetilde{\mathcal{V}}_k(P)$ as ideals in $\mathbb{C}[a]$.*

From Theorem 4.1 or Corollary 4.2, it is easy to see that we have the following criteria for Hessian nilpotency of formal power series $P(z) \in \mathbb{C}[[z]]$.

Theorem 4.3. *For any $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$, the following statements are equivalent.*

(1) *$P(z)$ is HN.*
(2) *$\Delta^m P^m = 0$ for any $m \geq 1$.*
(3) *$\Delta^m P^m = 0$ for any $1 \leq m \leq n$.*

Proof: By Theorem 4.1 or Corollary 4.2, we have, for any fixed $k \geq 1$, $\Delta^m P^m = 0$ $(1 \leq m \leq k)$ if and only if $\operatorname{Tr}\operatorname{Hes}^m(P) = 0$ $(1 \leq m \leq k)$. Then, the theorem follows directly from the following facts in linear algebra, namely, for any $n \times n$ matrix $A \in M_n(\mathbb{C})$, A is nilpotent if and only if $\operatorname{Tr} A^m = 0$ $(m \geq 1)$, and if and only if $\operatorname{Tr} A^m = 0$ $(1 \leq m \leq n)$. $\square$

Besides the criteria in Theorem 4.3, we believe the following one is also true.

Conjecture 4.4. *Let $P(z) \in \mathbb{C}[[z]]$ with $o(P(z)) \geq 2$. If $\Delta^m P^m(z) = 0$ for $m >> 0$, then $P(z)$ is HN.*

In the rest of this subsection, we fix the universal formal power series $P(z) = \sum_{I \in \mathbb{N}^n} a_I z^I$ with $o(P(z)) \geq 2$ and give a proof for Theorem 4.1. Note that all results proved in the previous sections also hold for formal power series over the $\mathbb{C}$-algebra $\mathbb{C}[a]$. In particular, they hold for our universal formal power series $P(z) \in \mathbb{C}[a][[z]]$.

We begin with the following two lemmas.

Lemma 4.5. *Let $Q_t(z) \in \mathbb{C}[a][[z,t]]$ be the deformed inversion pair of $P(z)$. Then, there exists a sequence $\{w_k(P)(z) \in \mathcal{U}_k(P)|k \geq 1\}$ such that*

$$\Delta Q_t(z) = \sum_{k=1}^{\infty} w_k(P)(z)\ t^{k-1}, \tag{4.3}$$

[1] Professor David Wright [Wr3] recently informed the author that he has obtained a different proof for this result.

and, for any $k \geq 1$,

$$w_k(P)(z) \equiv u_k(P)(z) \mod (\mathcal{U}_{k-1}(P)). \tag{4.4}$$

Proof: We set $N_t(z) = \nabla Q_t(z)$. First, by composing $G_t(z) = z + tN_t(z)$ from right to Eq. (2.5) in Lemma 2.5, we have

$$\Delta Q_t(z) = \sum_{i=1}^{\infty} u_i(P)(z + tN_t(z))\ t^{i-1}. \tag{4.5}$$

Now we write the Taylor expansion of $u_i(P)(z + tN_t(z))$ $(i \geq 1)$ at z as

$$u_i(P)(z + tN_t(z)) = u_i(P)(z) + \sum_{k=1}^{\infty} \sum_{\substack{\mathbf{s}\in\mathbb{N}^n \\ |\mathbf{s}|=k}} \frac{1}{\mathbf{s}!} \frac{\partial^{\mathbf{s}} u_i(P)}{\partial z^{\mathbf{s}}}(z) N_t^{\mathbf{s}}(z) t^k. \tag{4.6}$$

Next we want to write the RHS of Eq. (4.6) as a formal power series in t with coefficients in $\mathbb{C}[[a, z]]$. This can be done by first doing so for $N_t^{\mathbf{s}}(z)$ $(\mathbf{s} \in \mathbb{N}^n)$ in Eq. (4.6) and then re-arranging properly all the terms involved. Note that, $u_i(P)(z)$ or $\frac{\partial^{\mathbf{s}} u_i(P)}{\partial z^{\mathbf{s}}}(z)$ $(\mathbf{s} \in \mathbb{N}^n)$ do not depend on t and are in the ideal $\mathcal{U}_i(P)$. Also note that all the terms or products in the sum of Eq. (4.6) except the first one $u_i(P)(z)$ have positive degree in t due to the factors t^k $(k \geq 1)$. By using the observations above and keeping track the degree in t, it is easy to see that $u_i(P)(z + tN_t(z))$ can be written as

$$u_i(P)(z + tN_t(z)) = u_i(P)(z) + \sum_{j=1}^{\infty} A_{ij}(z)\, t^j \tag{4.7}$$

for some $A_{ij}(z) \in \mathcal{U}_i(P)(z)$ $(i, j \geq 1)$.

Now, by combining Eq. (4.5) and (4.7), we have

$$\begin{aligned}
\Delta Q_t(z) &= \sum_{i=1}^{\infty} \left(u_i(P)(z) + \sum_{j=1}^{\infty} A_{ij}(z) t^j \right) t^{i-1} \\
&= \sum_{k=1}^{\infty} (u_k(P) + \sum_{\substack{i+j=k \\ i,j\geq 1}} A_{ij}(z))\, t^{k-1} \\
&= \sum_{i=1}^{\infty} w_k(P)(z)\ t^{k-1},
\end{aligned}$$

where, for any $k \geq 1$,

$$w_k(P)(z) = u_k(P) + \sum_{\substack{i+j=k \\ i,j\geq 1}} A_{ij}(z). \tag{4.8}$$

Hence we get Eq. (4.3). By the fact that $A_{i,j}(z) \in \mathcal{U}_i(P)(z)$ for any $i, j \geq 1$, we see that each $A_{i,j}(z)$ in Eq. (4.8) lies in $\mathcal{U}_{k-1}(P)(z)$ since $i \leq k-1$. Therefore Eq. (4.4) also holds. □

Lemma 4.6. *For any $m, k \geq 1$ and $1 \leq l \leq k$, we have*

$$\frac{\partial^l}{\partial t^l}\Delta^m Q_t^m(z) \equiv \frac{\Delta^{m+l} Q_t^{m+l}}{2^l(m+1)\cdots(m+l)} \mod (\mathcal{U}_k(P), t^{k-l+1}). \tag{4.9}$$

Proof: We fix $k \geq 1$ and use the mathematical induction on l to show Eq. (4.9) holds for any $m \geq 1$.

By Eq. (2.11) with $k = m$ and Eq. (4.3), it is easy to see that Eq. (4.9) holds for any $m \geq 1$ when $l = 1$.

Now we assume that Eq. (4.9) holds for any $1 \leq l \leq k_0 < k$ and consider the case $l = k_0 + 1$. By applying $\frac{\partial}{\partial t}$ to Eq. (4.9) with $l = k_0$, we have

$$\frac{\partial^{k_0+1}\Delta^m Q_t^m}{\partial t^{k_0+1}} \equiv \frac{\frac{\partial}{\partial t}\Delta^{m+k_0} Q_t^{m+k_0}}{2^{k_0}(m+1)\cdots(m+k_0)} \mod (\mathcal{U}_k(P), t^{k-k_0}). \tag{4.10}$$

While, from Eq. (4.9) with $l = 1$, we have

$$\frac{\partial}{\partial t}\Delta^{m+k_0} Q_t^{m+k_0} \equiv \frac{\Delta^{m+k_0+1} Q_t^{m+k_0+1}}{2(m+k_0+1)} \mod (\mathcal{U}_k(P), t^k).$$

Since $k - k_0 \leq k$, hence we also have

$$\frac{\partial}{\partial t}\Delta^{m+k_0} Q_t^{m+k_0} \equiv \frac{\Delta^{m+k_0+1} Q_t^{m+k_0+1}}{2(m+k_0+1)} \mod (\mathcal{U}_k(P), t^{k-k_0}). \tag{4.11}$$

By combining Eq. (4.10) and (4.11), we have

$$\frac{\partial^{k_0+1}}{\partial t^{k_0+1}}\Delta^m Q_t^m \equiv \frac{\Delta^{m+k_0+1} Q_t^{m+k_0+1}}{2^{k_0+1}(m+1)(m+2)\cdots(m+k_0+1)} \mod (\mathcal{U}_k(P), t^{k-k_0} = t^{k-(k_0+1)+1})$$

which is Eq. (4.9) for $l = k_0 + 1$. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1: We use the mathematical induction on $k \geq 1$. Since $u_1(P) = \Delta P = v_1(P)$, hence, the theorem is true for $k = 1$.

Now, we assume that $\mathcal{U}_k(P) = \mathcal{V}_k(P)$ for some $k \geq 1$. By Eq. (4.9) with $m = 1$ and $l = k$, we have

$$\frac{\partial^k}{\partial t^k}\Delta Q_t \equiv \frac{\Delta^{k+1} Q_t^{k+1}}{2^k(k+1)!} \mod (\mathcal{U}_k(P), t).$$

In other words, we have

$$\frac{\partial^k}{\partial t^k}\Delta Q_t\bigg|_{t=0} \equiv \frac{\Delta^{k+1} Q_t^{k+1}}{2^k(k+1)!}\bigg|_{t=0} \mod (\mathcal{U}_k(P)). \tag{4.12}$$

On the other hand, by the initial condition in Eq. (2.8), we have

$$\frac{\Delta^{k+1} Q_t^{k+1}}{2^k(k+1)!}\bigg|_{t=0} = \frac{\Delta^{k+1} P^{k+1}}{2^k(k+1)!} = \frac{v_{k+1}(P)}{2^k(k+1)!}.$$

By Eq. (4.3) and (4.4), we have

$$\frac{\partial^k}{\partial t^k}\Delta Q_t\bigg|_{t=0} = k!\, w_{k+1}(P) \equiv k! u_{k+1}(P) \mod (\mathcal{U}_k(P)).$$

Therefore, by Eq. (4.12) and the two equations above, we have

$$u_{k+1}(P) \equiv \frac{1}{2^k k!(k+1)!} v_{k+1}(P) \mod (\mathcal{U}_k(P)).$$

Since $\mathcal{U}_k(P) = \mathcal{V}_k(P)$, hence we have $\mathcal{U}_{k+1}(P) = \mathcal{V}_{k+1}(P)$. □

5. **Hessian Nilpotent Polynomials**

In this section, we first derive in Subsection 5.1 a criterion for Hessian nilpotency of homogeneous polynomials by using a fundamental theorem (See Theorem 5.2) of harmonic polynomials. We then give in Subsection 5.2 some examples of HNP's (Hessian nilpotent polynomials) and HNS (Hessian nilpotent formal power series).

5.1. **A Criterion for Hessian Nilpotency of Homogeneous Harmonic Polynomials.**

For any $n \geq 1$, we let $X(\mathbb{C}^n)$ or simply X denote the affine variety defined by $\sum_{i=1}^n z_i^2 = 0$. For any $d \geq 0$, we denoted by $V_d(z)$ the vector space of homogeneous polynomials in z of degree $d \geq 0$. For any $\alpha \in \mathbb{C}^n$, we denote by $h_\alpha(z)$ the linear function $< \alpha, z >$ of $\mathbb{C}^n$.

The following identities are almost trivial but very useful for our later arguments. So we formulate them as a lemma without giving proofs.

Lemma 5.1. (*a*) *For any* $\alpha \in \mathbb{C}^n$ *and* $m \geq 1$, *we have*

$$Hes\,(h_\alpha^m)(z) = m(m-1)h_\alpha^{m-2}(z)\,\alpha \cdot \alpha^t \tag{5.1}$$

(*b*) *For any* $\alpha, \beta \in X(\mathbb{C}^n)$ *and* $m, k \geq 1$, *we have*

$$\Delta(h_\alpha^m(z)h_\beta^k(z)) = 2mk < \alpha, \beta > h_\alpha^{m-1}(z)h_\beta^{k-1}(z) \tag{5.2}$$

By Remark 2.4, we know that any HNS $P(z)$ is automatically harmonic, i.e. $\Delta P(z) = 0$. For harmonic polynomials, we have the following fundamental theorem.

Theorem 5.2. *For any homogeneous harmonic polynomial* $P(z)$ *of degree* $d \geq 2$, *we have*

$$P(z) = \sum_{i=1}^{k} c_i h_{\alpha_i}^d(z) \tag{5.3}$$

for some $c_i \in \mathbb{C}$ *and* $\alpha_i \in X(\mathbb{C}^n)$ $(1 \leq i \leq k)$.

For the proof of this theorem, see, for example, [H] and [T].

Note that, by replacing α_i by $c_i^{-\frac{1}{d}}\alpha_i$ $(1 \leq i \leq k)$ in Eq. (5.3), we see that any homogeneous harmonic polynomial $P(z)$ of degree $d \geq 2$ can be written as

$$P(z) = \sum_{i=1}^{k} h_{\alpha_i}^d(z) \tag{5.4}$$

for some $\alpha_i \in X(\mathbb{C}^n)$ $(1 \leq i \leq k)$.

In the rest of this subsection, we fix a homogeneous harmonic polynomial $P(z) \in V_d(z)$ of degree $d \geq 2$ and assume that $P(z)$ is given by Eq. (5.4) for some $\alpha_i \in X(\mathbb{C}^n)$ $(1 \leq i \leq k)$. We also assume $\{h_\alpha^d(z) | \alpha \in \mathbb{C}^n\}$ are linearly independent in $V_d(z)$.

We first define the following matrices associated with $P(z)$.

$$A_P = (< \alpha_i, \alpha_j >)_{k \times k}, \tag{5.5}$$

$$\Psi_P = (< \alpha_i, \alpha_j > h_{\alpha_j}^{d-2}(z))_{k \times k}. \tag{5.6}$$

The main result of this section is the following proposition.

Proposition 5.3. *Let* $P(z) \in V_d(z)$ *be given by Eq. (5.4). Then, for any* $m \geq 1$, *we have*

$$Tr\,Hes^m(P) = (d(d-1))^m\,Tr\,\Psi_P^m. \tag{5.7}$$

In particular, $P(z)$ *is HN if and only if the matrix* Ψ_P *is nilpotent.*

Proof: First, by Eq (5.4) and (5.1), we can write $\mathrm{Hes}\,(P)$ explicitly as

$$\mathrm{Hes}\,(P) = d(d-1)\sum_{i=1}^{k} h_{\alpha_i}^{d-2}(z)\ \alpha_i \cdot \alpha_i^t. \tag{5.8}$$

For any $m \geq 1$, we set $c_m = (d(d-1))^m$. By Eq. (5.8), we have

$$\begin{aligned}
\mathrm{Hes}^{\,m}(P) &= c_m \sum_{i_1,i_2,\cdots,i_m=1}^{k} h_{\alpha_{i_1}}^{d-2}(z)\cdots h_{\alpha_{i_m}}^{d-2}(z)(\alpha_{i_1}\cdot\alpha_{i_1}^t)\cdots(\alpha_{i_m}\cdot\alpha_{i_m}^t) \\
&= c_m \sum_{i_1,i_2,\cdots,i_m=1}^{k} \alpha_{i_1}(\alpha_{i_1}^t\cdot\alpha_{i_2})(\alpha_{i_2}^t\cdot\alpha_{i_3})\cdots(\alpha_{i_{m-1}}^t\cdot\alpha_{i_m})\alpha_{i_m}^t \\
&\qquad\qquad \cdot h_{\alpha_{i_1}}^{d-2}(z)\cdots h_{\alpha_{i_m}}^{d-2}(z) \\
&= c_m \sum_{i_1,i_2,\cdots,i_m=1}^{k} <\alpha_{i_1},\alpha_{i_2}><\alpha_{i_2},\alpha_{i_3}>\cdots<\alpha_{i_{m-1}},\alpha_{i_m}> \\
&\qquad\qquad \cdot h_{\alpha_{i_1}}^{d-2}(z)\cdots h_{\alpha_{i_m}}^{d-2}(z)\ \alpha_{i_1}\cdot\alpha_{i_m}^t \\
&= c_m \sum_{i_1,i_m=1}^{k} \left(\Psi_P^{m-1}\right)_{i_1,i_m} h_{\alpha_{i_1}}^{d-2}(z)\ \alpha_{i_1}\cdot\alpha_{i_m}^t.
\end{aligned}$$

By taking the trace of the matrices above, we get

$$\begin{aligned}
\mathrm{Tr}\,\mathrm{Hes}^{\,m}(P) &= c_m \sum_{i_1,i_m=1}^{k} \left(\Psi_P^{m-1}\right)_{i_1,i_m} \mathrm{Tr}\,(\alpha_{i_1}\cdot\alpha_{i_m}^t)\, h_{\alpha_{i_1}}^{d-2}(z) \\
&= c_m \sum_{i_1,i_m=1}^{k} \left(\Psi_P^{m-1}\right)_{i_1,i_m} <\alpha_{i_m},\alpha_{i_1}> h_{\alpha_{i_1}}^{d-2}(z) \\
&= c_m \sum_{i_1,i_m=1}^{k} \left(\Psi_P^{m-1}\right)_{i_1,i_m} (\Psi_P)_{i_m,i_1} \\
&= c_m \mathrm{Tr}\,\Psi_P^m.
\end{aligned}$$

Hence, we get Eq. (5.7). □

Corollary 5.4. *Let $P(z) \in V_d(z)$ be given by Eq. (5.4). Suppose that $P(z)$ is HN. Then the matrix A_P must be singular.*

Proof: By Proposition 5.3, we have Ψ_P is nilpotent. Therefore, we have

$$0 = \det \Psi_P = h_{\alpha_1}^{d-2}(z) h_{\alpha_2}^{d-2}(z) \cdots h_{\alpha_k}^{d-2}(z) \det A_P.$$

Hence, we have $\det A_P = 0$. $\square$

Corollary 5.5. *Let $P(z)$ be HN and given by Eq. (5.4). Then, for any $2 \leq m \leq d$, we have*

$$\sum_{i,j=1}^{k} < \alpha_i, \alpha_j >^m h_{\alpha_i}^{d-m}(z) h_{\alpha_j}^{d-m}(z) = 0. \tag{5.9}$$

In particular, we have

$$\sum_{i=1}^{k} P(\alpha_i) = 0. \tag{5.10}$$

Proof: First, note that, Eq. (5.10) follows directly from Eq. (5.9) with $m = d$.

To prove Eq. (5.9), we first consider the case $m = 2$, the LHS of Eq. (5.9) is just $\operatorname{Tr} \Psi_P^2$ up to a non-zero constant. Hence, by Proposition 5.3, Eq. (5.9) holds in this case.

Now consider the case $m > 2$. By Eq. (5.1), we have

$$\begin{aligned} \Delta^l(h_{\alpha_i}^{d-2}(z) h_{\alpha_j}^{d-2}(z)) = 2^l (d-2)^2 \cdots (d-l-1)^2 \cdot \\ \cdot < \alpha_i, \alpha_j >^l h_{\alpha_i}^{d-2-l}(z) h_{\alpha_j}^{d-2-l}(z) \end{aligned} \tag{5.11}$$

for any $1 \leq i, j \leq k$ and $1 \leq l \leq d-2$.

Now by applying Δ^{m-2} to Eq. (5.9) for the case $m = 2$ and applying Eq. (5.11), it is easy to see that Eq. (5.9) holds for any $3 \leq m \leq d-2$. $\square$

Actually, by Eq. (5.2), the LHS of Eq. (5.9) is also $\Delta^m P^2(z)$ up to a non-zero constant. Hence the corollary above also follows from Theorem 4.3, which implies that $\Delta^m P^2(z) = 0$ for any $m \geq 2$.

One remark is that, by applying similar arguments as above to the equations $\Delta^m P^m(z) = 0$ and $\operatorname{Tr} \Psi_P^m = 0$ $(m \geq 1)$, one can derive more explicit identities satisfied by certain powers of $h_{\alpha_i}(z)$ $(1 \leq i \leq k)$. But, in order to keep this paper in certain size, we skip them here. More study on homogeneous HNP's will be given in [Z3].

5.2. **Some Examples of HNP's and HNS.** In this subsection, we give some examples of HNS and HNP's.

First, let $\Xi = \{\beta_i | 1 \leq i \leq k\}$ be any non-empty subset of $X(\mathbb{C}^n)$ such that $< \beta_i, \beta_j >= 0$ for any $1 \leq i, j \leq k$. For any $d \geq 2$, we set

$$W_{[\Xi,\, d]}(z) = \sum_{i=1}^{k} h^d_{\beta_i}(z). \tag{5.12}$$

For convenience, we also set $W_{[\Xi,\, d]}(z) = 0$ for $\Xi = \emptyset$.

Now let $\widetilde{\Xi} = (\Xi_1, \Xi_2, \cdots, \Xi_m, \cdots)$ be a sequence of finite subsets of $X(\mathbb{C}^n)$ such that, for any $m_1, m_2 \geq 1$ and any $\beta_i \in \Xi_{m_i}$ $(i = 1, 2)$, we have $< \beta_1, \beta_2 >= 0$. We set

$$W_{\widetilde{\Xi}}(z) = \sum_{m=1}^{\infty} W_{[\Xi_m,\, m+1]}(z) = \sum_{m=1}^{\infty} \sum_{\beta_{m_i} \in \Xi_m} h^{m+1}_{\beta_{m_i}}(z). \tag{5.13}$$

A more general construction is as follows.

Let $w = (w_1, w_2, \cdots, w_k)$ be a sequence of commutative variables and $(\beta_1, \beta_2, \cdots, \beta_k)$ a sequence of elements of $\mathbb{C}^n$ with $< \beta_i, \beta_j >= 0$ $(1 \leq i, j \leq k)$. For any formal power series $g(w) \in \mathbb{C}[[w]]$, we define $U_g(z) \in \mathbb{C}[[z]]$ by

$$U_g(z) = g(h_{\beta_1}(z), h_{\beta_2}(z), \cdots, h_{\beta_k}(z)). \tag{5.14}$$

One special case of the construction above is as follows. We introduce new commutative variables $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$. For any $g(z) \in \mathbb{C}[[z]]$, we set

$$P_g(u, v) = g(u_1 + \sqrt{-1}v_1, u_2 + \sqrt{-1}v_2, \cdots, u_n + \sqrt{-1}v_n). \tag{5.15}$$

Note that, by setting $w_i = z_i$ and $\beta_i \in X(\mathbb{C}^{2n})$ such that $h_{\alpha_i}(u, v) = u_i + \sqrt{-1}v_i$ $(1 \leq i \leq n)$, we have $U_g(u, v) = P_g(u, v)$.

The following lemma is easy to check directly by using Lemma 5.1.

Lemma 5.6. *For any $P(z) \in \mathbb{C}[[z]]$ given by Eq. (5.13), Eq. (5.14), we have*

$$\Delta P^m(z) = 0. \tag{5.16}$$

In particular, $P(z)$ is HN.

Note that, by choosing $\widetilde{\Xi}$ in Eq. (5.13) and $g(w) \in \mathbb{C}[[w]]$ in Eq. (5.14) properly, we can construct many HNS, HNP's and homogeneous HNP's. Unfortunately, all these HNS or HNP's $P(z)$ are of "trivial type" in the sense that their deformed inversion pair $Q_t(z) = P(z)$. This can be easily seen from Eq. (5.16) and Corollary 3.9. A family of non-trivial HNP's was given in [BE1] which was constructed as follows.

Let $H(z) = (H_1(z), H_2(z), \cdots, H_n(z)) \in \mathbb{C}[z]^{\times n}$. Let u, v as defined before Eq.(5.15) and set

$$P_H(u,v) = \sum_{i=1}^{n} v_i H_i(u_1 + \sqrt{-1}v_1, u_2 + \sqrt{-1}v_2, \cdots, u_n + \sqrt{-1}v_n).$$

It was shown in Lemma 1.2 in [BE1] that $P_H(u,v) \in \mathbb{C}[u,v]$ is HN if and only if $JH(z)$ is nilpotent.

6. **More Properties of HN Polynomials**

In this section, we derive more properties of HNS (Hessian nilpotent formal power series) and HNP's (Hessian nilpotent polynomials). We prove an identity in Proposition 6.1 and an equivalence of certain vanishing properties in Theorem 6.2 of $\{\Delta^k P^m(z) | k, m \geq 1\}$ for HNS $P(z) \in \mathbb{C}[[z]]$. In Subsection 6.3, we study an isotropic property of $\{\Delta^k P^m(z) | k, m \geq 1\}$ for homogeneous HNP's $P(z)$.

6.1. **An Identity of HN Formal Power Series.** Let $P(z) \in \mathbb{C}[[z]]$ be a HNS. For any $k \geq 0$ and $\alpha \geq 1$, we set

$$u_{k,\alpha}(P) = \frac{\alpha!}{2^k k!(k+\alpha)!} \Delta^k P^{k+\alpha}. \tag{6.1}$$

Proposition 6.1. *For any $\alpha, \beta \geq 1$ and $m \geq 0$, we have*

$$u_{m,\alpha+\beta}(P) = \sum_{\substack{k+l=m \\ k,l \geq 0}} u_{k,\alpha}(P) u_{l,\beta}(P). \tag{6.2}$$

More explicitly, we have

(6.3)

$$\Delta^m P^{m+\alpha+\beta} = \binom{\alpha+\beta}{\alpha}^{-1} \sum_{\substack{k+l=m \\ k,l \geq 0}} \binom{m}{k} \binom{m+\alpha+\beta}{k+\alpha} (\Delta^k P^{k+\alpha})(\Delta^l P^{l+\beta}).$$

Proof: First, it is easy to see that Eq. (6.3) follows directly from Eq. (6.2) and (6.1). So we only need prove Eq. (6.2).

By Eq. (3.8) and (6.1), we have

$$Q_t^{\gamma}(z) = \sum_{m=0}^{\infty} u_{m,\gamma}(P) t^m \tag{6.4}$$

for any $\gamma \geq 1$. By comparing the coefficients of t^m of both sides of the equation $Q_t^{\alpha+\beta}(z) = Q_t^{\alpha}(z) Q_t^{\beta}(z)$, we see that Eq. (6.2) holds. $\square$

6.2. **A Vanishing Property of HN Formal Power Series.**

Theorem 6.2. *For any HNS $P(z) \in \mathbb{C}[[z]]$, the following statements are equivalent.*

(1) *For any $k \geq 1$, $\Delta^m P^{m+k} = 0$ when $m >> 0$.*
(2) *There exists $k_0 \geq 1$, $\Delta^m P^{m+k_0} = 0$ when $m >> 0$.*
(3) *$\Delta^m P^{m+1} = 0$ when $m >> 0$.*

Proof: (1) $\Rightarrow$ (2) is trivial. To show (2) $\Rightarrow$ (3), we assume that $\Delta^m P^{m+k_0} = 0$ when $m > M_0$ for some $M_0 \geq 1$. For any $m > M_0 + k_0 - 1$, we have

$$\Delta^m P^{m+1} = \Delta^{k_0-1}(\Delta^{m-k_0+1} P^{(m-k_0+1)+k_0}) = 0.$$

Hence (3) holds in this case.

Now we consider (3) $\Rightarrow$ (1). Since $P(z)$ is HN, Eq. (3.8) in Theorem 3.4 holds for any $k \geq 1$. In particular, $Q_t(z)$ is a polynomial in t with coefficients in $\mathbb{C}[[z]]$ by our assumption of (3). Therefore, for any $k \geq 1$, $Q_t^k(z)$ is also a polynomial in t with coefficients in $\mathbb{C}[[z]]$. By Eq. (3.8) again, we see that (1) holds. $\square$

We believe that Theorem 7.2 is still true without the Hessian nilpotency condition. Actually, if Conjecture 4.4 is true, it is certainly the case. More precisely, suppose that one of the statements, say (3), of Theorem 7.2 holds for some $P(z) \in \mathbb{C}[[z]]$. Then we have

$$\Delta^{m+1} P^{m+1} = \Delta(\Delta^m P^{m+1}) = 0$$

when $m >> 0$. If Conjecture 4.4 is true, then $P(z)$ is HN. Hence all other statements of Theorem 7.2 also hold.

Later we will show in Theorem 7.2 that the Jacobian conjecture is equivalent to saying that one of the statements in Theorem 6.2 holds for HNP's $P(z)$.

6.3. **Isotropic Properties of Homogeneous HN Polynomials.**

For any $1 \leq i \leq n$, we set $D_i = \frac{\partial}{\partial z_i}$ and $D = (D_1, D_2, \cdots, D_n)$. We define a $\mathbb{C}$-bilinear map $\{\cdot,\cdot\} : \mathbb{C}[z] \times \mathbb{C}[z] \to \mathbb{C}[z]$ by setting $\{f, g\} = f(D)g(z)$ for any $f, g \in \mathbb{C}[z]$. The $\mathbb{C}$-bilinear map $\{\cdot,\cdot\}$ defined above is closely related with the following commonly used Hermitian inner product of $\mathbb{C}[z]$. See, for example, [ABR], [H] and [KR].

$$\begin{aligned}(\cdot,\cdot) : \mathbb{C}[z] \times \mathbb{C}[z] &\to \mathbb{C},\\ (f, \quad g) \quad &\to (f(D)\bar{g})(0),\end{aligned}$$

where, $\bar{g}(z) = \sum_{I\in\mathbb{N}^n} \bar{a}_{\mathbf{s}} z^{\mathbf{s}}$ if $g(z) = \sum_{I\in\mathbb{N}^n} a_{\mathbf{s}} z^{\mathbf{s}}$. In particular, for any homogeneous polynomials $f, g \in \mathbb{C}[z]$ of the same degree, we have $\{f, \bar{g}\} = (f, g)$.

Actually, the Hermitian inner product $(\cdot,\cdot)$ plays an very important role in the study of classical harmonic polynomials (See [ABR] and [H].). Due to the connection of $\{\cdot,\cdot\}$ with the Hermitian inner product $(\cdot,\cdot)$ described above, we refer to the properties of HNP's derived in this subsection as certain isotropic properties.

The main result of this subsection is the following theorem.

Theorem 6.3. *Let $P(z)$ be a homogeneous HNP of degree $d \geq 3$ and $\mathcal{I}(P)$ the ideal of $\mathbb{C}[z]$ generated by $\sigma^2 := \sum_{i=1}^n z_i^2$ and $\frac{\partial P}{\partial z_i}$ $(1 \leq i \leq n)$. Then, for any $f(z) \in \mathcal{I}(P)$ and $m \geq 0$, we have*

$$\{f, \Delta^m P^{m+1}\} = f(D)\Delta^m P^{m+1} = 0. \tag{6.5}$$

To prove this theorem, we first need the following two lemmas.

Lemma 6.4. *For any homogeneous polynomial $f(z)$ of degree $k \geq 1$, we have*

$$\begin{aligned}
&\sum_{i_1,i_2,\cdots,i_k=1}^{n} \frac{\partial^k f(z)}{\partial z_{i_1}\partial z_{i_2}\cdots\partial z_{i_k}} \frac{\partial^k}{\partial z_{i_1}\partial z_{i_2}\cdots\partial z_{i_k}} \\
&= \sum_{\substack{\mathbf{s}\in\mathbb{N}^n \\ |\mathbf{s}|=k}} \binom{k}{\mathbf{s}} \frac{\partial^k f}{\partial z^{\mathbf{s}}} \frac{\partial^k}{\partial z^{\mathbf{s}}} \\
&= k! f(D),
\end{aligned} \tag{6.6}$$

where $\binom{k}{\mathbf{s}} = \frac{k!}{s_1! s_2! \cdots s_n!}$ for any $\mathbf{s} = (s_1, s_2, \cdots, s_n) \in \mathbb{N}^n$ with $|\mathbf{s}| = k$.

Proof: Since Eq. (6.6) is linear on $f(z)$, we may assume that $f(z)$ is a single monomial, say, $f(z) = z_1^{l_1} z_2^{l_2} \cdots z_n^{l_n}$ with $l_i \geq 0$ and $\sum_{i=1}^n l_i = k$. Now, we consider

$$\begin{aligned}
&\sum_{i_1,i_2,\cdots,i_k=1}^{n} \left(\frac{\partial^k}{\partial z_{i_1}\partial z_{i_2}\cdots\partial z_{i_k}} z_1^{l_1} z_2^{l_2} \cdots z_n^{l_n} \right) \frac{\partial^k}{\partial z_{i_1}\partial z_{i_2}\cdots\partial z_{i_k}} \\
&= \left(\frac{k!}{l_1! l_2! \cdots l_n!} \frac{\partial^k}{\partial z_1^{l_1}\partial z_2^{l_2}\cdots\partial z_n^{l_n}} z_1^{l_1} z_2^{l_2} \cdots z_n^{l_n} \right) \frac{\partial^k}{\partial z_1^{l_1}\partial z_2^{l_2}\cdots\partial z_n^{l_n}} \\
&= k! \left(\frac{\partial}{\partial z_1}\right)^{l_1} \left(\frac{\partial}{\partial z_2}\right)^{l_2} \cdots \left(\frac{\partial}{\partial z_n}\right)^{l_n} \\
&= k! f(D).
\end{aligned}$$

□

Lemma 6.5. *For any $f(z), g(z) \in \mathbb{C}[[z]]$ and $l \geq 1$, we have*

(6.7)
$$\Delta^l(gf) = \sum_{\substack{k_1+k_2+k_3=l \\ k_1,k_2,k_3\geq 0}} 2^{k_2}\binom{l}{k_1,k_2,k_3} \sum_{i_1,i_2,\cdots,i_{k_2}=1}^{n} \frac{\partial^{k_2}\Delta^{k_1}g(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2}}}\frac{\partial^{k_2}\Delta^{k_3}f(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2}}}$$
$$= \sum_{\substack{k_1+k_2+k_3=l \\ k_1,k_2,k_3\geq 0}} 2^{k_2}\binom{l}{k_1,k_2,k_3} \sum_{\substack{\mathbf{s}\in\mathbb{N}^n \\ |\mathbf{s}|=k_2}} \binom{k_2}{\mathbf{s}} \frac{\partial^{k_2}\Delta^{k_1}g}{\partial z^{\mathbf{s}}}\frac{\partial^{k_2}\Delta^{k_3}f}{\partial z^{\mathbf{s}}}.$$

where $\binom{l}{k_1,k_2,k_3} = \frac{l!}{k_1!k_2!k_3!}$ for any $k_1, k_2, k_3 \geq 0$.

Proof: We use the mathematical induction on $l \geq 1$. When $l = 1$, by the Leibniz's rule, it is easy to check that

$$\Delta(gf)(z) = (\Delta g(z))f(z) + 2\sum_{i=1}^{n}\frac{\partial g(z)}{\partial z_i}\frac{\partial f(z)}{\partial z_i} + g(z)\Delta f(z) \tag{6.8}$$

which is exactly Eq. (6.7) with $l = 1$.

Now we assume Eq. (6.7) holds for $l = l_0 \geq 0$. By using Eq. (6.8) and Eq. (6.7) with $l = l_0$, we have

$$\Delta^{l_0+1}(gf) = \Delta(\Delta^{l_0}(gf))$$
$$= \sum_{\substack{k_1+k_2+k_3=l \\ k_1,k_2,k_3\geq 0}} 2^{k_2}\binom{l}{k_1,k_2,k_3} \sum_{i_1,i_2,\cdots,i_{k_2}=1}^{n} \Delta\left(\frac{\partial^{k_2}\Delta^{k_1}g(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2}}}\frac{\partial^{k_2}\Delta^{k_3}f(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2}}}\right)$$
$$= \sum_{\substack{k_1+k_2+k_3=l \\ k_1,k_2,k_3\geq 0}} 2^{k_2}\binom{l}{k_1,k_2,k_3} \sum_{i_1,i_2,\cdots,i_{k_2}=1}^{n} \frac{\partial^{k_2}\Delta^{k_1+1}g(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2}}}\frac{\partial^{k_2}\Delta^{k_3}f(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2}}}$$
$$+ \sum_{\substack{k_1+k_2+k_3=l \\ k_1,k_2,k_3\geq 0}} 2^{k_2+1}\binom{l}{k_1,k_2,k_3} \sum_{i_1,i_2,\cdots,i_{k_2+1}=1}^{n} \frac{\partial^{k_2+1}\Delta^{k_1}g(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2+1}}}\frac{\partial^{k_2+1}\Delta^{k_3}f(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2+1}}}$$
$$+ \sum_{\substack{k_1+k_2+k_3=l \\ k_1,k_2,k_3\geq 0}} 2^{k_2}\binom{l}{k_1,k_2,k_3} \sum_{i_1,i_2,\cdots,i_{k_2}=1}^{n} \frac{\partial^{k_2}\Delta^{k_1}g(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2}}}\frac{\partial^{k_2}\Delta^{k_3+1}f(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2}}}$$

By shifting indices and using the convention that $\binom{l}{k_1,k_2,k_3} = 0$ if k_1, k_2 or $k_3 < 0$:

$$= \sum_{\substack{k_1+k_2+k_3=l+1 \\ k_1,k_2,k_3\geq 0}} 2^{k_2} \left(\binom{l}{k_1-1,k_2,k_3} + \binom{l}{k_1,k_2-1,k_3} + \binom{l}{k_1,k_2,k_3-1} \right) \sum_{i_1,i_2,\cdots,i_{k_2}=1}^{n} \frac{\partial^{k_2}\Delta^{k_1} g(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2}}} \frac{\partial^{k_2}\Delta^{k_3} f(z)}{\partial z_{i_1}\cdots\partial z_{i_{k_2}}}.$$

Hence, we only need show that

$$\binom{l+1}{k_1,k_2,k_3} = \binom{l}{k_1-1,k_2,k_3} + \binom{l}{k_1,k_2-1,k_3} + \binom{l}{k_1,k_2,k_3-1}$$

for any $k_1, k_2, k_3 \geq 0$. But this identity follows directly from the binomal expression of $(x+y+z)^k$ for $k = l, l+1$ and the identity

$$(x+y+z)^{l+1} = (x+y+z)(x+y+z)^l.$$

□

One immediate consequence of Eq. (6.7) is the following corollary.

Corollary 6.6. *For $g(z), f(z) \in \mathbb{C}[[z]]$ with $g(z)$ harmonic, we have*

$$\Delta^l(gf) = \sum_{k=0}^{l} 2^k \binom{l}{k} \sum_{\substack{\mathbf{s}\in\mathbb{N}^n \\ |\mathbf{s}|=k}} \binom{k}{\mathbf{s}} \frac{\partial^k g}{\partial z^{\mathbf{s}}} \frac{\partial^k \Delta^{l-k} f}{\partial z^{\mathbf{s}}}. \tag{6.9}$$

Now we are ready to prove Theorem 6.3.

Proof of Theorem 6.3: Since $P(z)$ is HN, by Theorem 4.3, we know that $\Delta^m P^m = 0$ for any $m \geq 1$. Therefore, we have,

$$\sigma^2(D)\Delta^m P^{m+1} = \Delta^{m+1} P^{m+1} = 0.$$

Hence it will be enough to prove the theorem for $f(z) = \frac{\partial P}{\partial z_i}(z)$ for any fixed $1 \leq i \leq n$.

Since $d \geq 3$, we have $m+d-1 \geq m+2$ and

$$\Delta^{m+d-1} P^{m+2} = 0. \tag{6.10}$$

By applying $\frac{\partial}{\partial z_i}$ to the equation above, we get

$$\Delta^{m+d-1} \frac{\partial P}{\partial z_i} P^{m+1} = \Delta^{m+d-1}(f P^{m+1}) = 0. \tag{6.11}$$

But, on the other hand, by Eq. (6.9), we have

$$\Delta^{m+d-1}\left(fP^{m+1}\right) = \sum_{k=0}^{m+d-1} 2^k \binom{m+d-1}{k} \sum_{\substack{\mathbf{s}\in\mathbb{N}^n \\ |\mathbf{s}|=k}} \binom{k}{\mathbf{s}} \frac{\partial^k f}{\partial z^{\mathbf{s}}} \frac{\partial^k \Delta^{m+d-1-k} P^{m+1}}{\partial z^{\mathbf{s}}}. \tag{6.12}$$

Furthermore, we also have the following equations.

$$\frac{\partial^{\mathbf{s}} f}{\partial z^{\mathbf{s}}} = 0 \qquad \text{for any } \mathbf{s}\in\mathbb{N}^n \text{ with } |\mathbf{s}| = k > d-1,$$
$$\Delta^a P^{m+1} = 0 \qquad \text{for any } a \geq m+1.$$

Hence all the terms in the RHS of Eq. (6.12) except the one with $k = d-1$ are zero. Therefore, by Eq. (6.6) in Lemma 6.4, we have

$$\begin{aligned}\Delta^{m+d-1}(fP^{m+1}) &= 2^{d-1} \sum_{\substack{\mathbf{s}\in\mathbb{N}^n \\ |\mathbf{s}|=d-1}} \binom{d-1}{\mathbf{s}} \frac{\partial^{d-1} f}{\partial z^{\mathbf{s}}} \frac{\partial^{d-1}\Delta^m P^{m+1}}{\partial z^{\mathbf{s}}} \\ &= 2^{d-1}(d-1)! f(D)\Delta^m P^{m+1}.\end{aligned}$$

Hence, by Eq. (6.11) and the equation above, we see that the theorem holds for $f(z) = \frac{\partial P}{\partial z_i}(z)$ $(1 \leq i \leq n)$. □

Corollary 6.7. *Let $P(z)$ be a homogeneous HNP of degree $d \geq 3$ and $Q_t(z)$ its deformed inversion pair. Then, for any $k, l \geq 0$ with $k > l$, we have*

$$(\Delta^l P^k)(D) Q_t(z) = 0. \tag{6.13}$$

More precisely, we have

$$(\Delta^l P^k)(D)(\Delta^m P^{m+1}(z)) = 0 \tag{6.14}$$

for any $k, l \geq 0$ with $k > l$.

In particular, we have

$$P(D) Q_t(z) = 0. \tag{6.15}$$

Proof: First note that Eq. (6.15) follows from Eq. (6.13) by setting $l = 0$ and $k = 1$. By Eq. (3.8), we see that Eq. (6.13) and Eq. (6.14) are equivalent to each other. Hence, it is enough to show Eq. (6.14). Furthermore, by Theorem 6.3, it will be enough to show that $\Delta^l P^k(z)$ for any $k, l \geq 0$ with $k > l$ lies in the ideal $\tilde{\mathfrak{J}}(P)$ generated by $\frac{\partial P}{\partial z_i}(z)$ $(1 \leq i \leq n)$.

By Euler's lemma, we have $P(z) = \frac{1}{d}\sum_{i=1}^n z_i \frac{\partial P}{\partial z_i}(z)$. Hence, for any $k \geq 1$, $P^k(z) \in \tilde{\mathfrak{J}}(P)$ and Eq. (6.14) holds when $l = 0$. Now we consider

the case $l > 0$. Note that Δ^l is a sum of the differential operators of the form $\frac{\partial^2}{\partial z_{i_1}^2}\frac{\partial^2}{\partial z_{i_2}^2}\cdots\frac{\partial^2}{\partial z_{i_l}^2}$ with $1 \le i_1, i_2, \cdots, i_l \le n$. When we distribute the $2l$ derivations $\frac{\partial}{\partial z_{i_j}}$ $(1 \le j \le l)$ of the differential operator above to k copies $P(z)$ of $P^k(z)$, there is always at least one copy $P(z)$ of $P^k(z)$ receives none or one derivation. Otherwise, we would have $2k \le 2l$, which contradicts to our condition $k > l$. Since we have already shown $P(z) \in \tilde{\mathfrak{J}}(P)$ above, hence we have $\Delta^l P^k(z) \in \tilde{\mathfrak{J}}(P)$ for any $k, l \ge 0$ with $k > l > 0$. $\square$

Theorem 6.3 and Corollary 6.7 do not hold for homogeneous HNP's $P(z)$ of degree $d = 2$. But, by similar arguments as the proof of Theorem 6.3 starting from $\Delta^{m+2}P^{m+2} = \Delta^{m+2}(P \cdot P^{m+1}) = 0$ instead of Eq. (6.11), one can show the following proposition.

Proposition 6.8. *Let $P(z)$ be a homogeneous HNP of degree $d = 2$ and $\mathfrak{J}(P)$ the ideal of $\mathbb{C}[z]$ generated by $P(z)$ and $\sigma^2 = \sum_{i=1}^n z_i^2$. Then, for any $f(z) \in \mathfrak{J}(P)$ and $m \ge 0$, we have*

$$f(D)\Delta^m P^{m+1} = 0. \tag{6.16}$$

In particular, we have

$$P(D)Q_t(z) = 0. \tag{6.17}$$

7. Vanishing Conjectures of HN Polynomials

In this section, we propose some conjectures on certain vanishing properties of polynomials $\{\Delta^k P^m(z) | m, k \ge 1 \text{ and } m > k\}$ for HNP's (Hessian Nilpotent polynomials) $P(z)$. We also show that these so-called *vanishing conjectures* are equivalent to the well-known Jacobian conjecture.

Conjecture 7.1. (Vanishing Conjecture)
For any HN (not necessarily homogeneous) polynomial $P(z)$ of degree $d \ge 2$, its deformed inversion pair $Q_t(z)$ is a polynomial in both t and z. More precisely, $\Delta^k P^{k+1} = 0$ when $k >> 0$.

Theorem 7.2. *The following statements are equivalent.*

1. *The vanishing conjecture for homogeneous HNP of degree $d = 4$.*
2. *The vanishing conjecture for homogeneous HNP of degree $d \ge 2$.*
3. *The vanishing conjecture.*
4. *The Jacobian conjecture.*

Proof: First, it is easy to see that (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are trivial. By the gradient reduction in [BE1] and the homogeneous reduction in [BCW], [Y] on the Jacobian conjecture, we know that

the Jacobian conjecture will be true if it is true for polynomial maps $F(z) = z - \nabla P(z)$ with $P(z)$ being homogeneous HNP of degree $d = 4$. Therefore, (4) $\Leftrightarrow$ (1) follows directly from Eq. (3.8) in Theorem 3.4. Hence we only need show (4) $\Rightarrow$ (3).

Now we assume the Jacobian conjecture and let $P(z)$ be a HNP of degree $d \geq 2$. Let $F_t(z) = z - t\nabla P(z)$ and $G_t(z) = z + t\nabla Q_t(z)$ as before. Consider the formal map $U(z,t) = (F_t(z), t)$ from $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. It is easy to check that the Jacobian of the map $U(z,t)$ with respect to (z,t) is also identically equal to 1 and the formal inverse $V(z,t)$ is given by $V(z,t) = (G_t(z), t)$. Since we have assumed the Jacobian conjecture, $V(z,t)$ must be a polynomial in (z,t). Hence so is $G_t(z)$. By Eq. (3.8) again, we see that $\Delta^m P^{m+1}(z)$ must vanish when $m >> 0$. $\square$

Next, by using the upper bound given in [BCW], Corollary 1.4, for the degrees of inverse maps of polynomial automorphisms of $\mathbb{C}^n$, we show that Conjecture 7.1 for homogeneous HNP's can actually be reformulated more precisely as follows.

Conjecture 7.3. (Homogeneous Vanishing Conjecture)
For any homogeneous HNP $P(z)$ of degree $d \geq 2$, we have

(1) $\Delta^m P^{m+1} = 0$ *for any* $m > \alpha_{[n,d]} := \frac{1}{d-2}((d-1)^{n-1} - (d-1))$.
(2) *For any* $k \geq 1$, $\Delta^m P^{m+k} = 0$ *for any* $m > k\alpha_{[n,d]}$.

Proposition 7.4. (a) *For any homogeneous HNP $P(z)$ of degree $d \geq 2$, the statements* (1) *and* (2) *in Conjecture 7.3 are equivalent.*

(b) *Conjecture 7.3 for $d \geq 2$ and Conjecture 7.3 for $d = 4$ are both equivalent to the Jacobian conjecture.*

Proof: (a) First, (2) $\Rightarrow$ (1) is trivial. Now we show that (1) $\Rightarrow$ (2). By Eq. (3.8) with $k = 1$, we see that the degree $\deg_t Q_t(z)$ of $Q_t(z)$ with respect to t is less or equal to $\alpha_{n,d}$, i.e. $\deg_t Q_t(z) \leq \alpha_{[n,d]}$. Therefore, for any $k \geq 1$, we have, $\deg_t Q_t^k(z) \leq k\alpha_{[n,d]}$. By Eq. (3.8) again, we see that (2) holds for any $k \geq 1$.

(b) By theorem 7.2, it is easy to see that Conjecture 7.3 for $d \geq 2$ or Conjecture 7.3 with $d = 4$ implies the Jacobian conjecture. Therefore it will be enough to show that the Jacobian conjecture implies Conjecture 7.3. By Corollary 1.4 in [BCW], we know that, for any polynomial automorphism $F(z)$ of $\mathbb{C}^n$ with $\deg F(z) = d - 1$, $\deg G(z) \leq (d-1)^{n-1}$. By applying this result to the polynomial automorphism $F(z) = z - \nabla P(z)$ and its inverse $G(z) = z - \nabla Q(z)$, we get $\deg Q(z) \leq (d-1)^{n-1} + 1$. Furthermore, by Eq. (3.8) with $k = 1$ and the fact

$\deg \Delta^m P^{m+1}(z) = (m+1)d - 2m$ $(m \geq 1)$, we get $\Delta^m P^{m+1}(z) = 0$ if

$$(m+1)d - 2m > (d-1)^{n-1} + 1.$$

By separating m from the inequality above, we get $\Delta^m P^{m+1}(z) = 0$ whenever $m > \alpha_{[n,d]}$. $\square$

Finally, by translating certain known results on the Jacobian conjecture, we know that the vanishing conjectures, Conjecture 7.1 or 7.3, are true for the following cases.

- *S. Wang* [Wa] *proved that the Jacobian conjecture holds for any polynomial map $F(z)$ of* $\deg F(z) \leq 2$. *Hence Conjecture 7.1 and 7.3 hold for any HNP $P(z)$ of degree $d \leq 3$.*
- *For any symmetric polynomial map $F(z) = z - H(z)$ with $o(H(z)) \geq 2$ and $JH(z)$ nilpotent, A. van den Essen and S. Washburn* [EW] *showed that the Jacobian conjecture holds when $n \leq 4$ and $H(z)$ is homogeneous. Later, M. de Bondt and A. van den Essen* [BE2]-[BE4] *further proved that the Jacobian conjecture holds either $n \leq 4$ without $H(z)$ being homogeneous, or $n = 5$ with $H(z)$ being homogeneous. (For an exposition discussion on these results, see* [BE5].*). From the results above, we see that, Conjecture 7.1 has an affirmative answer when $n \leq 4$, and Conjecture 7.3 is true when $n \leq 5$.*
- *By Theorem* 4.1 *in* [EW] *and similar arguments there, it is easy to show that the only HNP's (not necessarily homogeneous) $P(z) \in \mathbb{R}[z]$ $(o(P(z)) \geq 2)$ with real coefficients are $P(z) = 0$. Hence, Conjecture 7.1 and 7.3 hold trivially in this case.*
- *Recently, D. Wright* [Wr1] *showed that the Jacobian conjecture holds for any symmetric polynomial map $F(z) = z - H(z)$ with $H(z)$ homogeneous and $JH^3(z) = 0$. Hence Conjecture 7.3 holds for any homogeneous HNP $P(z)$ with $Hes^3(P(z)) = 0$.*

Furthermore, by Corollary 3.9, we also know that Conjecture 7.1 holds for any HNP $P(z)$ such that $\Delta P^2(z) = 0$. In particular, it is true for HNP's constructed by Eq.(5.13) and Eq.(5.14).

References

[Ar] V. I. Arnord, *Mathematical Methods of Classical Mechanics,* Springe-Verlag New York, Inc. 1978. [MR 0690288].

[ABR] S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory.* Second edition. Graduate Texts in Mathematics, 137. Springer-Verlag, New York, 2001. [MR1805196].

[BCW] H. Bass, E. Connell, D. Wright, *The Jacobian conjecture, reduction of degree and formal expansion of the inverse*. Bull. Amer. Math. Soc. **7**, (1982), 287–330. [MR83k:14028], [Zbl.539.13012].

[BE1] M. de Bondt and A. van den Essen, *A Reduction of the Jacobian Conjecture to the Symmetric Case*, Report No. 0308, University of Nijmegen, June, 2003. To appear in *Proc. of the AMS.*.

[BE2] M. de Bondt and A. van den Essen, *Nilpotent Symmetric Jacobian Matrices and the Jacobian Conjecture*, Report No. 0307, University of Nijmegen, June, 2003. To appear in *J. Pure and Appl, Alg.*.

[BE3] M. de Bondt and A. van den Essen, *Singular Hessians*, Report No. 0317, University of Nijmegen, October, 2003.

[BE4] M. de Bondt and A. van den Essen, *Nilpotent Symmetric Jacobian Matrices and the Jacobian Conjecture II*, Report No. 0318, University of Nijmegen, November, 2003.

[BE5] M. de Bondt and A. van den Essen, *Hesse and the Jacobian Conjecture*, Report No. 0321, University of Nijmegen, December, 2003.

[E] A. van den Essen, *P*olynomial automorphisms and the Jacobian conjecture. Progress in Mathematics, 190. Birkhuser Verlag, Basel, 2000. [MR1790619].

[EW] A. van den Essen and S. Washburn, *The Jacobian Conjecture for Symmetric Jacobian Matrices*, J. Pure Appl. Algebra, **189** (2004), no. 1-3, 123–133. [MR2038568]

[H] H. Henryk, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, 17. American Mathematical Society, Providence, RI, 1997. [MR1474964].

[Ke] O. H. Keller, *Ganze Gremona-Transformation*, Monats. Math. Physik **47** (1939), 299-306.

[KR] V. Kac and A. K. Raina, *Bombay Lectures on Highest Weight Representations of Infinite Dimensional Lie Algebras*, Advanced Series in Mathematical Physics, 2. World Scientific Publishing Co., Inc., Teaneck, NJ, 1987. [MR1021978].

[M] G. Meng, *Legendre Transform, Hessian Conjecture and Tree Formula*, math-ph/0308035.

[R] J. Rauch, *Partial Differential Equations*, Springer-Verlag New York Inc., 1991. [MR 1223093].

[S] S. Smale, *Mathematical Problems for the Next Century*, Math. Intelligencer 20, No. 2, 7-15, 1998. [MR1631413 (99h:01033)].

[T] M. Takeuchi, *Modern spherical functions*, Translations of Mathematical Monographs, **135**. American Mathematical Society, Providence, RI, 1994. [MR 1280269].

[Wa] S. Wang, *A Jacobian criterion for Separability*, J. Algebra **65** (1980), 453-494. [MR 83e:14010].

[Wr1] D. Wright, *Ideal Membership Questions Relating to the Jacobian Conjecture.* To appear.

[Wr2] D. Wright, *The Jacobian Conjecture: Ideal Membership Questions and recent advances*, To appear.

[Wr3] D. Wright, *Personal communications.*

[Y] A. V. Jagžev, *On a problem of O.-H. Keller.* (Russian) Sibirsk. Mat. Zh. 21 (1980), no. 5, 141–150, 191. [MR0592226].

[Z1] W. Zhao, *Recurrent Inversion Formulas*, math. CV/0305162.

[Z2] W. Zhao, *Inversion Problem, Legendre Transform and the Inviscid Burgers' Equation*, math. CV/0403020. Submitted.

[Z3] W. Zhao, *Some Properties and Open Problems of Hessian Nilpotent polynomials*, In preparation.

Department of Mathematics, Illinois State University, Normal, IL 61790-4520.

E-mail: wzhao@ilstu.edu.