arXiv:math/0509138v2 [math.CO] 9 Sep 2005

$\mathcal{N}$CS SYSTEMS OVER DIFFERENTIAL OPERATOR ALGEBRAS AND THE GROSSMAN-LARSON HOPF ALGEBRAS OF LABELED ROOTED TREES

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Abstract. Let K be any unital commutative $\mathbb{Q}$-algebra and W any non-empty subset of $\mathbb{N}^+$. Let $z = (z_1, \ldots, z_n)$ be commutative or noncommutative free variables and t a formal central parameter. Let $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ ($\alpha \geq 1$) be the unital algebra generated by the differential operators of $K\langle\langle z\rangle\rangle$ which increase the degree in z by at least $\alpha - 1$ and $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ the group of automorphisms $F_t(z) = z - H_t(z)$ of $K[[t]]\langle\langle z\rangle\rangle$ with $o(H_t(z)) \geq \alpha$ and $H_{t=0}(z) = 0$. First, we study a connection of the $\mathcal{N}$CS systems Ω_{F_t} ($F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$) ([Z5], [Z6]) over the differential operators algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ and the $\mathcal{N}$CS system $\Omega_{\mathbb{T}}^W$ ([Z8]) over the Grossman-Larson Hopf algebra $\mathcal{H}_{GL}^W$ ([GL], [F1], [F2]) of W-labeled rooted trees. We construct a Hopf algebra homomorphism $\mathcal{A}_{F_t} : \mathcal{H}_{GL}^W \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ ($F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$) such that $\mathcal{A}_{F_t}^{\times 5}(\Omega_{\mathbb{T}}^W) = \Omega_{F_t}$. Secondly, we generalize the tree expansion formulas for the inverse map ([BCW], [Wr]), the D-Log and the formal flow ([WZ]) of F_t in the commutative case to the noncommutative case. Thirdly, we prove the injectivity of the specialization $\mathcal{T} : \mathcal{N}Sym \to \mathcal{H}_{GL}^{\mathbb{N}^+}$ ([Z8]) of NCSF's (noncommutative symmetric functions) ([GKLLRT]). Finally, we show the family of the specializations $\mathcal{S}_{F_t}$ of NCSF's with all $n \geq 1$ and the polynomial automorphisms $F_t = z - H_t(z)$ with $H_t(z)$ homogeneous and the Jacobian matrix JH_t strictly lower triangular can distinguish any two different NCSF's. The graded dualized versions of the main results above are also discussed.

2000 *Mathematics Subject Classification.* Primary: 05E05, 14R10, 16W30; Secondary: 16W20, 06A11.

Key words and phrases. $\mathcal{N}$CS systems, noncommutative symmetric functions, quasi-symmetric functions, specializations, formal automorphisms in commutative or non-commutative variables, D-Log's, the formal flows, tree expansion formulas, the Grossman-Larson Hopf algebra, the Connes-Kreimer Hopf algebra, labeled rooted trees, the (strict) order polynomials of posets.

1. Introduction

Let K be any unital commutative $\mathbb{Q}$-algebra and A a unital associative but not necessarily commutative K-algebra. A $\mathcal{N}CS$ (*noncommutative symmetric*) *system* over A (see Definition 2.1) by definition is a 5-tuple $\Omega \in A[[t]]^{\times 5}$ which satisfies the defining equations (see Eqs. (2.1)–(2.5)) of the NCSF's (noncommutative symmetric functions) first introduced and studied in the seminal paper [GKLLRT]. When the base algebra K is clear in the context, the ordered pair (A, Ω) is also called a $\mathcal{N}CS$ *system*. In some sense, a $\mathcal{N}$CS system over an associative K-algebra can be viewed as a system of analogs in A of the NCSF's defined by Eqs. (2.1)–(2.5). For some general discussions on the $\mathcal{N}$CS systems, see [Z5]. For more studies on NCSF's, see [T], [KLT], [DKKT], [KT1], [KT2] and [DFT]. One immediate but probably the most important example of the $\mathcal{N}$CS systems is $(\mathcal{N}Sym, \Pi)$ formed by the generating functions of the NCSF's defined in [GKLLRT] by Eqs. (2.1)–(2.5) over the free K-algebra $\mathcal{N}Sym$ of NCSF's (see Section 2.1). It serves as the universal $\mathcal{N}$CS system over all associative K-algebra (see Theorem 2.3). More precisely, for any $\mathcal{N}$CS system (A, Ω), there exists a unique K-algebra homomorphism $\mathcal{S} : \mathcal{N}Sym \to A$ such that $\mathcal{S}^{\times 5}(\Pi) = \Omega$ (here we have extended the homomorphism $\mathcal{S}$ to $\mathcal{S} : \mathcal{N}Sym[[t]] \to A[[t]]$ by the base extension). In [Z8] and [Z6], some families of $\mathcal{N}$CS systems over differential operator algebras and the Grossman-Larson Hopf algebra ([GL], [F1], [F2]) of labeled rooted trees have been constructed, respectively. Consequently, by the universal property of the $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$, one obtains two families of specializations of NCSF's by differential operators and labeled rooted trees (see Sections 2.2 and 2.3 for a brief review of the results above).

In the first part of this paper, we study a connection of the $\mathcal{N}$CS systems in [Z8] over the Grossman-Larson Hopf algebras of labeled rooted trees and the $\mathcal{N}$CS systems in [Z6] over differential operator algebras. We construct a Hopf algebra homomorphism from the former algebra to the later which maps the former $\mathcal{N}$CS system to the later. In the second part of this paper, we first apply the connection above to derive tree expansion formulas for the D-Log's, the formal flows and the inverse maps of formal automorphisms in commutative or noncommutative variables, which generalize the tree expansion formulas obtained in [BCW], [Wr] and [WZ] for commutative variables to the noncommutative case. Secondly, by combining the connection above with some results obtained in [Z6] and [Z8], we prove more properties for the specializations of NCSF's by differential operator and labeled rooted trees obtained in [Z6] and [Z8], respectively.

To be more precise, let $z = (z_1, \ldots, z_n)$ be commutative or noncommutative free variables and t a formal central parameter. Denote uniformly for both commutative and noncommutative variables z by $K\langle\langle z\rangle\rangle$ (resp. $K\langle z\rangle$) the formal power series (resp. polynomial) algebra of z over K. For any $\alpha \geq 1$, let $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ (resp. $\mathcal{D}^{[\alpha]}\langle z\rangle$) be the unital algebra generated by the differential operators of $K\langle\langle z\rangle\rangle$ (resp. $K\langle z\rangle$) which increase the degree in z by at least $\alpha - 1$ and $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ the group of automorphisms $F_t(z) = z - H_t(z)$ of $K[[t]]\langle\langle z\rangle\rangle$ with $o(H_t(z)) \geq \alpha$ and $H_{t=0}(z) = 0$. In [Z6], associated with each automorphism $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, a $\mathcal{N}$CS system ([Z5]) Ω_{F_t} over the differential operator algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ has been constructed. Consequently, by the universal property of the $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$, one obtains a families of specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ of NCSF's by differential operators. In [Z8], for any non-empty $W \subseteq \mathbb{N}^+$, a $\mathcal{N}$CS system $\Omega_{\mathbb{T}}^W$ over the Grossman-Larson Hopf algebra $\mathcal{H}_{GL}^W$ were given explicitly. Hence, one also gets a specialization $\mathcal{T}_W : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ of NCSF's by W-labeled rooted trees.

In the first part of this paper, for any fixed $\alpha \geq 1$, $\emptyset \neq W \subseteq \mathbb{N}^+$ and $F_t(z) = z - H_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ such that $H_t(z)$ can be written as $\sum_{m\in W} t^m H_{[m]}(z)$ for some $H_{[m]}(z) \in K\langle\langle z\rangle\rangle^{\times n}$ $(m \in W)$, we construct a Hopf algebra homomorphism $\mathcal{A}_{F_t} : \mathcal{H}_{GL}^W \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ such that $\mathcal{A}_{F_t}^{\times 5}(\Omega_{\mathbb{T}}^W) = \Omega_{F_t}$ (see Theorem 3.5 and 3.6). Furthermore, we also show in Proposition 3.9 that the specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ and $\mathcal{T}_W : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ of NCSF's are connected by $\mathcal{S}_{F_t} = \mathcal{T}_W \circ \mathcal{A}_{F_t}$. Note that, it has been shown in [Z6] that $\mathcal{S}_{F_t}$ is a graded Hopf algebra homomorphism from $\mathcal{N}Sym$ to the Hopf subalgebra $\mathcal{D}^{[\alpha]}\langle z\rangle \subset \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ iff $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ has the form $F_t(z) = t^{-1}F(tz)$ for some automorphism $F(z)$ of $K\langle\langle z\rangle\rangle$. By taking the graded duals of the results above, we get the corresponding commutative diagrams (see Proposition 3.12) for the Hopf algebras $\mathcal{Q}Sym$ of quasi-symmetric functions ([Ge], [MR], [St2]), the Connes-Kreimer Hopf algebra $\mathcal{H}_{CK}^W$ of W-labeled rooted forests ([CM], [Kr], [CK], [F1], [F2]), and the graded dual $\mathcal{D}^{[\alpha]}\langle z\rangle^*$ of differential operator algebra $\mathcal{D}^{[\alpha]}\langle z\rangle$ of $K\langle z\rangle$.

In the second part of this paper, by applying the Hopf algebra homomorphism $\mathcal{A}_{F_t} : \mathcal{H}_{GL}^W \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ described above, we first derive tree expansion formulas for the D-Log, the formal flow and the inverse map of F_t (see Theorem 4.1 and Corollaries 4.3, 4.4). Since the proofs given here do not depend on the commutativity of the free variables z, the formulas derived here can be viewed as some natural generalizations to the noncommutative case of the tree expansion formulas derived in [BCW], [Wr] and [WZ] in the commutative case. Finally,

we apply the Hopf algebra homomorphism $\mathcal{A}_{F_t} : \mathcal{H}_{GL}^W \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ above combining with some results already obtained in [Z6] and [Z8] to study more properties of the specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ and $\mathcal{T}_W : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ of NCSF's. In Theorem 5.1, we show that, when $W = \mathbb{N}^+$, the specialization $\mathcal{T}_W : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ actually embeds $\mathcal{N}Sym$ into $\mathcal{H}_{GL}^W$ as a graded K-Hopf subalgebra. By taking the graded duals, we get a surjective graded Hopf algebra homomorphism $\mathcal{T}_W^* : \mathcal{H}_{CK}^W \to \mathcal{Q}Sym$ from the Connes-Kreimer Hopf algebra $\mathcal{H}_{CK}^W$ ([CM], [Kr], [CK], [F1], [F2]) of W-labeled rooted forests onto the Hopf algebra $\mathcal{Q}Sym$ of quasi-symmetric functions (see Proposition 5.2). In Theorem 5.3, we show the family of the differential operator specializations $\mathcal{S}_{F_t}$ of NCSF's with all $n \geq 1$ and polynomial automorphisms $F_t = z - H_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ such that, $H_t(z)$ is homogeneous and the Jacobian matrix JH is strictly lower triangular, can distinguish any two different NCSF's.

Considering the length of paper, we give a more detailed arrangement of the paper as follows.

In Section 2, we mainly fix some necessary notations and recall some main results of [Z6] and [Z8] that will be needed throughout this paper. In Subsection 2.1, we recall the definition of general $\mathcal{N}$CS systems and the universal $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$ from NCSF's. In Subsection 2.2, we recall the $\mathcal{N}$CS systems Ω_{F_t} constructed in [Z6] over the differential operator algebras $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ and the resulted differential operator specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ of NCSF's. In Subsection 2.3, we recall the definition of the $\mathcal{N}$CS systems $\mathcal{H}_{GL}^W$ ($\emptyset \neq W \subseteq \mathbb{N}^+$) constructed in [Z8] over the Grossman-Larson Hopf algebra of W-labeled rooted trees. In Section 3, for any $F_t(z) = z - H_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ and any non-empty $W \subseteq \mathbb{N}^+$ such that $H_t(z) = \sum_{m\in W} t^m H_{[m]}(z)$ for some $H_{[m]}(z) \in K\langle\langle z\rangle\rangle^{\times n}$ ($m \in W$), we constructed a Hopf algebra homomorphisms $\mathcal{A}_{F_t} : \mathcal{H}_{GL}^W \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ such that $\mathcal{A}_{F_t}^{\times 5}(\Omega_{\mathbb{T}}^W) = \Omega_{F_t}$ (see Theorem 3.5 and 3.6). Furthermore, we also show in Proposition 3.9 that the specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ and $\mathcal{T}_W : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ of NCSF's are connected by $\mathcal{S}_{F_t} = \mathcal{T}_W \circ \mathcal{A}_{F_t}$. The graded dualized versions of the main results above are also discussed. In Section 4, by applying the Hopf algebra homomorphism $\mathcal{A}_{F_t} : \mathcal{H}_{GL}^W \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ constructed in the previous section, we derive tree expansion formulas for the D-Log, formal flow and the inverse map of F_t (see Theorems 4.1 and Corollaries 4.3, 4.4). In Section 5, we study more properties of the specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ and $\mathcal{T}_W : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ of NCSF's. First, we show in Theorem 5.1 that, when $W = \mathbb{N}^+$, the specialization $\mathcal{T}_W : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ is actually an injective graded K-Hopf

algebra homomorphism. By taking the graded duals, we get a surjective graded Hopf algebra homomorphism $\mathcal{T}_W^* : \mathcal{H}_{CK}^W \to \mathcal{Q}Sym$ (see Proposition 5.2). Secondly, we show in Theorem 5.3 that the family of the specializations $\mathcal{S}_{F_t}$ of NCSF's with all $n \geq 1$ and the polynomial automorphisms $F_t = z - H_t(z)$ with $H_t(z)$ homogeneous and the Jacobian matrix JH_t strictly lower triangular can distinguish any two different NCSF's.

2. $\mathcal{N}$CS Systems

Let K be any unital commutative $\mathbb{Q}$-algebra and A any unital associative but not necessarily commutative K-algebra. Let t be a formal central parameter, i.e. it commutes with all elements of A, and $A[[t]]$ the K-algebra of formal power series in t with coefficients in A. First let us recall the following notion formulated in [Z5].

Definition 2.1. *For any unital associative K-algebra A, a 5-tuple $\Omega = (f(t),\ g(t),\ d(t),\ h(t),\ m(t)) \in A[[t]]^{\times 5}$ is said to be a $\mathcal{N}CS$ (noncommutative symmetric) system over A if the following equations are satisfied.*

$$f(0) = 1 \tag{2.1}$$

$$f(-t)g(t) = g(t)f(-t) = 1, \tag{2.2}$$

$$e^{d(t)} = g(t), \tag{2.3}$$

$$\frac{dg(t)}{dt} = g(t)h(t), \tag{2.4}$$

$$\frac{dg(t)}{dt} = m(t)g(t). \tag{2.5}$$

When the base algebra K is clear in the context, we also call the ordered pair (A, Ω) a $\mathcal{N}CS$ *system*. Since $\mathcal{N}$CS systems often come from generating functions of certain elements of A that are under concern, the components of Ω will also be refereed as the *generating functions* of their coefficients.

Since all K-algebras that we are going to work on in this paper K-Hopf algebras ([A], [Knu], [Mo]), the following result proved in [Z5] later will be useful to our later arguments.

Proposition 2.2. *Let (A, Ω) be a $\mathcal{N}CS$ system as above. Suppose A is further a K-bialgebra. T*

(a) *The coefficients of $f(t)$ form a divided power series of A.*
(b) *The coefficients of $g(t)$ form a divided power series of A.*
(c) *One (hence also all) of $d(t)$, $h(t)$ and $m(t)$ has all its coefficients primitive in A.*

In this section, we briefly recall in Subsection 2.1 the $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$ formed by generating functions of some of the NCSF's defined in [GKLLRT] and its universal property (see Theorem 2.3). In Subsection 2.2, we recall the $\mathcal{N}$CS systems constructed in [Z6] over differential operator algebras. Finally, in Subsection 2.3, we recall the $\mathcal{N}$CS systems constructed in [Z8] over the Grossman-Larson Hopf algebra of labeled rooted trees.

2.1. **The Universal $\mathcal{N}$CS System from Noncommutative Symmetric Functions.** Let $\Lambda = \{\Lambda_m \,|\, m \geq 1\}$ be a sequence of noncommutative free variables and $\mathcal{N}Sym$ the free associative algebra generated by Λ over K. For convenience, we also set $\Lambda_0 = 1$. We denote by $\lambda(t)$ the generating function of Λ_m $(m \geq 0)$, i.e. we set

$$\lambda(t) := \sum_{m\geq 0} t^m \Lambda_m = 1 + \sum_{k\geq 1} t^m \Lambda_m. \tag{2.6}$$

In the theory of NCSF's ([GKLLRT]), Λ_m $(m \geq 0)$ is the noncommutative analog of the mth classical (commutative) elementary symmetric function and is called the mth (*noncommutative*) *elementary symmetric function.*

To define some other NCSF's, we consider Eqs. (2.2)–(2.5) over the free K-algebra $\mathcal{N}Sym$ with $f(t) = \lambda(t)$. The solutions for $g(t)$, $d(t)$, $h(t)$, $m(t)$ exist and are unique, whose coefficients will be the NCSF's that we are going to define. Following the notation in [GKLLRT] and [GKLLRT], we denote the resulted 5-tuple by

$$\Pi := (\lambda(t),\, \sigma(t),\, \Phi(t),\, \psi(t),\, \xi(t)) \tag{2.7}$$

and write the last four generating functions of Π explicitly as follows.

$$\sigma(t) = \sum_{m\geq 0} t^m S_m, \tag{2.8}$$

$$\Phi(t) = \sum_{m\geq 1} t^m \frac{\Phi_m}{m} \tag{2.9}$$

$$\psi(t) = \sum_{m\geq 1} t^{m-1} \Psi_m, \tag{2.10}$$

$$\xi(t) = \sum_{m\geq 1} t^{m-1} \Xi_m. \tag{2.11}$$

Following [GKLLRT], we call S_m $(m \geq 1)$ the mth (*noncommutative*) *complete homogeneous symmetric function* and Φ_m (resp. Ψ_m) the mth

power sum symmetric function of the second (*resp. first*) *kind*. Following [Z5], we call $\Xi_m \in \mathcal{N}Sym$ ($m \geq 1$) the mth (*noncommutative*) *power sum symmetric function of the third kind*.

Next, let us recall the following graded K-Hopf algebra structure of $\mathcal{N}Sym$. It has been shown in [GKLLRT] that $\mathcal{N}Sym$ is the universal enveloping algebra of the free Lie algebra generated by Ψ_m ($m \geq 1$). Hence, it has a Hopf K-algebra structure as all other universal enveloping algebras of Lie algebras do. Its co-unit $\epsilon : \mathcal{N}Sym \to K$, co-product Δ and antipode S are uniquely determined by

$$\epsilon(\Psi_m) = 0, \tag{2.12}$$

$$\Delta(\Psi_m) = 1 \otimes \Psi_m + \Psi_m \otimes 1, \tag{2.13}$$

$$S(\Psi_m) = -\Psi_m, \tag{2.14}$$

for any $m \geq 1$.

Next, we introduce the *weight* of NCSF's by setting the weight of any monomial $\Lambda_{m_1}^{i_1}\Lambda_{m_2}^{i_2}\cdots\Phi_{m_k}^{i_k}$ to be $\sum_{j=1}^{k} i_j m_j$. For any $m \geq 0$, we denote by $\mathcal{N}Sym_{[m]}$ the vector subspace of $\mathcal{N}Sym$ spanned by the monomials of Λ of weight m. Then it is easy to see that

$$\mathcal{N}Sym = \bigoplus_{m\geq 0} \mathcal{N}Sym_{[m]}, \tag{2.15}$$

which provides a grading for $\mathcal{N}Sym$.

Note that, it has been shown in [GKLLRT], for any $m \geq 1$, the NCSF's $S_m, \Phi_m, \Psi_m \in \mathcal{N}Sym_{[m]}$. By the facts above and Eqs. (2.12)–(2.14), it is also easy to check that, with the grading given in Eq. (2.15), $\mathcal{N}Sym$ forms a graded K-Hopf algebra. Its graded dual is given by the space $\mathcal{Q}Sym$ of quasi-symmetric functions, which were first introduced by I. Gessel [Ge] (also see [MR] and [St2] for more discussions).

Now we come back to our discussions on the $\mathcal{N}$CS systems. From the definitions of the NCSF's above, we see that $(\mathcal{N}Sym, \Pi)$ obviously forms a $\mathcal{N}$CS system. More importantly, as shown in Theorem 2.1 in [Z5], we have the following important theorem on the $\mathcal{N}$CS system $(\mathcal{N}Sym, \Pi)$.

Theorem 2.3. *Let A be a K-algebra and Ω a $\mathcal{N}CS$ system over A. Then,*

(*a*) *There exists a unique K-algebra homomorphism $\mathcal{S} : \mathcal{N}Sym \to A$ such that $\mathcal{S}^{\times 5}(\Pi) = \Omega$.*

(*b*) *If A is further a K-bialgebra* (*resp. K-Hopf algebra*) *and one of the equivalent statements in Proposition 2.2 holds for the $\mathcal{N}CS$ system Ω, then $\mathcal{S} : \mathcal{N}Sym \to A$ is also a homomorphism of K-bialgebras* (*resp. K-Hopf algebras*).

Remark 2.4. *By applying the similar arguments as in the proof of Theorem 2.3, or simply taking the quotient over the two-sided ideal generated by the commutators of Λ_m's, it is easy to see that, over the category of commutative K-algebras, the universal $\mathcal{N}CS$ system is given by the generating functions of the corresponding classical* (*commutative*) *symmetric functions* [Ma].

2.2. **$\mathcal{N}$CS Systems over Differential Operator Algebras.** In this subsection, we briefly recall the $\mathcal{N}$CS systems constructed in [Z6] over the differential operator algebras in commutative or noncommutative free variables. The construction of this $\mathcal{N}$CS is mainly motivated by the studies in [Z3] and [Z4] on the deformations of formal analytic maps and their applications to the inversion problem ([BCW], [E4], [Z7]).

First, let us fix the following notation.

Let K be any unital commutative $\mathbb{Q}$-algebra as before and $z = (z_1, z_2, ..., z_n)$ commutative or noncommutative free variables.[1] Let t be a formal central parameter, i.e. it commutes with z and elements of K. We denote by $K\langle\langle z\rangle\rangle$ and $K[[t]]\langle\langle z\rangle\rangle$ the K-algebras of formal power series in z over K and $K[[t]]$, respectively. We denote by $\mathcal{D}er_K\langle\langle z\rangle\rangle$ or $\mathcal{D}er\langle\langle z\rangle\rangle$, when the base algebra K is clear from the context, the set of all K-derivations of $K\langle\langle z\rangle\rangle$. The unital subalgebra of $\mathrm{End}_k(K\langle\langle z\rangle\rangle)$ generated by all K-derivations of $K\langle\langle z\rangle\rangle$ will be denoted by $\mathcal{D}_K\langle\langle z\rangle\rangle$ or $\mathcal{D}\langle\langle z\rangle\rangle$. Elements of $\mathcal{D}_K\langle\langle z\rangle\rangle$ will be called (*formal*) *differential operators* in the commutative and noncommutative variables z.

For any $\alpha \geq 1$, we denote by $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ the set of the K-derivations of $K\langle\langle z\rangle\rangle$ which increase the degree in z by at least $\alpha - 1$. The unital subalgebra of $\mathcal{D}\langle\langle z\rangle\rangle$ generated by elements of $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ will be denoted by $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Note that, by the definitions above, the operators of scalar multiplications are also in $\mathcal{D}\langle\langle z\rangle\rangle$ and $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. When the base algebra is $K[[t]]$ instead of K itself, the notation $\mathcal{D}er\langle\langle z\rangle\rangle$, $\mathcal{D}\langle\langle z\rangle\rangle$, $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ and $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ will be denoted by $\mathcal{D}er_t\langle\langle z\rangle\rangle$, $\mathcal{D}_t\langle\langle z\rangle\rangle$, $\mathcal{D}er_t^{[\alpha]}\langle\langle z\rangle\rangle$ and $\mathcal{D}_t^{[\alpha]}\langle\langle z\rangle\rangle$, respectively. For example, $\mathcal{D}er_t^{[\alpha]}\langle\langle z\rangle\rangle$ stands for the set of all $K[[t]]$-derivations of $K[[t]]\langle\langle z\rangle\rangle$ which increase the degree in z by at least $\alpha - 1$. Note that, $\mathcal{D}er_t^{[\alpha]}\langle\langle z\rangle\rangle = \mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle[[t]]$ and $\mathcal{D}_t^{[\alpha]}\langle\langle z\rangle\rangle = \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[[t]]$.

[1]Since most of the results as well as their proofs in this paper do not depend on the commutativity of the free variables z, we will not distinguish the commutative and the noncommutative case, unless stated otherwise, and adapt the notations for noncommutative variables uniformly for the both cases.

For any $1 \leq i \leq n$ and $u(z) \in K\langle\langle z\rangle\rangle$, we denote by $\left[u(z)\frac{\partial}{\partial z_i}\right]$ the K-derivation which maps z_i to $u(z)$ and z_j to 0 for any $j \neq i$. For any $\vec{u} = (u_1, u_2, \cdots, u_n) \in K\langle\langle z\rangle\rangle^{\times n}$, we set

$$[\vec{u}\frac{\partial}{\partial z}] := \sum_{i=1}^{n}[u_i\frac{\partial}{\partial z_i}]. \tag{2.16}$$

Note that, in the noncommutative case, we in general do **not** have $\left[u(z)\frac{\partial}{\partial z_i}\right] g(z) = u(z)\frac{\partial g}{\partial z_i}$ for all $u(z), g(z) \in K\langle\langle z\rangle\rangle$. This is the reason that we put a bracket $[\cdot]$ in the notation above for the K-derivations. With the notation above, it is easy to see that any K-derivations δ of $K\langle\langle z\rangle\rangle$ can be written uniquely as $\sum_{i=1}^{n}\left[f_i(z)\frac{\partial}{\partial z_i}\right]$ with $f_i(z) = \delta z_i \in K\langle\langle z\rangle\rangle$ $(1 \leq i \leq n)$.

Note that, the differential operator algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ $(\alpha \geq 1)$, as the universal enveloping algebra of Lie algebra $\mathcal{D}er^{[\alpha]}\langle\langle z\rangle\rangle$ with the commutator bracket, has a Hopf algebra structure as all other enveloping algebras of Lie algebras do. In particular, Its coproduct Δ, antipode S and co-unit ϵ are uniquely determined respectively by the properties

$$\Delta(\delta) = 1 \otimes \delta + \delta \otimes 1, \tag{2.17}$$

$$S(\delta) = -\delta, \tag{2.18}$$

$$\epsilon(\delta) = \delta \cdot 1, \tag{2.19}$$

for any $\delta \in \mathcal{D}er\langle\langle z\rangle\rangle$.

For any $\alpha \geq 1$, let $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ be the set of all the automorphism $F_t(z)$ of $K[[t]]\langle\langle z\rangle\rangle$ over $K[[t]]$, which have the form $F(z) = z - H_t(z)$ for some $H_t(z) \in K[[t]]\langle\langle z\rangle\rangle^{\times n}$ with $o(H_t(z)) \geq \alpha$ and $H_{t=0}(z) = 0$. Note that, for any $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ as above, its inverse map $G_t := F_t^{-1}$ can always be written uniquely as $G_t(z) = z + M_t(z)$ for some $M_t(z) \in K[[t]]\langle\langle z\rangle\rangle^{\times n}$ with $o(M_t(z)) \geq \alpha$ and $M_{t=0}(z) = 0$.

Now we recall the $\mathcal{N}$CS systems constructed in [Z6] over the differential operator algebras $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ $(\alpha \geq 1)$.

We fix an $\alpha \geq 1$ and an arbitrary $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$. We will always let $H_t(z)$, $G_t(z)$ and $M_t(z)$ be determined as above. The $\mathcal{N}$CS system

$$\Omega_{F_t} = (f(t),\, g(t),\, d(t),\, h(t),\, m(t)) \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[[t]]^{\times 5}. \tag{2.20}$$

are determined as follows.

The last two components are given directly as

$$h(t) := \left[\frac{\partial M_t}{\partial t}(F_t)\frac{\partial}{\partial z}\right], \tag{2.21}$$

$$m(t) := \left[\frac{\partial H_t}{\partial t}(G_t)\frac{\partial}{\partial z}\right]. \tag{2.22}$$

The first three components are given by the following proposition which was proved in Section 3.2 in [Z6].

Proposition 2.5. *There exist unique $f(t), g(t), d(t) \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle[[t]]$ with $f(0) = 1$ and $d(0) = 0$ such that, for any $u_t(z) \in K[[t]]\langle\langle z\rangle\rangle$, we have*

$$f(-t)\, u_t(z) = u_t(F_t), \tag{2.23}$$

$$g(t)\, u_t(z) = u_t(G_t), \tag{2.24}$$

$$e^{d(t)}\, u_t(z) = u_t(G_t), \tag{2.25}$$

where, as usual, the exponential in Eq. (2.25) is given by

$$e^{d(t)} = \sum_{m\geq 0} \frac{d(t)^m}{m!}. \tag{2.26}$$

Note that, when we write $d(t)$ above as $d(t) = -\left[a_t(z)\frac{\partial}{\partial z}\right]$ for some $a_t(z) \in tK[[t]]\langle\langle z\rangle\rangle$, then we get the so-called *D-Log* $a_t(z)$ of the automorphism $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, which has been studied in [E1]-[E3], [Z1] and [WZ] for the commutative case.

Theorem 2.6. ([Z6]) *For any $\alpha \geq 1$ and $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, we have,*

(a) the 5-tuple Ω_{F_t} defined in Eq. (2.20) forms a $\mathcal{N}CS$ system over the K-algebra $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$.

(b) there exists a unique homomorphism $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ of K-Hopf algebras such that $\mathcal{S}_{F_t}^{\times 5}(\Pi) = \Omega_{F_t}$.

2.3. **A $\mathcal{N}$CS System over the Grossman-Larson Hopf Algebra of Labeled Rooted Trees.** In this subsection, we recall the $\mathcal{N}$CS system $(\mathcal{H}_{GL}^W, \Omega_{\mathbb{T}}^W)$ constructed in [Z8] over the Grossman-Larson Hopf algebra $\mathcal{H}_{GL}^W$ of rooted trees labeled by positive integers of an non-empty $W \subseteq \mathbb{N}^+$. First, let us fix the following notation which will be used throughout the rest of this paper.

Notation:

By a *rooted tree* we mean a finite 1-connected graph with one vertex designated as its *root.* For convenience, we also view the empty set $\emptyset$ as a rooted tree and call it the *emptyset* rooted tree. The rooted tree with a single vertex is called the *singleton* and denoted by $\circ$. There are natural ancestral relations between vertices. We say a vertex w is a *child* of vertex v if the two are connected by an edge and w lies further

from the root than v. In the same situation, we say v is the *parent* of w. A vertex is called a *leaf* if it has no children.

Let $W \subseteq \mathbb{N}^+$ be a non-empty subset of positive integers. A W-*labeled rooted tree* is a rooted tree with each vertex labeled by an element of W. If an element $m \in W$ is assigned to a vertex v, then m is called the *weight* of the vertex v. When we speak of isomorphisms between unlabeled (resp. W-labeled) rooted trees, we will always mean isomorphisms which also preserve the root (resp. the root and also the labels of vertices). We will denote by $\mathbb{T}$ (resp. $\mathbb{T}^W$) the set of isomorphism classes of all unlabeled (resp. W-labeled) rooted trees. A disjoint union of any finitely many rooted trees (resp. W-labeled rooted trees) is called a *rooted forest* (resp. W-labeled *rooted forest*). We denote by $\mathbb{F}$ (resp. $\mathbb{F}^W$) the set of unlabeled (resp. W-labeled) rooted forests.

With these notions in mind, we establish the following notation.

(1) For any rooted tree $T \in \mathbb{T}^W$, we set the following notation:
- rt_T denotes the root vertex of T and $O(T)$ the set of all the children of rt_T. We set $o(T) = |O(T)|$ (the cardinal number of the set $O(T)$).
- $E(T)$ denotes the set of edges of T.
- $V(T)$ denotes the set of vertices of T and $v(T) = |V(T)|$.
- $L(T)$ denotes the set of leaves of T and $l(T) = |L(T)|$
- For any $v \in V(T)$ define the *height* of v to be the number of edges in the (unique) geodesic connecting v to rt_T. The *height* of T is defined to be the maximum of the heights of its vertices.
- For any $T \in \mathbb{T}^W$ and $T \neq \emptyset$, $|T|$ denotes the sum of the weights of all vertices of T. When $T = \emptyset$, we set $|T| = 0$.
- For any $T \in \mathbb{T}^W$, we denote by $\mathrm{Aut}(T)$ the automorphism group of T and $\alpha(T)$ the cardinal number of $\mathrm{Aut}(T)$.

(2) Any subset of $E(T)$ is called a *cut* of T. A cut $C \subseteq E(T)$ is said to be *admissible* if no two different edges of C lie in the path connecting the root and a leaf. We denote by $\mathcal{C}(T)$ the set of all admissible cuts of T. Note that, the empty subset $\emptyset$ of $E(T)$ and $C = \{e\}$ for any $e \in E(T)$ are always admissible cuts. We will identify any edge $e \in E(T)$ with the admissible cut $C := \{e\}$ and simply say the edge e itself is an admissible cut of T.

(3) For any $T \in \mathbb{T}^W$ with $T \neq \circ$, let $C \in \mathcal{C}(T)$ be an admissible cut of T with $|C| = m \geq 1$. Note that, after deleting the edges in C from T, we get a disjoint union of $m+1$ rooted trees, say T_0,

T_1, ..., T_m with $\mathrm{rt}(T) \in V(T_0)$. We define $R_C(T) = T_0 \in \mathbb{T}^W$ and $P_C(T) \in \mathbb{F}^W$ the rooted forest formed by T_1, ..., T_m.

(4) For any disjoint admissible cuts C_1 and C_2, we say "C_1 lies above C_2", and write $C_1 \succ C_2$, if $C_2 \subseteq E(R_{C_1}(T))$. This merely says that all edges of C_2 remain when we remove all edges of C_1 and $P_{C_1}(T)$. Note that this relation is not transitive. When we write $C_1 \succ \cdots \succ C_r$ for $C_1, \ldots, C_r \in \mathcal{C}(T)$, we will mean that $C_i \succ C_j$ whenever $i < j$.

(5) For any $T \in \mathbb{T}^W$, we say T is a *chain* if its underlying rooted tree is a rooted tree with a single leaf. We say T is a *shrub* if its underlying rooted tree is a rooted tree of height 1. We say T is *primitive* if its root has only one child. For any $m \geq 1$, we set $\mathbb{H}_m$, $\mathbb{S}_m$ and $\mathbb{P}_m$ to be the sets of the chains, shrubs and primitive rooted trees T of weight $|T| = m$, respectively. $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$ are set to be the unions of $\mathbb{H}_m$, $\mathbb{S}_m$ and $\mathbb{P}_m$, respectively, for all $m \geq 1$.

Let K be any unital commutative $\mathbb{Q}$-algebra and W a non-empty subset of positive integers. First, let us recall the Connes-Kreimer Hopf algebras $\mathcal{H}_{CK}^W$ of labeled rooted forests.

As a K-algebra, the Connes-Kreimer Hopf algebra $\mathcal{H}_{CK}^W$ is the free commutative algebra generated by formal variables $\{X_T \,|\, T \in \mathbb{T}^W\}$. Here, for convenience, we will still use T to denote the variable X_T in $\mathcal{H}_{CK}^W$. The K-algebra product is given by the disjoint union. The identity element of this algebra, denoted by 1, is the free variable $X_\emptyset$ corresponding to the emptyset rooted tree. The coproduct $\Delta : \mathcal{H}_{CK}^W \to \mathcal{H}_{CK}^W \otimes \mathcal{H}_{CK}^W$ is uniquely determined by setting

$$\Delta(1) = 1 \otimes 1, \tag{2.27}$$

$$\Delta(T) = T \otimes 1 + \sum_{C \in \mathcal{C}(T)} P_C(T) \otimes R_C(T). \tag{2.28}$$

The co-unit $\epsilon : \mathcal{H}_{CK}^W \to K$ is the K-algebra homomorphism which sends $1 \in \mathcal{H}_{CK}^W$ to $1 \in K$ and T to 0 for any $T \in \mathbb{T}^W$ with $T \neq \emptyset$. With the operations defined above and the grading given by the weight, the vector space $\mathcal{H}_{CK}^W$ forms a graded commutative bi-algebra, hence there is a unique antipode $S : \mathcal{H}_{CK}^W \to \mathcal{H}_{CK}^W$ that makes $\mathcal{H}_{CK}^W$ a Hopf algebra.

Next we recall the Grossman-Larson Hopf algebra of labeled rooted trees. First we need define the following operations for labeled rooted forests. For any labeled rooted forest F which is disjoint union of labeled rooted trees T_1, T_2, ... , T_m, we set $B_+(T_1, T_2, \cdots, T_m)$ the rooted tree obtained by connecting roots of T_i $(1 \leq i \leq m)$ to a newly added

root. We will keep the labels for the vertices of $B_+(T_1, T_2, \cdots, T_m)$ from T_i's, but for the root, we label it by 0.

Furthermore, we fix the following convention for the operation B_+. First, for any $T_i \in \mathbb{T}^W$ $(1 \leq i \leq m)$ and $j_i \geq 1$, the notation $B_+(T_1^{j_1}, T_2^{j_2}, \cdots, T_m^{j_m})$ denotes the rooted tree obtained by applying the operation B_+ to j_1-copies of T_1; j_2-copies of T_2; and so on. Secondly, for any $m \geq 1$, we will extend the operation B_+ multi-linearly to a linear map B_+ from $\left(\mathcal{H}_{CK}^W\right)^{\times m}$ to the vector spaces spanned by the resulted rooted trees.

Now, we set $\bar{\mathbb{T}}^W := \{B_+(F) \,|\, F \in \mathbb{F}^W\}$. Then, $B_+ : \mathbb{F}^W \to \bar{\mathbb{T}}^W$ becomes a bijection. We denote by $B_- : \bar{\mathbb{T}}^W \to \mathbb{F}^W$ the inverse map of B_+. More precisely, for any $T \in \bar{\mathbb{T}}^W$, $B_-(T)$ is the W-labeled rooted forest obtained by cutting off the root of T as well as all edges connecting to the root in T.

Note that, precisely speaking, elements of $\bar{\mathbb{T}}^W$ are not W-labeled trees for $0 \notin W$. But, if we set $\bar{W} = W \cup \{0\}$, then we can view $\bar{\mathbb{T}}^W$ as a subset of $\bar{W}$-labeled rooted trees T with the root rt_T labeled by 0 and all other vertices labeled by non-zero elements of $\bar{W}$. We extend the definition of the weight for elements of $\mathbb{F}^W$ to elements of $\bar{\mathbb{T}}^W$ by simply counting the weight of roots by zero. We set $\bar{\mathbb{S}}_m^W := B_+(\mathbb{S}_m^W)$ $(m \geq 1)$ and $\bar{\mathbb{S}}^W := B_+(\mathbb{S}^W)$. We also define $\bar{\mathbb{H}}_m^W$, $\bar{\mathbb{P}}_m^W$, $\bar{\mathbb{H}}^W$ and $\bar{\mathbb{P}}^W$ in the similar way.

The Grossman-Larson Hopf algebra $\mathcal{H}_{GL}^W$ as a vector space is the vector space spanned by elements of $\bar{\mathbb{T}}^W$ over K. For any $T \in \bar{\mathbb{T}}^W$, we will still denote by T the vector in $\mathcal{H}_{GL}^W$ that is corresponding to T. The algebra product is defined as follows. For any $T, S \in \bar{\mathbb{T}}^W$ with $T = B_+(T_1, T_2, \cdots, T_m)$, we set $T \cdot S$ to be the sum of the rooted trees obtained by connecting the roots of T_i $(1 \leq i \leq m)$ to vertices of S in all possible $m^{v(S)}$ different ways. Note that, the identity element with respect to this algebra product is given by the singleton $\circ = B_+(\emptyset)$. But we will denote it by 1.

To define the co-product $\Delta : \mathcal{H}_{GL}^W \to \mathcal{H}_{GL}^W \otimes \mathcal{H}_{GL}^W$, we first set

$$\Delta(\circ) = \circ \otimes \circ. \tag{2.29}$$

Now let $T \in \bar{\mathbb{T}}^W$ with $T \neq \circ$, say $T = B_+(T_1, T_2, \cdots, T_m)$ with $m \geq 1$ and $T_i \in \mathbb{T}^W$ $(1 \leq i \leq m)$. For any non-empty subset $I \subseteq \{1, 2, \cdots, m\}$, we denote by $B_+(T_I)$ the rooted tree obtained by applying the B_+ operation to the rooted trees T_i with $i \in I$. For convenience, when $I = \emptyset$, we set $B_+(T_I) = 1$. With these notation fixed,

the co-product for T is given by

$$\Delta(T) = \sum_{I\sqcup J=\{1,2,\cdots,m\}} B_+(T_I)\otimes B_+(T_J). \tag{2.30}$$

Note that, a rooted tree in $\bar{\mathbb{T}}^W$ is a primitive element of the Hopf algebra $\mathcal{H}^W_{GL}$ iff it is a primitive rooted tree in the sense that we defined before, namely the root of T has one and only one child.

Remark 2.7. *Note that, for any $S\in\bar{\mathbb{T}}^W$ and $T\in\mathbb{F}^W$, we also can define a "product" still denoted by $S\cdot T$ in the exact same way as we define the product of elements of $\bar{\mathbb{T}}^W$. By the linear extention, this "product" makes $\mathcal{H}^W_{CK}$ a K-algebra module of $\mathcal{H}^W_{GL}$.*

The following results later will be very useful in our later arguments.

Theorem 2.8. (a) *The Hopf algebras $\mathcal{H}^W_{GL}$ and $\mathcal{H}^W_{CK}$ are graded dual to each other. The pairing is given by, for any $T\in\bar{\mathbb{T}}^W$ and $S\in\mathbb{F}^W$,*

$$< T, F >= \begin{cases} 0, & \text{if } T\not\simeq B_+(F),\\ \alpha(T), & \text{if } T\simeq B_+(F).\end{cases} \tag{2.31}$$

(b) *$\mathcal{H}^W_{GL}$ as a Hopf algebra is isomorphic to the universal enveloping algebra of the Lie algebra formed by its primitive elements, which are exactly linear combinations of the primitive rooted trees. In particular, $\mathcal{H}^W_{GL}$ as an K-algebra is generated by the primitive rooted trees.*

For a proof of (a), see [H] and [F2]. (b) follows directly from the well-known Milnor-Moore's Theorem ([MM]), since $\mathcal{H}^W_{GL}$ is a connected graded and cocommutative Hopf algebra.

Next, let us recall the following lemma proved in [Z8] that will be crucial for our later arguments.

Let $\vec{C}=(C_1,\ldots,C_r)\in\mathcal{C}(T)^{\times r}$ be a sequence of admissible cuts with $C_1\succ\cdots\succ C_r$. We define a sequence of $T_{\vec{C},1},\ldots,T_{\vec{C},r+1}\in\bar{\mathbb{T}}^W$ as follows: we first set $T_{\vec{C},1}=B_+(P_{C_1}(T))$ and let $S_1=R_{C_1}(T)$. Note that $C_2,\ldots,C_r\in\mathcal{C}(S_1)$. We then set $T_{\vec{C},2}=B_+(P_{C_2}(S_1))$ and $S_2=R_{C_2}(S_1)$ and repeat this procedure until we get $S_r=R_{C_r}(S_{r-1})$ and then set $T_{\vec{C},r+1}=S_r$. In the case that, each C_i $(1\le i\le r)$ consists of a single edge, say $e_i\in E(T)$, we simply denote $T_{\vec{C},i}$ by T_{e_i}.

Lemma 2.9. *For any $r\ge 1$, $y=\{y^{(i)}_T\,|\,1\le i\le r;\,,T\in\bar{\mathbb{T}}^W\}$ be a collection of commutative formal variables. Then, we have,*

$$\sum_{(T_1,\ldots,T_r)\in(\bar{\mathbb{T}}^W)^r}\left[y^{(1)}_{T_1}\mathcal{V}_{T_1}\right]\cdots\left[y^{(r)}_{T_r}\mathcal{V}_{T_r}\right] \tag{2.32}$$

$$= \sum_{T\in\bar{\mathbb{T}}^W} \sum_{\substack{\vec{C}=(C_1,\dots,C_r)\in\mathcal{C}(T)^r \\ C_1\succ\cdots\succ C_r}} y^{(1)}_{T_{\vec{C},1}}\cdots y^{(r)}_{T_{\vec{C},r}}\mathcal{V}_T.$$

Now we recall the $\mathcal{NCS}$ system $\Omega^W_{\mathbb{T}}$ constructed in [Z8] over the Grossman-Larson Hopf algebra $\mathcal{H}^W_{GL}$.

First, let us define the following constants for the rooted trees in $\bar{\mathbb{T}}^W$:

- We set β_T to be the weight of the unique leaf of T if $T\in\bar{\mathbb{H}}^W$ and 0 otherwise.
- We set γ_T to be the weight of the unique child of the root of T if $T\in\bar{\mathbb{P}}^W$ and 0 otherwise.
- We set θ_T to be the coefficient of s of the order polynomial $\Omega(B_-(T),s)$ of the underlying unlabeled rooted forest of $B_-(T)$.

For general studies on the order polynomials $\Omega(P,s)$ of finite posets P, see [St1]. For an interpretation of the constant $\phi_T=(-1)^{v(T)-1}\varphi_T=(-1)^{v(T)-1}\theta_{B_+(T)}$ in terms of the numbers of chains with fixed lengths in the lattice of the ideals of the poset T, see Lemma 2.8 in [SWZ].

The following results[2] on θ_T ($T\in\bar{\mathbb{T}}^W$) proved in Section 5 in [Z8] will be needed later.

Proposition 2.10. (1) *For the singleton $\circ$ and any non-primitive rooted tree $T\in\mathbb{T}$, i.e. $o(T)>1$, we set $\theta_\circ=\theta_T=0$.*

(2) *For $T=B_+(\circ)$, we set $\theta_T=1$.*

(3) *For any primitive $T\in\mathbb{P}$ with $v(T)\geq 3$, we define θ_T inductively by*

$$\theta_T = 1-\sum_{m\geq 2}\frac{1}{m!}\sum_{\substack{\vec{e}=(e_1,\dots,e_{m-1})\in E(T)^{m-1} \\ e_1\succ\cdots\succ e_{m-1}}}\theta_{T_{e_1}}\theta_{T_{e_2}}\cdots\theta_{T_{e_m}}. \tag{2.33}$$

We will also need the following proposition proved in [Z8] for the constants θ_T ($T\in\mathbb{T}$).

Proposition 2.11. *For any $T\in\mathbb{P}$, we have*

(2.34)

$$\nabla\Omega(T,s)=\theta_T\,s+\sum_{k=2}^{v(T)}\frac{s^k}{k!}\sum_{\substack{\vec{e}=(e_1,\dots,e_{k-1})\in E(T)^{k-1} \\ e_1\succ\cdots\succ e_{k-1}}}\theta_{B_-(T_{\vec{e},1})}\cdots\theta_{B_-(T_{\vec{e},k-1})}\theta_{T_{\vec{e},k}},$$

where $\nabla: K[s]\to K[s]$ is the linear operator that maps any $f(s)\in K[s]$ to $f(s)-f(s-1)$.

[2]Note that, the approach to θ_T's in [Z8] is different from the one we adapt here. But it was shown there that the constants determined by the properties in Proposition 2.10 are same as the θ_T's we defined here.

Now we consider the following generating functions of $T \in \bar{\mathbb{T}}^W$.

$$\tilde{f}(t) := \sum_{T\in\bar{\mathbb{S}}^W} (-1)^{o(T)} t^{|T|}\mathcal{V}_T = 1 + \sum_{\substack{T\in\bar{\mathbb{S}}^W\\ T\neq\circ}} (-1)^{o(T)} t^{|T|}\mathcal{V}_T, \tag{2.35}$$

$$\tilde{g}(t) := \sum_{T\in\bar{\mathbb{T}}^W} t^{|T|}\mathcal{V}_T = 1 + \sum_{\substack{T\in\bar{\mathbb{T}}^W\\ T\neq\circ}} t^{|T|}\mathcal{V}_T, \tag{2.36}$$

$$\tilde{d}(t) := \sum_{T\in\bar{\mathbb{P}}^W} t^{|T|}\theta_T\mathcal{V}_T. \tag{2.37}$$

$$\tilde{h}(t) := \sum_{T\in\bar{\mathbb{H}}^W} t^{|T|-1}\beta_T\mathcal{V}_T, \tag{2.38}$$

$$\widetilde{m}(t) := \sum_{T\in\bar{\mathbb{P}}^W} t^{|T|-1}\gamma_T\mathcal{V}_T, \tag{2.39}$$

where, for any $T \in \bar{\mathbb{T}}^W$, $\mathcal{V}_T := \frac{1}{\alpha(T)}T$. We further set

$$\Omega_{\mathbb{T}}^W := (\,\tilde{f}(t),\, \tilde{g}(t),\, \tilde{d}\,(t),\, \tilde{h}(t), \widetilde{m}(t)\,). \tag{2.40}$$

Theorem 2.12. ([Z8]) *For any non-empty set $W \subseteq \mathbb{N}^+$, we have*

(a) the 5-tuple $\tilde{\Omega}_{F_t}$ defined in Eq. (2.40) forms a $\mathcal{N}CS$ system over the Grossman-Larson Hopf algebra $\mathcal{H}_{GL}^W$.

(b) there exists a unique homomorphism $\mathcal{T}_W : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ of graded K-Hopf algebras such that $\mathcal{T}_W^{\times 5}(\Pi) = \tilde{\Omega}_{F_t}$.

3. A Hopf Algebra Homomorphisms from $\mathcal{H}_{GL}^W$ to $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$

In this section, for any non-empty $W \subseteq \mathbb{N}^+$, $\alpha \geq 1$ and $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ satisfying Eq. (3.5) below, we construct a K-Hopf algebra homomorphism $\mathcal{A} : \mathcal{H}_{GL}^W \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ such that $\mathcal{A}^{\times 5}$ maps the $\mathcal{N}$CS system $\Omega_{\mathbb{T}}^W$ in Theorem 2.12 to the $\mathcal{N}$CS system Ω_{F_t} in Theorem 2.6.

Let K, z and t be as given in Subsection 2.2. We will also use the notation fixed in Sections 3 freely throughout this section.

Let us start with the introducing of the following two operations for the K-derivations of $K\langle\langle z\rangle\rangle$.

First, for any $\phi, \delta \in \mathcal{D}er\langle\langle z\rangle\rangle$ with $\delta = \left[f(z)\frac{\partial}{\partial z}\right]$, we set

$$\phi \triangleright \delta := \left[(\phi f)(z)\frac{\partial}{\partial z}\right]. \tag{3.1}$$

To define the second operation, let $w = (w_1, w_2, \cdots, w_n)$ be another n free variables which are independent with the free variables z. For

any K-derivations $\delta_i = \left[\vec{v}_i(z)\frac{\partial}{\partial z}\right]$ $(1 \leq i \leq m)$ with $\vec{v}_i(z) \in K\langle\langle z\rangle\rangle^{\times n}$, we define $B_+(\delta_1, \delta_2, \cdots, \delta_m)$ by setting, for any $u(z) \in K\langle\langle z\rangle\rangle$,

$$B_+(\delta_1, \delta_2, \cdots, \delta_m)u(z) := \left[\vec{v}_1(w)\frac{\partial}{\partial z}\right]\left[\vec{v}_2(w)\frac{\partial}{\partial z}\right]\cdots\left[\vec{v}_m(w)\frac{\partial}{\partial z}\right]u(z)\Bigg|_{w=z}. \tag{3.2}$$

Furthermore, for any $k_i \geq 0$ $(1 \leq i \leq m)$, we let $B_+(\delta_1^{k_1}, \delta_2^{k_2}, \cdots, \delta_m^{k_m})$ denote the operator obtained by applying B_+ to the multi-set of j_1-copies of δ_1; j_2-copies of δ_2, ..., j_m-copies of δ_m.

Note that $B_+(\delta_1, \delta_2, \cdots, \delta_m)$ is multi-linear and symmetric on δ_i $(1 \leq i \leq m)$. When $m = 1$, $B_+(\delta_1) = \delta_1$.

The following two lemmas have been proved in Section 3 in [Z6].

Lemma 3.1. *(a) Let $\delta_i \in \mathcal{D}er\langle\langle z\rangle\rangle$ $(1 \leq i \leq m)$. Then, for any $\phi \in \mathcal{D}\langle\langle z\rangle\rangle$, we have*

$$\phi \cdot B_+(\delta_1, \delta_2, \cdots, \delta_m) = B_+(\phi, \delta_1, \delta_2, \cdots, \delta_m) + \sum_{i=1}^{m} B_+(\delta_1, \cdots, \phi \triangleright \delta_i, \cdots \delta_m). \tag{3.3}$$

(b) For any $\delta_i \in \mathcal{D}er\langle\langle z\rangle\rangle$ $(1 \leq i \leq m)$, $B_+(\delta_1, \delta_2, \cdots, \delta_m) \in \mathcal{D}\langle\langle z\rangle\rangle$.

Lemma 3.2. *In terms of the B_+ operation defined above, $f(t) \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ defined in Proposition (2.5) is given by*

$$f(t) = \sum_{k\geq 0} \frac{(-1)^k}{k!} B_+\left(\left[H_t(z)\frac{\partial}{\partial z}\right]^k\right). \tag{3.4}$$

Now, we fix a non-empty subset $W \subseteq \mathbb{N}^+$ and $\alpha \geq 1$. Let $F_t(z) = z - H_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ such that

$$H_t(z) = \sum_{m\in W} t^m H_{[m]}(z), \tag{3.5}$$

for some $H_{[m]}(z) \in K\langle\langle z\rangle\rangle^{\times n}$ $(m \in W)$.

First, we assign $P_T(z) \in K\langle\langle z\rangle\rangle^{\times n}$ for each $T \in \mathbb{T}^W$ inductively as follows.

(1) For the singleton $\circ$ labeled by $m \in W$, denoted by $\circ_m$, we set $P_{\circ_m}(z) := H_{[m]}(z)$.
(2) For any non-singleton $T \in \mathbb{T}^W$ with rt_T labeled by $m \in W$, write $T = B_+(T_1, T_2, ..., T_d)$ with $T_i \in \mathbb{T}^W$ $(1 \leq i \leq d)$ and

$$P_T(z) := B_+\left(\left[P_{T_1}(z)\frac{\partial}{\partial z}\right], ..., \left[P_{T_m}(z)\frac{\partial}{\partial z}\right]\right) H_{[m]}(z). \tag{3.6}$$

Let $\mathcal{H}_{CK}^W[1]$ be the vector subspace of $\mathcal{H}_{CK}^W$ spanned by $T \in \mathbb{T}^W$. We define a linear map $\mathcal{U}_{F_t} : \mathcal{H}_C^W[1] \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ by setting

$$\begin{aligned} \mathcal{U}_{F_t} : \mathcal{H}_{CK}^W[1] &\to K\langle\langle z\rangle\rangle^{\times n} \\ T &\to P_T(z). \end{aligned} \tag{3.7}$$

Next we assign $D_T(z) \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ for each $T \in \bar{\mathbb{T}}^W = B_+(\mathbb{T}^W)$ as follows.

(1) For the singleton $T = \circ$, we set $D_T = id$.
(2) For any rooted tree $T = B_+(T_1, T_2, \cdots, T_m)$ with $T_i \in \mathbb{T}^W$, we set

$$D_T(z) := B_+\left(\left[P_{T_1}(z)\frac{\partial}{\partial z}\right], \left[P_{T_2}(z)\frac{\partial}{\partial z}\right], ..., \left[P_{T_m}(z)\frac{\partial}{\partial z}\right]\right). \tag{3.8}$$

Note that, from the definition of $P_T(z)$'s above, it is easy to see inductively that, for any $T \in \mathbb{T}^W$, $o(P_T(z)) \geq \alpha$. Hence we do have $D_T \in \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ for any $T \in \bar{\mathbb{T}}^W$.

Now we define a linear map $\mathcal{A}_{F_t} : \mathcal{H}_{GL}^W \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ by setting

$$\begin{aligned} \mathcal{A}_{F_t} : \mathcal{H}_{GL}^W &\to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle \\ T &\to D_T(z). \end{aligned} \tag{3.9}$$

When $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ is clear in the context, $\mathcal{U}_{F_t}$ and $\mathcal{A}_{F_t}$ will also be simply written as $\mathcal{U}$ and $\mathcal{A}$, respectively.

From the definitions above, the following lemma follows immediately.

Lemma 3.3. *(a) For any $T \in \mathbb{T}^W$ with the root labeled by $m \in W$, we have*

$$\mathcal{A}(B_+(T))H_{[m]}(z) = D_T. \tag{3.10}$$

(b) For any $m \geq 1$, we have

$$\mathcal{A} \circ B_+ = B_+ \circ \mathcal{A}^{\times m} \circ B_+^{\times m}, \tag{3.11}$$

as maps from $(\mathbb{T}^W)^{\times m}$ to $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$.

Now let us prove the following technic lemma.

Lemma 3.4. *For any $S \in \bar{\mathbb{P}}^W$ and $T \in \mathbb{T}^W$, we have*

$$D_S \cdot P_T = \mathcal{U}(S \cdot T), \tag{3.12}$$

where $S \cdot T$ is the "product" of S and T as in Remark (2.7).

Proof: We use the induction on the number $v(T)$ of vertices of T.

First, when $v(T) = 1$, then T is the singleton labeled by some $m \in W$. Then, by Eqs. (3.6) and (3.8), it is easy to see that both sides of Eq. (3.11) in this case are $P_S(z)$.

Now, assume $v(T) \geq 2$. We write $S = B_+(S')$ and $T = (T_1, \cdots, T_d)$ with $S, T_i \in \mathbb{T}^W$ $(1 \leq i \leq d)$. Let $m \in W$ be the label of the root of T. Set $\phi := D_S$ and $\delta_i := D_{T_i}$ $(1 \leq i \leq d)$. Note that, by our conditions, ϕ and $\delta_i = D_{T_i}$ $(1 \leq i \leq d)$ are all K-derivations. Now by Eqs. (3.6) and (3.3), we have

$$\begin{aligned} D_S \cdot P_T &= \phi \cdot B_+(\delta_1,\, \delta_2, \cdots \delta_d)\, H_{[m]}(z) \\ &= B_+(\phi,\, \delta_1,\, \delta_2,\, \cdots \delta_d)\, H_{[m]}(z) \\ &\qquad + \sum_{1 \leq i \leq d} B_+(\delta_1, \cdots,\, \phi \triangleright \delta_i, \cdots,\, \delta_d)\, H_{[m]}(z) \end{aligned}$$

Applying Eq. (3.1) and the induction assumption :

$$\begin{aligned} &= B_+(\phi,\, \delta_1,\, \delta_2,\, \cdots \delta_d)\, H_{[m]}(z) \\ &\qquad + \sum_{1 \leq i \leq d} B_+(\delta_1, \cdots,\, \left[P_{S' \cdot T_i}(z) \frac{\partial}{\partial z}\right], \cdots,\, \delta_d)\, H_{[m]}(z) \end{aligned}$$

Applying Eq. (3.11):

$$\begin{aligned} &= \mathcal{A}\left(B_+(S', T_1,\, \cdots, T_2)\right) H_{[m]}(z) \\ &\quad + \sum_{1 \leq i \leq d} \mathcal{A}\left(\, B_+(T_1, \cdots,\, S' \cdot T_i, \cdots,\, T_d)\right)\, H_{[m]}(z) \end{aligned}$$

Applying Eq. (3.10):

$$\begin{aligned} &= \mathcal{U}\left(B_+(S', T_1,\, \cdots, T_2)\right) \\ &\qquad + \sum_{1 \leq i \leq d} \mathcal{U}\left(B_+(T_1, \cdots,\, S' \cdot T_i, \cdots,\, T_d)\right) \end{aligned}$$

Note that, by the definition of $S \cdot T$ (see Remark 2.7), we have

(3.13)

$$\begin{aligned} S \cdot T &= S \cdot B_+(T_1,\, T_2,\, \cdots,\, T_2) \\ &= B_+(S', T_1,\, \cdots, T_2) + \sum_{1 \leq i \leq d} B_+(T_1, \cdots,\, S' \cdot T_i, \cdots,\, T_d). \end{aligned}$$

Combining the two equations above and using the linearity of $\mathcal{U}$, we get Eq. (3.12). $\square$

Now, we can formulate and prove the first main result of this section.

Theorem 3.5. *The linear map $\mathcal{A}_{F_t} : \mathcal{H}_{GL}^W \to \mathcal{D}^{[\alpha]}\langle\langle z \rangle\rangle$ is a homomorphisms of K-Hopf algebras.*

Proof: Let us first show the linear map $\mathcal{A}$ is a homomorphism of K-algebras. By the definition of D_T's and $\mathcal{A}$, $\mathcal{A}$ maps the identity element $\circ \in \mathcal{H}_{GL}^W$ to the identity element $D_\circ = 1$ of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. So we only need show, for any $S, T \in \tilde{\mathbb{T}}^W$, we have

$$D_S \cdot D_T = D_{S\cdot T}, \tag{3.14}$$

where $\cdot$ denotes the both algebra product of $\mathcal{H}_{GL}^W$ and $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$.

By the fact pointed in (b) of Theorem 2.8, we may assume that S is primitive, in which case D_S is a derivation. Now, assume $v(T) \geq 2$. (When $v(T) = 1$, Eq.(3.14) is trivial). We write $S = B_+(S')$ and $T = (T_1, \cdots, T_d)$ with $S, T_i \in \mathbb{T}^W$ $(1 \leq i \leq d)$. Let $m \in W$ be the label of the root of T. Set $\phi := D_S$ and $\delta_i := D_{T_i}$ $(1 \leq i \leq d)$. Now by Eqs. (3.8) and (3.3), we have

$$\begin{aligned} &D_S \cdot D_T = \phi \cdot B_+(\delta_1,\, \delta_2, \cdots \delta_d) \\ &= B_+(\phi,\, \delta_1,\, \delta_2,\, \cdots \delta_d) + \sum_{1\leq i\leq d} B_+(\delta_1, \cdots,\, \phi \triangleright \delta_i, \cdots,\, \delta_d) \end{aligned}$$

Applying Eqs. (3.1) and (3.12):

$$= B_+(\phi,\, \delta_1,\, \delta_2,\, \cdots \delta_d) + \sum_{1\leq i\leq d} B_+(\delta_1, \cdots,\, \left[P_{S'\cdot T_i}(z)\frac{\partial}{\partial z}\right], \cdots,\, \delta_d)$$

Applying Eq. (3.11) and the linearity of $\mathcal{A}$:

$$= \mathcal{A}\left(B_+(S', T_1,\, \cdots, T_2) + \sum_{1\leq i\leq d} B_+(T_1, \cdots,\, S'\cdot T_i, \cdots,\, T_d\right)$$

Applying Eq. (3.13):

$$\begin{aligned} &= \mathcal{A}(S\cdot T) \\ &= D_{S\cdot T}. \end{aligned}$$

Hence, we have proved that $\mathcal{A}$ is a K-algebra homomorphism. To see that it is also a homomorphism of Hopf algebras, note that, again, by the fact in (b) of Theorem 2.8, we know that $\mathcal{H}_{GL}^W$ as a Hopf algebra is isomorphic to the universal enveloping algebra of the Lie sub-algebra $\mathcal{P}(\mathcal{H}_{GL}^W)$ generated by the primitive rooted trees. On the other hand, the Hopf algebra structure of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ is also obtained by viewing it as the universal enveloping algebra of the Lie algebra $\mathcal{D}er\langle\langle z\rangle\rangle$. Since the Lie brackets of $\mathcal{P}(\mathcal{H}_{GL}^W)$ and $\mathcal{D}er\langle\langle z\rangle\rangle$ are both given respectively by the commutator brackets of the algebra products of $\mathcal{H}_{GL}^W$ and $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$, the restriction of $\mathcal{A} : \mathcal{P}(\mathcal{H}_{GL}^W) \to \mathcal{D}er\langle\langle z\rangle\rangle$ is a homomorphism of Lie

algebras. Consequently, $\mathcal{A}$ also preserves the Hopf algebras of the corresponding enveloping algebras, which are $\mathcal{H}_{GL}^{W}$ and $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. $\square$

Our second main result of this section is the following theorem.

Theorem 3.6. *Let Ω_{F_t} and Ω_W the $\mathcal{N}CS$ systems in Theorems 2.6 and 2.12, respectively, and $\mathcal{A}_{F_t} : \mathcal{H}_{GL}^{W} \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ be the K-Hopf algebras homomorphism in Theorem 3.5. Then we have $\mathcal{A}_{F_t}^{\times 5}(\Omega_{\mathbb{T}}^{W}) = \Omega_{F_t}$.*

Before we prove the theorem above, we need the following lemma proved in [Z8], which gives a different way to look at the generating function $\tilde{f}(t)$ defined in Eq. (2.35).

Lemma 3.7. *For any $m \in W$, let κ_m denote the singleton labeled by m and set*

$$\kappa(t) := \sum_{m\in W} t^m \kappa_m. \tag{3.15}$$

Then, we have

$$\tilde{f}(t) = 1 + \sum_{d\geq 1} \frac{(-1)^d}{d!} B_+(\,\kappa(t)^d\,), \tag{3.16}$$

where $B_+(\,\kappa(t)^d\,)$ denotes the term obtained by applying B_+ to d-copies of $\kappa(t)$.

<u>Proof of Theorem 3.6</u>: By Corollary 2.8 in [Z5], it will be enough to show that $\mathcal{A}$ maps one component of $\Omega_{\mathbb{T}}^{W}$ to the component of Ω_{F_t} at the same location. Below we will show $\mathcal{A}(\tilde{f}(t)) = f(t)$.

First, for any $m \in W$, let κ_m denote the singleton labeled by m and set $\kappa(t) := \sum_{m\in W} t^m \kappa_m$.

By Eq. (3.5) and the definition of $\mathcal{A}$ in Eq. (3.9), we have

$$\begin{aligned} \mathcal{A}B_+(\kappa(t))) &= \sum_{m\in W} t^m \mathcal{A}(B_+(\kappa(t))) \\ &= \sum_{m\in W} t^m \left[H_{[m]}(z)\frac{\partial}{\partial z}\right] \\ &= \left[H_t(z)\frac{\partial}{\partial z}\right] \end{aligned} \tag{3.17}$$

By Eq. (3.11) and the equation above, we have for any $d \geq 1$,

$$\begin{aligned} (\mathcal{A}\circ B_+)(\kappa(t))^d) &= (B_+ \circ \mathcal{A}^d \circ B_+^d)(\kappa(t)^d) \\ &= B_+(\left[H_t(z)\frac{\partial}{\partial z}\right]^d). \end{aligned} \tag{3.18}$$

Therefore, by Eq. (3.16) in Lemma 3.7 and the equation above, we have

$$\begin{aligned}\mathcal{A}\left(\tilde{f}(t)\right) &= 1+\sum_{d\geq 1}\frac{(-1)^d}{d!}\mathcal{A}\left(B_+(\,\kappa(t)^d\,)\right)\\ &= 1+\sum_{d\geq 1}\frac{(-1)^d}{d!}B_+(\left[H_t(z)\frac{\partial}{\partial z}\right]^d)\end{aligned}$$

By Eq. (3.4) in Lemma 3.2:

$$= f(t).$$

□

By applying Lemma 3.4 and Theorem 3.5, it is easy to see the following proposition also holds.

Proposition 3.8. *Let $\tilde{\mathcal{L}} : \mathcal{H}^W_{GL}\times\mathcal{H}^W_{CK}[1]\to\mathcal{H}^W_{CK}[1]$ be the action induced by the natural action of elements $S\in\bar{\mathbb{T}}^W$ on elements $T\in\mathbb{T}^W$ (see Remark 2.7) and $\mathcal{L}:\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle\times K\langle\langle z\rangle\rangle\to K\langle\langle z\rangle\rangle^{\times n}$ the natural action of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ on $K\langle\langle z\rangle\rangle^{\times n}$. With W and $F_t\in\mathbb{A}^{[\alpha]}_t\langle\langle z\rangle\rangle$ as before, the following diagram commutes.*

$$\begin{array}{ccc}\mathcal{H}^W_{GL}\times\mathcal{H}^W_{CK}[1] & \xrightarrow{\tilde{\mathcal{L}}} & \mathcal{H}^W_{CK}[1]\\ {\scriptstyle \mathcal{A}_{F_t}\times\mathcal{U}_{F_t}}\downarrow & & {\scriptstyle \mathcal{U}_{F_t}}\downarrow\\ \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle\times K\langle\langle z\rangle\rangle^{\times n} & \xrightarrow{\mathcal{L}} & K\langle\langle z\rangle\rangle^{\times n}\end{array}\tag{3.19}$$

Combining Theorems 2.6, 2.12 and 3.5, we have the following proposition.

Proposition 3.9. *For any $\alpha\geq 1$, let $W\subseteq\mathbb{N}^+$ and $F_t\in\mathbb{A}^{[\alpha]}_t\langle\langle z\rangle\rangle$ fixed as before, we have the following commutative diagrams of K-Hopf algebra homomorphisms.*

$$\begin{array}{ccc}\mathcal{N}Sym & \xrightarrow{\mathcal{T}_W} & \mathcal{H}^W_{GL}\\ {\scriptstyle \mathcal{S}_{F_t}}\downarrow & & {\scriptstyle \mathcal{A}_{F_t}}\downarrow\\ \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle & = & \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle\end{array}\tag{3.20}$$

Proof: By Theorems 2.6, 2.12 and 3.6, it is easy to see that

$$\left(\mathcal{A}_{F_t}\circ\mathcal{T}_W\right)^{\times 5}(\Pi)=\mathcal{S}^{\times 5}_{F_t}(\Pi)=\Omega_{F_t}.\tag{3.21}$$

In particular, we have $\mathcal{A}_{F_t} \circ \mathcal{T}_W(\lambda(t)) = \mathcal{S}_{F_t}(\lambda(t)) = f(t)$. Hence, for any $m \geq 1$, we have $\mathcal{A}_{F_t} \circ \mathcal{T}_W(\Lambda_m) = \mathcal{S}_{F_t}(\Lambda_m)$. Since $\mathcal{N}Sym$ is the free K-algebra generated by Λ_m $(m \geq 1)$, we have $\mathcal{A}_{F_t} \circ \mathcal{T}_W = \mathcal{S}_{F_t}$. $\square$

Next, let us consider the question when we can dualize the commutative diagram in the proposition above. First, we have to know when the Hopf algebra homomorphism $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ preserves the gradings of $\mathcal{S}_{F_t}$ and $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Note that, precisely speaking, $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ is not graded in the usual sense, for some infinite sums are allowed in $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. But we can consider the following graded subalgebras of $\mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$.

Let $\mathcal{D}\langle z\rangle$ be the differential operator algebra of the polynomial algebra $K\langle z\rangle$, i.e. $\mathcal{D}\langle z\rangle$ is the unital subalgebra of $\mathrm{End}_K(K\langle z\rangle)$ generated by all K-derivations of $K\langle z\rangle$. For any $m \geq 0$, let $\mathcal{D}_{[m]}\langle z\rangle$ be the set of all differential operators U such that, for any homogeneous polynomial $h(z) \in K\langle z\rangle$ of degree $d \geq 0$, $Uh(z)$ either is zero or is homogeneous of degree $m+d$. For any $\alpha \geq 1$, set $\mathcal{D}^{[\alpha]}\langle z\rangle := \mathcal{D}\langle z\rangle \cap \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$. Then, we have the grading

$$\mathcal{D}^{[\alpha]}\langle z\rangle = \bigoplus_{m\geq\alpha-1} \mathcal{D}_{[m]}\langle z\rangle, \tag{3.22}$$

with respect to which $\mathcal{D}^{[\alpha]}\langle z\rangle$ becomes a graded K-Hopf algebra.

Now, for any $\alpha \geq 2$, we let $\mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$ be the set of all automorphisms $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ such that $F_t(z) = t^{-1}F(tz)$ for some automorphism $F(z)$ of $K\langle\langle z\rangle\rangle$. It is easy to check that $\mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$ is a subgroup of $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$. Then we have the following proposition proved in [Z6].

Proposition 3.10. *For any $\alpha \geq 2$ and $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, the differential operator specialization $\mathcal{S}_{F_t}$ is a graded K-Hopf algebra homomorphism $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle z\rangle \subset \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ iff $F_t \in \mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$.*

Now, for any $F_t \in \mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$ $(\alpha \geq 2)$, by the proposition above, we can take the graded dual of the graded K-Hopf algebra homomorphism $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle z\rangle$ and get the following corollary.

Corollary 3.11. *For any $\alpha \geq 2$ and $F_t \in \mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$, let $\mathcal{D}^{[\alpha]}\langle z\rangle^*$ be the graded dual of the graded K-Hopf algebra $\mathcal{D}^{[\alpha]}\langle z\rangle$. Then,*

$$\mathcal{S}_{F_t}^* : \mathcal{D}^{[\alpha]}\langle z\rangle^* \to \mathcal{Q}Sym$$

is a homomorphism of graded K-Hopf algebras.

By combining Proposition 3.11 and Proposition 3.9 above, we have the following proposition.

Proposition 3.12. *For any $\alpha \geq 2$ and $F_t \in \mathbb{G}_t^{[\alpha]}\langle\langle z\rangle\rangle$, we have the following commutative diagrams of graded K-Hopf algebra homomorphisms.*

$$\begin{array}{ccc} \mathcal{Q}Sym & \xleftarrow{\mathcal{T}^*} & \mathcal{H}_{CK}^W \\ {\scriptstyle \mathcal{S}_{F_t}^*}\uparrow & & {\scriptstyle \mathcal{A}_{F_t}^*}\uparrow \\ \mathcal{D}^{[\alpha]}\langle z\rangle^* & = & \mathcal{D}^{[\alpha]}\langle z\rangle^* \end{array} \tag{3.23}$$

4. Tree Expansion Formulas for D-Log's and Formal Flows

In this section, we apply the K-Hopf algebra homomorphisms $\mathcal{A}_{F_t}: \mathcal{H}_{GL}^W \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ ($F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$) constructed in Section 3 to derive tree expansion formulas for the D-Log, the formal flow and the inverse map of F_t. Note that, for the commutative case, the tree expansion formula Eq. (4.22) for the inverse map was first given in [BCW] and [Wr]. Later, it was generalized in [WZ] to the D-Log's and the formal flows (see Eqs. (4.20) and (4.21)). The proofs given here do not depend on the commutativity of the free variables. It not only generalizes the tree expansion formulas in [BCW], [Wr] and [WZ] for the inverse maps, the D-Log's and the formal flows to the noncommutative case, but also provides some new understandings to these formulas from the $\mathcal{N}$CS system point view.

First, we let K, z and t as before and fix an automorphism $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ ($\alpha \geq 1$). As before, we fix the following notation for F_t and its inverse map $G_t := F_t^{-1}$:

$$F_t(z) = z - H_t(z), \tag{4.1}$$

$$G_t(z) = z + tN_t(z), \tag{4.2}$$

with $H_t(z), N_t(z) \in K[t]\langle\langle z\rangle\rangle^{\times n}$.

Note that, in terms of the notation in Section 3, we have $M_t(z) = tN_t(z)$. Recall that, the D-Log of F_t by definition is the unique $a_t(z) \in K[[t]]\langle\langle z\rangle\rangle$ such that, for any $u_t(z) \in K[[t]]\langle\langle z\rangle\rangle$,

$$e^{\left[a_t(z)\frac{\partial}{\partial z}\right]} \cdot u_t(z) = u_t(F_t). \tag{4.3}$$

By the comments after Eq. (2.25), the relation of the D-Log $a_t(z)$ of F_t with the third component $d(t)$ of the $\mathcal{N}$CS system Ω_{F_t} in Section 2.2 is given by

$$d(t) = -\left[a_t(z)\frac{\partial}{\partial z}\right]. \tag{4.4}$$

Now let s be another central parameter, i.e. it commutes with z and t. We define

$$F_t(z,s) := e^{s\left[a_t(z)\frac{\partial}{\partial z}\right]}\, z = e^{-s\, d(t)}\, z. \tag{4.5}$$

Note that, since $d(0) = 0$, the exponential above is always well-defined. Actually it is easy to see $F_t(z,s) \in (K[s][[t]])\langle\langle z\rangle\rangle^{\times n}$. Therefore, for any $s_0 \in K$, $F_t(z,s_0)$ makes sense.

Following its analog in [E1]-[E3] and [WZ] in the commutative case, we call $F_t(z,s)$ the *formal flow* generated by $F_t(z)$.

Two remarks on the formal flow defined above are as follows.

First, it is well-known that the exponential of a derivation of any K-algebra A, when it makes sense, is always an automorphism of the algebra, so in our case, for any $s_0 \in K$, $e^{s_0\left[a_t(z)\frac{\partial}{\partial z}\right]}$ is also an automorphism of $K[[t]]\langle\langle z\rangle\rangle$ over $K[[t]]$ which maps z to $F_t(z,s_0)$. From Eq. (4.5), it is clear that this automorphism also lies in $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ since $o(a_t(z)) \geq \alpha$.

Secondly, by Eq. (4.5) and the remark above, the formal flow $F_t(z,s)$ has the following properties:

$$F_t(z,0) = z, \tag{4.6}$$

$$F_t(z,1) = F_t(z), \tag{4.7}$$

$$F_t(F_t(z,s_2),s_1) = F_t(z,s_1+s_2), \tag{4.8}$$

for any $s_1, s_2 \in K$.

In other words, $F_t(z,s)$ forms an one-parameter subgroup of the group $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$. Therefore, for any integer $m \in \mathbb{Z}$, $F_t(z,m)$ gives the mth (composing) power of F_t as an element of the group $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$. In particular, by setting $m = -1$, we get the inverse map G_t of F_t, i.e. $F_t(z,-1) = G_t(z)$.

Now we consider the tree expansion formulas for the D-Log and formal flow of $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$. For convenience, we first introduce the following short notations. For any labeled rooted trees $T \in \mathbb{T}^W$ and $T' \in \bar{\mathbb{T}}^W$, we set

$$\mathcal{P}_T(z) = \mathcal{U}(\mathcal{V}_{T'}) = \frac{1}{\alpha(T)} P_T(z), \tag{4.9}$$

$$\mathcal{D}_{T'}(z) = \mathcal{A}(\mathcal{V}_{T'}) = \frac{1}{\alpha(T')} D_{T'}(z). \tag{4.10}$$

By Eqs. (3.6) and (3.8), it is easy to see that, for any primitive $T \in \bar{\mathbb{T}}^W$, we have

$$\mathcal{D}_T \cdot z = \mathcal{P}_{B_-(T)}(z). \tag{4.11}$$

The main results of this subsection is the following theorem.

Theorem 4.1. *For any $\alpha \geq 1$ and $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, let W be the set of positive integers such that F_t can be written as in Eq. (3.5). Then*

(a) The D-Log of $F_t(z)$ is given by

$$a_t(z) = \sum_{T\in\mathbb{T}^W} t^{|T|-1}\varphi_T \mathcal{P}_T(z), \tag{4.12}$$

where, for any $T \in \mathbb{T}^W$, φ_T is the coefficient of s in the order polynomial $\Omega(T,s)$ of T as an unlabeled rooted tree.

(b) The formal flow $F_t(z,s)$ generated by $F_t(z)$ is given by

$$\begin{aligned} F_t(z,s) &= z + \sum_{T\in\mathbb{T}^W} t^{|T|}\,\Omega(T,-s)\,\mathcal{P}_T(z) \\ &= z + \sum_{T\in\mathbb{T}^W} (-1)^{v(T)} t^{|T|}\,\bar{\Omega}(T,s)\,\mathcal{P}_T(z), \end{aligned} \tag{4.13}$$

where $\bar{\Omega}(T,s)$ is the strict order polynomial of T as an unlabeled rooted tree.

Note that, by the definition of θ_T ($T \in \bar{\mathbb{T}}^W$) (see page 15), we have

$$\varphi_T = \theta_{B_+(T)}. \tag{4.14}$$

For some discussions on the order polynomials $\Omega(T,s)$ and the strict order polynomials $\bar{\Omega}(T,s)$, see [St1] or Section 5 in [Z8]. We will need the following results on the (strict) order polynomials in the proof of the theorem above.

Proposition 4.2. *For any rooted tree T, we have*

$$\bar{\Omega}(T,s) = (-1)^{v(T)}\Omega(T,-s), \tag{4.15}$$

$$\nabla\Omega(T,s) = \Omega(B_-(T),s), \tag{4.16}$$

where $\nabla : K[s] \to K[s]$ is the linear operator that maps any $f(s) \in K[s]$ to $f(s) - f(s-1)$.

Eq. (4.15) is a special case of the well-known *Reciprocity Relation* of the strict order polynomials and order polynomials of finite posets. For a proof of this remarkable result, see Corollary 4.5.15 in [St1]. Eq. (4.16) was first proved by J. Shareshian (unpublished). It can be proved by using the definition of the strict order polynomials. For a proof of a similar property of the order polynomials, see Theorem 4.5 in [WZ]. Eq. (4.16) can be proved by a similar argument as the proof there. For more studies on these properties of the (strict) order polynomials, see [Z2] and [SWZ].

Proof of Theorem 4.1: (a) By Eq. (4.4) and Theorem (3.6) and (2.37), we have

$$\begin{aligned} -\left[a_t(z)\frac{\partial}{\partial z}\right] &= d(t) \\ &= \mathcal{A}(\tilde{d}(t)) \\ &= \sum_{T\in\bar{\mathbb{P}}^W} t^{|T|}\theta_T\mathcal{D}_T. \end{aligned}$$

Therefore, by the equation above and Eq. (4.11), we have

$$\begin{aligned} a_t(z) &= \sum_{T\in\bar{\mathbb{P}}^W} t^{|T|-1}\theta_T\mathcal{D}_T\cdot z \\ &= \sum_{T\in\bar{\mathbb{P}}^W} t^{|T|-1}\theta_T\mathcal{P}_{B_-(T)}(z) \end{aligned}$$

Replacing the summation index $T\in\bar{\mathbb{P}}^W$ by $B_+(T)$ with $T\in\mathbb{T}^W$ and noting that $|T|=|B_-(T)|$ for any $T\in\bar{\mathbb{P}}^W$:

$$= \sum_{T\in\mathbb{T}^W} t^{|T|-1}\theta_{B_+(T)}\mathcal{P}_T(z)$$

Applying Eq. (4.14):

$$= \sum_{T\in\mathbb{T}^W} t^{|T|-1}\varphi_T\mathcal{P}_T(z)\,.$$

Hence, we get Eq. (4.12).

(b) First, let us consider the exponential $e^{s\tilde{d}(t)}\in\mathcal{H}^W_{GL}[[t]]$.

$$\begin{aligned} e^{s\tilde{d}(t)} &= \sum_{m\geq 0}\frac{s^m}{m!}\tilde{d}(t)^m \\ &= 1+\sum_{m\geq 1}\frac{s^m}{m!}\left(\sum_{T\in\bar{\mathbb{P}}^W} t^{|T|}\theta_T\mathcal{V}_T\right)^m \end{aligned}$$

Applying Lemma 2.9:

$$= 1+\sum_{m\geq 1}\frac{s^m}{m!}\sum_{T\in\bar{\mathbb{T}}^W} t^{|T|}\sum_{\substack{\vec{e}=(e_1,\ldots,e_{m-1})\in E(T)^{m-1}\\ e_1\succ\cdots\succ e_{m-1}}}\theta_{T_{\vec{e},1}}\cdots\theta_{T_{\vec{e},m}}\mathcal{V}_T$$

$$= 1 + \sum_{T\in\bar{\mathbb{T}}^W} \left(\sum_{m=1}^{v(T)} \frac{s^m}{m!} \sum_{\substack{\vec{e}=(e_1,\dots,e_{m-1})\in E(T)^{m-1}\\ e_1\succ\cdots\succ e_{m-1}}} \theta_{T_{\vec{e},1}}\cdots\theta_{T_{\vec{e},m}} \right) t^{|T|}\mathcal{V}_T.$$

Now we apply $\mathcal{A}$ to the equation above. Note that $\mathcal{A}$ maps $\exp(s\tilde{d}(t))$ to $\exp(sd(t))$ since, by Theorem 3.5, $\mathcal{A} : \mathcal{H}^W_{GL} \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ is a K-algebra homomorphism. So we have

$$e^{sd(t)} = 1 + \sum_{T\in\bar{\mathbb{T}}^W} \left(\sum_{m=1}^{v(T)} \frac{s^m}{m!} \sum_{\substack{\vec{e}=(e_1,\dots,e_{m-1})\in E(T)^{m-1}\\ e_1\succ\cdots\succ e_{m-1}}} \theta_{T_{\vec{e},1}}\cdots\theta_{T_{\vec{e},m}} \right) t^{|T|}\mathcal{D}_T.$$

Applying the equation above to z and noting that $\mathcal{D}_T\cdot z = \mathcal{P}_{B_-(T)}(z)$ if $T\in\bar{\mathbb{P}}^W$ and 0 otherwise, we get

$$e^{sd(t)}z = 1 + \sum_{T\in\bar{\mathbb{P}}^W} \left(\sum_{m=1}^{v(T)} \frac{s^m}{m!} \sum_{\substack{e_1\succ\cdots\succ e_{m-1}\\ e_i\in E(T)}} \theta_{T_{\vec{e},1}}\cdots\theta_{T_{\vec{e},m}} \right) t^{|T|}\mathcal{P}_T(z)$$

Applying Eq. (2.34) in Proposition 2.11:

$$= 1 + \sum_{T\in\bar{\mathbb{P}}^W} \nabla\Omega(T,s)\mathcal{P}_T(z).$$

Now, replacing s by $-s$ in equation above, by Eq. (4.5), we get

$$F_t(z,s) = z + \sum_{T\in\bar{\mathbb{P}}^W} \nabla\Omega(T,-s)\ t^{|T|}\mathcal{P}_T(z)$$

Applying Eq. (4.16):

$$= z + \sum_{T\in\mathbb{P}^W} \Omega(B_-(T),-s)t^{|T|}\mathcal{P}_{B_-(T)}(z).$$

Changing the summation index $T\in\bar{\mathbb{P}}^W$ by $B_+(T)$ with $T\in\mathbb{T}^W$:

$$= z + \sum_{T\in\mathbb{T}^W} \Omega(T,-s)t^{|T|}\mathcal{P}_T(z)$$

Applying Eq. (4.15):

$$= z + \sum_{T\in\mathbb{T}^W} (-1)^{v(T)}t^{|T|}\Omega(T,s)\mathcal{P}_T(z). \tag{4.17}$$

Hence, we get Eq. (4.13). □

As we mentioned early (see page 25), for any $m \in \mathbb{Z}$, $F_t(z,m)$ is the mth (composing) power denoted by $F_t^{[m]}$ of the automorphism $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$. Hence, by plugging in m for s in Eq. (4.13), we get the following formulas.

Corollary 4.3. *For any $\alpha \geq 1$, $m \in \mathbb{Z}$ and $F_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$, we have*

$$F_t^{[m]}(z) = \sum_{T\in\mathbb{T}^W} t^{|T|}\,\Omega(T,-m)\,\mathcal{P}_T(z), \tag{4.18}$$

In particular, by letting $m=-1$, we get the following tree expansion formula for the inverse map $G_t(z)$ of $F_t(z)$.

$$G_t(z) = \sum_{T\in\mathbb{T}^W} t^{|T|}\,\mathcal{P}_T(z), \tag{4.19}$$

Proof: Note that, we only need prove Eq. (4.19). But it follows from Eq. (4.19) and the well-known fact that $\Omega(P,1)=1$ for any finite posets. □

One remark on the tree expansion formulas derived in Theorem 4.1 and Corollary 4.3 is as follows. Note that, one of the conditions we have required on $F_t(z) = z - H_t(z)$ is that $H_{t=0}(z) = 0$. But for the automorphisms $F(z)$ of $K\langle\langle z\rangle\rangle$ of the form $F(z) = z - H(z)$ with $H(z) \in K\langle\langle z\rangle\rangle^{\times n}$ and $o(H(z)) \geq \alpha$, all formulas derived can still be applied to $F(z)$ as follows.

First, we consider the deformation $F_t(z) = z - tH(z)$ which does lie in $\mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$. Actually, it can be viewed as an automorphism of $K[t]\langle\langle z\rangle\rangle$, instead of $K[[t]]\langle\langle z\rangle\rangle$, over the polynomial algebra $K[t]$. Therefore, all the formulas above with $W=\{1\}$ still apply to $F_t(z)$. Secondly, by the fact that $H_t(z) \in K[t]\langle\langle z\rangle\rangle^{\times n}$, it is easy to check that, for any $t_0 \in K$, the D-Log $a_{t=t_0}(z)$ of $F_{t=t_0}(z)$ and the inverse map $G_{t=t_0}(z)$ all make sense. In particular, by setting $t=1$, we recover the D-Log and the formal flow of the original automorphism $F(z)$. Thirdly, in the case $W=\{1\}$, the weight $|T|$ ($T \in \mathbb{T}^W$) of T is same as the number $v(T)$ of the vertices of T, and the set $\mathbb{T}^W$ of W-labeled trees can be identify with the set of unlabeled rooted trees $\mathbb{T}$.

By the discussions above, it is easy to see that we have the following tree expansion formulas for the automorphisms of $K\langle\langle z\rangle\rangle$.

Corollary 4.4. *For any automorphism $F(z)$ of the form $F(z) = z - H(z)$ with $o(H(z)) \geq 2$, we have*

(a) *The D-Log $a(z)$ of $F(z)$ is given by*

$$a(z) = \sum_{T\in\mathbb{T}} \varphi_T \mathcal{P}_T(z), \tag{4.20}$$

where, φ_T is the coefficient of s in the order polynomial $\Omega(T,s)$ of the rooted tree T.

(b) *The formal flow $F(z,s)$ generated by $F(z)$ is given by*

$$F(z,s) = z + \sum_{T\in\mathbb{T}} (-1)^{v(T)}\, \Omega(T,s)\, \mathcal{P}_T(z) \tag{4.21}$$

where $\bar{\Omega}(T,s)$ is the strict order polynomial of T as an unlabeled rooted tree.

In particular, we have the following tree expansion inversion formula:

$$G(z) := F^{-1}(z) = \sum_{T\in\mathbb{T}} \mathcal{P}_T(z), \tag{4.22}$$

5. More Properties of the Specializations $\mathcal{S}$ and $\mathcal{T}$ of NCSF's

Let K, z, t, $\alpha \geq 1$ and $W \subseteq \mathbb{N}^+$ as before. In this subsection, we study more properties of the specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ ($F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$) in Theorem 2.6 and $\mathcal{T}_W : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ in Theorem 2.12. We first show in Theorem 5.1 that, when $W = \mathbb{N}^+$, the specialization $\mathcal{T} : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ of NCSF's is actually an embedding. Then, in Theorem 5.3, we use Theorem 4.6 in [Z6] and improve it to the family of the specializations $\mathcal{S}_{F_t} : \mathcal{N}Sym \to \mathcal{D}^{[\alpha]}\langle\langle z\rangle\rangle$ with all $n \geq 1$ and $F_t = z - H_t(z) \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ such that $H_t(z)$ is homogeneous and the Jacobian matrix JH is strictly lower triangular.

Let us start with the following theorem.

Theorem 5.1. *When $W = \mathbb{N}^+$, the graded K-Hopf algebra homomorphism $\mathcal{T}_W : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ in Theorem 2.12 is an embedding.*

Proof: Let $P \in \mathcal{N}Sym$ be any non-zero NCSF. By Theorem 4.6 in [Z6], there exist $F_t \in \mathbb{A}_t^{[\alpha]}\langle\langle z\rangle\rangle$ such that $\mathcal{S}_{F_t}(P) \neq 0$. By Proposition 3.9, we have $\mathcal{A}_{F_t}(\mathcal{T}_W(P)) = \mathcal{S}_{F_t}(P) \neq 0$. Hence $\mathcal{T}_W(P) \neq 0$. □

Combining Corollary 3.11 and Theorem 5.1 above, we get the following corollary.

Corollary 5.2. *For any non-empty $W \subseteq \mathbb{N}^+$, let $\mathcal{T} : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ be the specialization of NCSF in Theorem 2.12 and $\mathcal{T}^*$ its graded dual map. Then $\mathcal{T}^* : \mathcal{H}_{CK}^W \to \mathcal{Q}sym$ is a homomorphism of graded K-Hopf algebras. Furthermore, when $W = \mathbb{N}^+$, $\mathcal{T}^*$ is also onto.*

To formulate next main theorem of this section. Let us first introduce the following notations.

For any z and $\alpha \geq 1$ as before, we let $\mathbb{B}_t^{[\alpha]}\langle z\rangle$ be the set of automorphisms $F_t = z - H_t(z)$ of the polynomial algebra $K[t]\langle z\rangle$ over $K[t]$ such that the following conditions are satisfied.

- $H_{t=0}(z) = 0$.
- $H_t(z)$ is homogeneous in z of degree $d \geq \alpha$.
- With a proper permutation of the free variables z_i's, the Jacobian matrix $JH_t(z)$ becomes strictly lower triangular.

Our next main result of this section is the following theorem.

Theorem 5.3. *In both commutative and noncommutative cases, the following statement holds.*

For any fixed $\alpha \geq 1$ and non-zero $P \in \mathcal{N}Sym$, there exist $n \geq 1$ (the number of the free variable z_i's) and $F_t(z) \in \mathbb{B}_t^{[\alpha]}\langle z\rangle$ such that $\mathcal{S}_{F_t}(P) \neq 0$.

To prove the theorem above, we first need the following lemma, which is also interesting in its own right.

Lemma 5.4. *Let $\alpha \geq 1$ and $W \subseteq \mathbb{N}^+$ be fixed above. For any $T \in \mathbb{T}^W$, there exist $n \geq 1$ and $F_t(z) \in \mathbb{B}_t^{[\alpha]}\langle z\rangle$ such that $P_T \neq 0$ and $P_{T'}(z) = 0$ for any $T' \in \mathbb{T}^W$ with $|T'| \geq |T|$ but $T' \not\simeq T$.*

Note that, in the commutative case with $W = \{1\}$, the lemma is essentially same as Theorem 2.4 in [WZ]. The proof given below is also parallel to the proof there.

Proof: (a) For any fixed $T \in \mathbb{T}^W$, we construct automorphism $F_t \in \mathbb{B}_t^{[\alpha]}\langle z\rangle$ as follows. Let $n = v(T)$, the number of vertices of T, and $d \geq \alpha$ be a positive integer that is greater or equal to the number of children of any vertex of T. Let $z = (z_1, z_2, \cdots, z_n, z_{n+1})$ be free variables. We first label the edges by $e_2, \ldots, e_n$ in an order preserving way, i.e. for any $2 \leq i < j \leq n$, we have $e_j \succ e_i$. We then assign the variable z_i $(2 \leq i \leq n)$ to the edge e_i and label the vertices as follows: let $v_1 = \mathrm{rt}_T$, and for $i = 2, \ldots, n$ let v_i be the vertex of e_i which is further away from the root. Finally, we define $H_t(z) \in K\langle z\rangle^{\times n}$ as follows. First, for any $1 \leq i \leq n$ and $m \in W$, if v_i is not a leaf of T, we set $\tilde{H}_{[m],i}(z) \in K\langle z\rangle$ be the product in any fixed order of all the free variables assigned to the edges connecting v_i with its m-labeled children, if there are any, and 0 otherwise. Note that $\deg \tilde{H}_{[m],i}(z) \leq d$. We set $H_{[m],i}(z) = z_{n+1}^k \tilde{H}_{[m],i}(z)$ for some $k \geq 0$ such that $\deg H_{[m],i}(z) = d$. Now suppose v_i is a leaf of T, we simply set $H_{[m],i}(z) := z_{n+1}^D$ if $m = \min W$ and 0

otherwise. Finally, we set $H_{[m],n+1}(z) := 0$ for all $m \in W$. Next, we set $H_{t,i}(z) := \sum_{m\in W} t^m H_{[m],i}(z)$ for any $1 \le i \le n+1$ and $H_t(z) = (\, H_{t,1}(z), H_{t,2}(z), \cdots, H_{t,n+1}(z))$. Then the wanted automorphism F_t associated with the fixed $T \in \bar{\mathbb{T}}^W$ will be $F_t(z) := z - H_t(z)$.

From the construction of $H_t(z)$ above and the definition of $P_T(z)$ in Eq. (3.6), it is easy to check the following facts:

- $H_{t=0}(z) = 0$ and $H_t(z)$ is homogeneous in z of degree $d \ge \alpha$.
- For any $1 \le i \le n+1$, $H_{t,i}(z)$ only depends on the free variables z_j with $j > i$. Hence, the Jacobian matrix $JH_t(z)$ is strictly lower triangular.
- For any $T' \in \mathbb{T}^W$, $P_{T'}(z) \neq 0$ only if either $T' \simeq T$ or there exists an admissible cut $C \in \mathcal{C}(T)$ such that T' is isomorphic to one of connected components obtained by cutting off the edges of C from T.
- For the fixed $T \in \bar{\mathbb{T}}^W$ itself, $P_T(z) = c\, z_{n+1}^b$ for some $c, b \in \mathbb{N}^+$.

Actually, with a little bit more effect, one can show that the constants c and b above are given by $c = \alpha(T)$ and $b = nd - (n-1)$. But we do not need these facts in our proof here.

From the discussions above, we only need show the polynomial map $F_t(z)$ defined above is indeed a polynomial automorphism of $K[t]\langle z\rangle$ over $K[t]$, i.e. its inverse is also a polynomial map. But, from the third observation listed above and the tree expansion inversion formula Eq. (4.19), it is easy to see that $G_t(z)$ is also a polynomial map. Note that, this also follows from the following well-known result in the inversion problem, namely, any polynomial map $F_t(z)$ with the Jacobian matrix JF_t lower triangular and invertible is a polynomial automorphism. $\square$

Now we can prove Theorem 5.3 as follows.

Proof Theorem 5.3: Let $P \in \mathcal{N}Sym$ be any non-zero NCSF. We choose $W = \mathbb{N}^+$ and let $\mathcal{T} : \mathcal{N}Sym \to \mathcal{H}_{GL}^W$ be the specialization of NCSF in Theorem 2.12, which we have shown in Theorem 5.1 is an embedding. Therefore $\mathcal{T}(P) \neq 0$. We write $\mathcal{T}(P)$ as

$$\mathcal{T}(P) = \sum_{T\in\bar{\mathbb{T}}^W} c_T\, T, \tag{5.1}$$

with $c_T \in K$ being all but finitely many zero.

Let $k_0 \ge 1$ be the least positive integer such that $c_T = 0$ for any $T \in \bar{\mathbb{T}}^W$ with $|T| < k_0$. We choose and fix $S \in \bar{\mathbb{T}}^W$ such that $c_S \neq 0$ and $|S| = k_0$. We fix any $m \in W$ and, for any $T \in \bar{\mathbb{T}}^W$, denote by T_m the W-labeled rooted trees in $\mathbb{T}^W$ obtained by (re)labeling the root of T by m. Note that, for any $T, T' \in \bar{\mathbb{T}}^W$, $T \simeq T'$ in $\bar{\mathbb{T}}^W$ iff $T_m \simeq T'_m$ in

$\mathbb{T}^W$. Now we apply by Lemma 5.4 to $S_m \in \mathbb{T}^W$ and choose $F_t \in \mathbb{B}$ such that $P_{S_m}(z) \neq 0$ but $P_{T_m}(z) = 0$ for any $T \in \bar{\mathbb{T}}^W$ with $|T| \geq |S| = k_0$ and $T \not\simeq S$. Now we apply $\mathcal{A}_{F_t}$ to $\mathcal{T}(P)$. Note that, by Eq. (3.10), we have, $D_T \cdot H_{[m]}(z) = P_{T_m}$ for any $T \in \bar{\mathbb{T}}^W$. By using all the facts above and Eq. (5.1), it is easy to see that, we have

$$\begin{aligned}\mathcal{A}_{F_t}(\mathcal{T}(P)) \cdot H_{[m]}(z) &= \sum_{T \in \bar{\mathbb{T}}^W} c_T \, P_{T_m}(z) \\ &= P_{S_m}(z) \neq 0.\end{aligned}$$

Finally, by Theorem 3.9, we have $\mathcal{S}_{F_t}(P) = \mathcal{A}_{F_t}(\mathcal{T}(P)) \neq 0$. Hence, we have proved Theorem 5.3. $\square$

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