Semi-infinite forms and topological vertex operator algebras

Yi-Zhi Huang and Wenhua Zhao

Abstract

Semi-infinite forms on the moduli spaces of genus-zero Riemann surfaces with punctures and local coordinates are introduced. A partial operad for semi-infinite forms is constructed. Using semi-infinite forms and motivated by a partial suboperad of the partial operad for semi-infinite forms, topological vertex partial operads of type $k < 0$ and strong topological vertex partial operads of type $k < 0$ are constructed. It is proved that the category of (locally-)grading-restricted (strong) topological vertex operator algebras of type $k < 0$ and the category of (weakly) meromorphic $\mathbb{Z}\times\mathbb{Z}$-graded algebras over the (strong) topological vertex partial operad of type k are isomorphic. As an application of this isomorphism theorem, the following conjecture of Lian-Zuckerman and Kimura-Voronov-Zuckerman is proved: A strong topological vertex operator algebra gives a homotopy Gerstenhaber algebra. These results hold in particular for the tensor product of the moonshine module vertex operator algebra, the vertex algebra constructed from a rank 2 Lorentz lattice and the ghost vertex operator algebra, studied in detail first by Lian and Zuckerman.

Contents

1 Introduction

In the present paper, we give a geometric and operadic formulation of the notion of topological vertex operator algebra (satisfying certain additional axioms) and as a consequence, we prove a conjecture of Lian and Zuckerman [LZ2] and Kimura, Voronov and Zuckerman [KVZ]. This geometric and operadic formulation is an application of the geometric theory of vertex operator algebras developed in [H1], [H2] and [H5] and the theory of semi-infinite forms for the Virasoro algebra introduced by Feigin [Fe] and developed by Frenkel, Garland and Zuckerman [FGZ] and by Lian and Zuckerman [LZ1]. These results will be useful in the future mathematical study of string backgrounds and $N = 2$ superconformal field theories.

A notion of topological chiral algebra was first introduced by Lian and Zuckerman [LZ2]. Following the mathematical terminology, these algebras are later called topological vertex operator algebras by several authors (though there is a very subtle difference between topological chiral algebras in the sense of [LZ2] (or topological vertex operator algebras in the sense of [KVZ]) and topological vertex operator algebras in the sense of [H4]; see below). In [LZ2], Lian and Zuckerman showed that the cohomology of a (strong) topological chiral algebra has a natural structure of a Batalin-Vilkovisky ("coboundary Gerstenhaber algebra") structure. Based on calculations, they also conjectured that there must be a certain "homotopy Gerstenhaber algebra" structure on a topological chiral algebra. Identities satisfied by the corresponding operations in topological vertex operator algebras were studied by Akman [A]. After attempts by different authors, there is now a precise notion of homotopy Gerstenhaber algebra (see [V]). In [KVZ], Kimura, Voronov and Zuckerman developed a framework for the construction of homotopy Gerstenhaber algebras from genus-zero topological conformal field theories. Actually the notion of homotopy Gerstenhaber algebra used in [KVZ] is not the correct one because the construction in [GJ] justifying the definition contained an error. But appropriate corrections can be made so that with the correct definition of homotopy Gerstenhaber algebra, the statement in [KVZ] that a genus-zero topological conformal field theory in the sense of Segal [S] and Getzler [G] has a structure of a homotopy Gerstenhaber algebra is still correct. See [V] for more details. With the precise definition of homotopy Gerstenhaber algebra, the conjecture of Lian-Zuckerman can be formulated precisely as follows (cf. [KVZ]):

Conjecture 1.1 *A topological vertex operator algebra gives a homotopy Gerstenhaber algebra.*

Since it was already proved in [KVZ] and [V] that a genus-zero topological conformal field theory gives a homotopy Gerstenhaber algebra, it is natural to try to prove Conjecture 1.1 by proving the following conjecture also by Kimura, Voronov and Zuckerman in [KVZ]:

Conjecture 1.2 *An appropriate topological completion of a topological vertex operator algebra has a structure of a genus-zero topological conformal field theory.*

There is a very subtle issue which we must address here. In many works on vertex operator algebras, including [FLM2], [FHL], [H4] and [H5], vertex operator algebras or topological vertex operator algebras are required to satisfy the following grading-restriction conditions: For a vertex operator algebra $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $V_{(0)} = 0$ for n sufficiently small. But the topological chiral algebras or topological vertex operator algebras discussed by Lian and Zuckerman in [LZ2] and by Kimura, Voronov and Zuckerman in [KVZ] in general do not satisfy the grading-restriction conditions. In fact, the important examples from string backgrounds, including the tensor product of the moonshine module vertex operator algebra, the vertex algebra constructed from a rank 2 Lorentz lattice and the ghost vertex operator algebra (see [LZ3]), in general do not satisfy these conditions.

In the present paper, an algebra satisfying the original definition of Lian and Zuckerman in [LZ2] is called a topological vertex operator algebra, as is in [KVZ]. (We warn the reader that a topological vertex operator algebra in this sense is in general not a vertex operator algebra in the sense of [FLM2] and [FHL] because the grading-restriction conditions are not satisfied.) A topological vertex operator algebra in the sense of [H4] is called a grading-restricted topological vertex operator algebra. We also introduce local grading-restriction conditions. Grading-restricted topological vertex operator algebras and the examples from string backgrounds are locally-grading-restricted topological vertex operator algebras. We also introduce the notion of strong topological vertex operator algebra for which the square of a certain operator on the algebra is 0. We prove Conjecture 1.2 for a local-grading-restricted strong topological vertex operator algebra as a consequence of the main theorem of the present paper stating that a geometric formulation of the notion of locally-grading-restricted (strong) topological vertex operator algebra is equivalent to the algebraic formulation. As a consequence, Conjecture 1.1 is true for a locally-grading-restricted strong topological vertex operator algebra. In particular, Conjecture 1.1 and Conjecture 1.2 are true for grading-restricted strong topological vertex operator algebra and for the tensor product of the moonshine module vertex operator algebra, the vertex algebra constructed from a rank 2 Lorentz lattice and the ghost vertex operator algebra (see [LZ3]). In the case that the topological vertex operator algebra is not strong, the same construction does not give a simple geometric structure.

The geometric theory of vertex operator algebras was initialized by Frenkel [Fr]. A geometric and operadic formulation of the notion of vertex operator algebra in the sense of [FLM2] and [FHL] was given by the first author in [H1], [H2] and [H5] (see also [H3]) and by Lepowsky and the first author in

[HL1] and [HL2]. The corresponding isomorphism theorem was proved by the first author in [H1], [H2] and [H5].

In [H4], the first author gave a geometric and operadic formulation of the notion of topological vertex algebra (which may not have a Virasoro element) and an isomorphism theorem was proved. This isomorphism theorem for topological vertex algebras combined with a theorem of Cohen [C] or Getzler [G] gives a geometric construction of the Gerstenhaber or Batalin-Vilkovisky algebra structure on the cohomology of a topological vertex algebra (see [H4]).

It is natural to look for a geometric formulation of the notion of grading-restricted topological vertex operator algebra, and more generally, of the notion of locally-grading-restricted topological vertex operator algebra. Because topological vertex operator algebras and locally-grading-restricted topological vertex operator algebras have Virasoro elements and elements which give differentials and fermion gradings, the geometry underlying these algebras is certainly much more complicated than that underlying topological vertex algebras.

In the present paper, we find that the correct geometric objects underlying locally-grading-restricted topological vertex operator algebras are "universal coverings" of a partial operad constructed from what we call "semi-infinite forms on the moduli spaces of genus-zero Riemann surfaces with punctures and local coordinates." Semi-infinite forms and semi-infinite cohomologies for graded Lie algebras were introduced by Feigin [Fe] and developed in the context of string theory by Frenkel, Garland and Zuckerman [FGZ] and further by Lian and Zuckerman [LZ1]. Using semi-infinite forms for the Virasoro algebra, we construct semi-infinite forms on the moduli spaces above and a partial operad for semi-infinite forms. Using these semi-infinite forms and motivated by a partial suboperad of the partial operad for semi-infinite forms, we construct partial operads called "topological vertex partial operad of type k" and partial operads called "strong topological vertex partial operad of type k" for $k < 0$. We also introduce the notions of (strong) topological vertex operator algebra of type k and (strong) topological vertex operator algebras of type k for $k < 0$. The main theorem of the present paper states that for any $k < 0$, the category of meromorphic $\mathbb{Z} \times \mathbb{Z}$-graded algebras over the (strong) topological vertex partial operad of type k is isomorphic to the category of (strong) topological vertex operator algebras of type k, and the category of weakly meromorphic $\mathbb{Z} \times \mathbb{Z}$-graded algebras over the (strong) topological vertex partial operad of type k is isomorphic to the category of local-grading-restricted (strong) topological vertex operator algebras of type k. The proof of Conjecture 1.2 for local-grading-restricted strong topological vertex operator algebras follows easily because there is a morphism from the operad for genus-zero topological conformal field theories to strong topological vertex partial operads.

The material in this paper depends heavily on the monograph [H5].

The present paper is organized as follows: In Section 2, we review basic concepts and state the main results, Theorems 2.8, 2.9 and 2.10, of the present paper. The notions of topological vertex operator algebra in the sense of [LZ2] and in the sense of [H4] and related notions are recalled in Subsection 2.1. In Subsection 2.2, after recalling the notions of Gerstenhaber algebra, homotopy Gerstenhaber algebra and genus-zero topological conformal field theory and a result of Kimura-Voronov-Zuckerman and Voronov, we state our solutions, Theorems 2.8 and 2.9, to Conjectures 1.2 and 1.1, respectively, for local-grading-restricted strong topological vertex operator algebras and in particular for grading-restricted strong topological vertex operator algebras in the sense of [H4]. In Subsection 2.3, the main theorem, Theorem 2.10, of the present paper is stated. In Section 3, we construct and study semi-infinite forms on moduli spaces of spheres with tubes. In Subsection 3.1, we construct left invariant meromorphic vector fields on $\tilde{K}^c(1)$ (see [H5]) satisfying the Virasoro relations. They are needed in the construction of semi-infinite forms. Semi-infinite forms on $K(0)$ (see [H5]) are constructed and studied in Subsection 3.2. In Subsections 3.3 and 3.4, semi-infinite forms on $K(n)$ for $n \geq 0$ and a partial operad $\mathfrak{G}$ for semi-infinite forms, respectively, are constructed. In Section 4, we construct the (strong) topological vertex partial operad and prove the main results stated in Subsections 2.2 and 2.3. In Subsection 4.1, some properties of topological vertex operator algebras are proved. The topological vertex partial operad and the strong topological vertex partial operad are constructed in Subsection 4.2. The main theorem, Theorem 2.10, of the present paper is proved in Subsection 4.3 and Theorem 2.8 is proved in Subsection 4.4. Section 5 is an appendix, in which two types of examples of strong topological vertex operator algebras and local-grading-restricted strong topological vertex operator algebras are given. The examples obtained by tensoring the ghost vertex operator algebra with vertex algebras of central charge 26 are given in Subsection 5.1. The examples obtained by twisting $N = 2$ superconformal vertex operator superalgebras are given in Subsection 5.2.

Acknowledgment We are grateful to Sasha Voronov for explaining to us his correction of the main result in [KVZ] and for sending us a preliminary version of [V]. We also thank Gregg Zuckerman and Bong H. Lian for comments and suggestioins. Y.Z.H. is supported in part by by NSF grant DMS-9622961. W.Z. is grateful to the Department of Mathematics at University of Chicago for the financial supports.

2 The main results

2.1 Topological vertex operator algebras

We assume that the reader is familiar with the precise notions of vertex algebra and vertex operator algebra (see, for example, [B1], [FLM2], [FHL]). In particular, a vertex operator algebra $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ in this paper is $\mathbb{Z}$-graded and satisfies the following grading-restriction conditions: $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $V_{(n)} = 0$ for n sufficiently small. In the present paper, we need the following variants (see, for example, [H4]; cf. [DL]):

Definition 2.1 A $\mathbb{Z}\times\mathbb{Z}$*-graded vertex operator algebra without grading restrictions* is a $\mathbb{Z}\times\mathbb{Z}$-graded vector space V (graded by *weights* and by *fermion numbers*), that is,

$$V = \coprod_{m,n\in\mathbb{Z}} V_{(n)}^{(m)} = \coprod_{n\in\mathbb{Z}} V_{(n)} = \coprod_{m\in\mathbb{Z}} V^{(m)}$$

where

$$V_{(n)} = \coprod_{m\in\mathbb{Z}} V_{(n)}^{(m)}, \quad V^{(m)} = \coprod_{n\in\mathbb{Z}} V_{(n)}^{(m)},$$

equipped with a vertex operator map $Y : V \otimes V \to V[[x, x^{-1}]]$ which maps $V^{(m_1)} \otimes V^{(m_2)}$ to $V^{(m_1+m_2)}[[x, x^{-1}]]$, a vacuum $\mathbf{1}$ and a Virasoro element ω, satisfying the following axioms:

1. For $v \in V^{(m)}$, we say that v has fermion number m and use $|v|$ to denote m. Then for the vacuum $\mathbf{1}$ and the Virasoro element ω, $|\mathbf{1}| = |\omega| = 0$.

2. For $u, v \in V$, the (Cauchy-)Jacobi identity

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)&Y(u,x_1)Y(v,x_2)\\ &-(-1)^{|u||v|}x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)\\ = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)&Y(Y(u,x_0)v,x_2) \end{aligned}$$

holds.

3. All the other axioms for vertex operator algebras, except the grading-restriction conditions, hold.

The following notion of topological vertex operator algebra is the same as the notion of topological chiral algebra introduced by Lian and Zuckerman in [LZ2]:

Definition 2.2 A *topological vertex operator algebra* is a $\mathbb{Z}\times\mathbb{Z}$-graded vertex operator algebra without grading restrictions $(V, Y, \mathbf{1}, \omega)$ equipped with three additional distinguished elements $f \in V_{(1)}$, $q \in V_{(1)}^{(1)}$ and $g \in V_{(2)}^{(1)}$ satisfying the following axioms:

1. Let $Y(f, x) = \sum_{n\in\mathbb{Z}} f_n x^{-n-1}$. Then for any $v \in V^{(m)}$,

$$f_0 v = mv. \tag{2.1}$$

2. Let $Y(q, x) = \sum_{n\in\mathbb{Z}} q_n x^{-n-1}$ and $Q = q_0$. Then

$$\begin{aligned} L(n)q &= 0, \quad n > 0,\\ Q^2 &= 0. \end{aligned}$$

3. Let $Y(g, x) = \sum_{n\in\mathbb{Z}} g(n) x^{-n-2}$ and ω the Virasoro element of V. Then

$$\begin{aligned} L(n)g &= 0, \quad n > 0,\\ Qg &= \omega. \end{aligned}$$

A topological vertex operator algebra is *strong* if it satisfies the condition $g(0)^2 = 0$.

A topological vertex operator algebra is *grading-restricted* if the grading-restriction conditions are satisfied.

A topological vertex operator algebra is *locally grading-restricted* if the following additional axioms are satisfied:

1. For any element of the topological vertex operator algebra, the module $W = \coprod_{n\in\mathbb{Z}} W_{(n)}$ for the Virasoro algebra generated by this element satisfies the grading-restriction conditions, that is, $\dim W_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $W_{(n)} = 0$ for n sufficiently small.

2. The vertex subalgebra generated by ω, q, f, g satisfies the grading-restriction conditions.

Let k be a negative integer. A locally-grading-restricted (strong) topological vertex operator algebra is *of type* k if the weights of the homogeneous nonzero elements of its vertex subalgebra generated by ω, q, f and g are larger than or equal to k.

Homomorphisms and *isomorphisms* of topological vertex operator algebras are defined in the obvious way.

We denote the topological vertex operator algebra just defined by

$$(V, Y, \mathbf{1}, \omega, f, q, g)$$

or simply by V. Note that (2.1) implies $f \in V_{(1)}^{(0)}$. Also it is clear that if a (strong) topological vertex operator algebra is of type k, it is also of type $k + m$ for any $m < 0$.

Remark 2.3 We warn the reader that topological vertex operator algebras defined above in general are not $\mathbb{Z}\times\mathbb{Z}$-graded vertex operator algebras because the grading-restriction conditions might not be satisfied. Topological vertex operator algebras defined in [H4] are grading-restricted topological vertex operator algebras defined above. The reason why we use the terminology above in the present paper is to make our terminology agree with those in [LZ2] and [KVZ]. We prefer to call an algebra satisfying all the axioms for vertex operator operator algebras except the grading-restriction axioms a *conformal* *vertex algbera*. Then a vertex operator algebra is a *grading-restricted conformal vertex algebra* and a topological vertex operator algebra in the sense of Definition 2.2 is called a *topological conformal vertex algebra*. Similarly, other notions can be defined using conformal vertex algebras. For example, a *($\mathbb{Z}\times\mathbb{Z}$-graded) locally-grading-restricted conformal vertex algebra* is a ($\mathbb{Z}\times\mathbb{Z}$-graded) vertex operator algebra without grading restrictions but satisfying the local grading-restriction conditions. For simplicity, this terminology will be used in Subsections 4.3 and 5.1.

In the appendix (Section 5), we give two types of examples of (locally-)grading-restricted topological vertex operator algebras: The ones obtained by tensoring the ghost vertex operator algebra with vertex operator algebras without grading restrictions of central charge 26 satisfying the local grading restriction conditions, and the ones obtained by twisting $N=2$ superconformal vertex operator algebras. These examples are the main interesting (locally-)grading-restricted topological vertex operator algebras. They are all strong in the sense that $g^2(0)=0$.

2.2 Homotopy Gerstenhaber algebras and topological conformal field theories

Definition 2.4 A *Gerstenhaber algebra* is a graded commutative algebra A together with a bracket $[\cdot,\cdot]: A\otimes A\to A$ such that $[A^{(n)}, A^{(m)}]\subset A^{(n+m-1)}$ (where $A^{(n)}$, $A^{(m)}$, $n, m\in\mathbb{Z}$, are homogeneous components of A), satisfying

$$\begin{aligned}
[a,b] &= -(-1)^{(|a|-1)(|b|-1)}[b,a],\\
[a,[b,c]] &= [[a,b],c]+(-1)^{(|a|-1)(|b|-1)}[b,[a,c]],\\
[a,bc] &= [a,b]c+(-1)^{|a|(|b|-1)}b[a,c]
\end{aligned}$$

for any homogeneous elements a, b, c of A.

The definition of homotopy Gerstenhaber algebra is much more involved. Roughly speaking, a homotopy Gerstenhaber algebra means a "Gerstenhaber algebra up to homotopy together with homotopies for the homotopies." The precise notion of homotopy Gerstenhaber algebra we shall use is the one proposed by Voronov in [V]. This definition needs an operad E^1 of complexes called *homotopy Gerstenhaber operad*. Since the explicit definition of homotopy Gerstenhaber operad is involved and since we shall not need the details of this definition in the present paper, we only mention that E^1 is the first term of the spectral sequence corresponding to a stratification of a certain moduli space operad which is homotopy equivalent to the little disks operad. See [V] for details.

Definition 2.5 A *homotopy Gerstenhaber algebra* is an algebra over the homotopy Gerstenhaber operad E^1.

We need a corrected version by Voronov in [V] of a result of Kimura, Voronov and Zuckerman in [KVZ]. Recall the suboperad $K_{\mathfrak{H}_1}$ of the partial operad K discussed in Section 6.4 of [H5]. We also need the operad $\wedge TK_{\mathfrak{H}_1}$ of complexes of the exterior algebras of the tangent bundles of components of $K_{\mathfrak{H}_1}$: For any $n \geq 0$, let $TK_{\mathfrak{H}_1}(n)$ be the tangent bundle of $K_{\mathfrak{H}_1}(n)$ and $\wedge TK_{\mathfrak{H}_1}(n)$ the direct sum of all exterior powers of $TK_{\mathfrak{H}_1}(n)$. Then it is easy to see that the sequence $\wedge TK_{\mathfrak{H}_1} = \{\wedge TK_{\mathfrak{H}_1}(n)\}_{n\geq 0}$ has a natural structure of operad.

Let (H, Q) be a complex. Then the endomorphism operad

$$\mathcal{E}_H = \{\mathrm{Hom}(H^{\otimes n}, H)\}_{n\geq 0}$$

has a natural structure of operad of complexes. We shall still use Q to denote the differentials on $\mathrm{Hom}(H^{\otimes n}, H)$ for $n \geq 0$. A map μ_n from $\wedge TK_{\mathfrak{H}_1}(n)$ to $\mathrm{Hom}(H^{\otimes n}, H)$ is said to be *fiber-linear* if it is linear on the fiber of $\wedge TK_{\mathfrak{H}_1}(n)$. Then a fiber-linear smooth map μ_n from $\wedge TK_{\mathfrak{H}_1}(n)$ to $\mathrm{Hom}(H^{\otimes n}, H)$ can be viewed as a $\mathrm{Hom}(H^{\otimes n}, H)$-valued form on $K_{\mathfrak{H}_1}(n)$. In particular, the differential d on $\mathrm{Hom}(H^{\otimes n}, H)$-valued forms on $K_{\mathfrak{H}_1}(n)$ acts on μ_n. A morphism μ from $\wedge TK_{\mathfrak{H}_1}$ to a smooth operad is said to be *smooth* if μ_n, $n \geq 0$, are smooth and is said to be *fiber-linear* if μ_n, $n \geq 0$, are fiber-linear.

The following notion is due to Segal [S] and Getzler [G] (see also [KSV] and [KVZ]):

Definition 2.6 A *genus-zero topological conformal field theory* is a complex (H, Q) and a fiber-linear smooth morphism μ of operads from $\wedge TK_{\mathfrak{H}_1}$ to the endomorphism operad $\tilde{\mathcal{E}}_H$ such that

$$d\mu_n = Q\mu_n, \tag{2.2}$$

for $n \geq 0$, where as discussed above, d is the differential on $\mathrm{Hom}(H^{\otimes n}, H)$-valued forms on $K_{\mathfrak{H}_1}(n)$ and Q is the differential on $\mathrm{Hom}(H^{\otimes n}, H)$ induced from $Q : H \to H$.

The following corrected version of a result in [KVZ] is proved by Voronov in [V]:

Theorem 2.7 *A genus-zero topological conformal field theory has a structure of a homotopy Gerstenhaber algebra.*

In Subsection 4.4, the following result is proved:

Theorem 2.8 *For a locally-grading-restricted strong topological vertex operator algebra V, there is a locally convex completion H of V and an extension (still denoted Q) to H of the operator Q on V such that (H, Q) has a structure of a genus-zero topological conformal field theory. In particular, for a grading-restricted strong topological vertex operator algebra, the same conclusion holds.*

This theorem proves Conjecture 1.2 for locally-grading-restricted strong topological vertex operator algebras and in particular, for grading-restricted strong topological vertex operator algebras. It is easy to see that for locally-grading-restricted topological vertex operator algebras which are not strong, the same construction does not give genus-zero topological conformal field theories.

Combining Theorem 2.7 and Theorem 2.8, we obtain:

Theorem 2.9 *For a locally-grading-restricted strong topological vertex operator algebra V, there is a locally convex completion H of V such that the Gerstenhaber algebra structure on the cohomology of V is extended to a homotopy Gerstenhaber algebra structure on H. In particular, for a grading-restricted strong topological vertex operator algebra, the same conclusion holds.* $\square$

This theorem proves Conjecture 1.1 for locally-grading-restricted strong topological vertex operator algebras and in particular, for grading-restricted strong topological vertex operator algebras.

2.3 The main theorem

Theorem 2.8 and consequently Theorem 2.9 are corollaries of the main theorem, Theorem 2.10 below, of the present paper.

In Subsection 4.2, for $k < 0$, a topological vertex partial operad $\mathcal{T}^k$ of type k and a strong topological vertex partial operad $\bar{\mathcal{T}}^k$ of type k are constructed. There is a natural morphism of partial operads from the operad $\wedge TK_{\mathfrak{H}_1}$ to the partial operad $\mathcal{T}^k$ for any $k < 0$. The notions of (weakly) meromorphic algebra over $\mathcal{T}^k$ and $\bar{\mathcal{T}}^k$ are introduced in the same subsection. In Subsection 4.3, we prove the following main theorem of the present paper:

Theorem 2.10 *Let k be an integer less than 0. The category of meromorphic $\mathbb{Z} \times \mathbb{Z}$-graded algebras over $\mathcal{T}^k$ ($\bar{\mathcal{T}}^k$) is isomorphic to the category of grading-restricted (strong) topological vertex operator algebras of type k. The category of weakly meromorphic $\mathbb{Z} \times \mathbb{Z}$-graded algebras over $\mathcal{T}^k$ ($\bar{\mathcal{T}}^k$) is isomorphic to the category of locally-grading-restricted (strong) topological vertex operator algebras of type k.*

3 Semi-infinite forms

3.1 Left invariant meromorphic vector fields satisfying the Virasoro relations

For $c \in \mathbb{C}$, recall the $c/2$-th power $\tilde{K}^c(1)$ of the determinant line bundle over the moduli space $K(1)$ of spheres with tubes of type $(1, 1)$ constructed in [H5]. In this subsection, we construct left invariant meromorphic vector fields on $\tilde{K}^c(1)$ satisfying the Virasoro relations. These vector fields are needed in the construction of semi-infinite forms on the moduli spaces of spheres with tubes.

In [H5], the holomorphic line bundle $\tilde{K}^c(1)$ over $K(1)$ was identified with the space

$$\{(A^{(0)}, (a_0, A^{(1)}); C) \mid A^{(0)}, A^{(1)} \in H \subset \mathbb{C}^\infty, a_0 \in \mathbb{C}^\times, C \in \mathbb{C}\},$$

where H is the subset of $\mathbb{C}^\infty$ consisting of sequences $A = \{A_j\}_{j>0}$ such that $\exp\left(\sum_{j>0} A_j x^{j+1} \frac{d}{dx}\right) x$ is convergent in a neighborhood (depending on A) of 0. There is an associative partial operation ${}_1\widetilde{\infty}_0^c$ on $\tilde{K}^c(1)$ with identity $\tilde{I} = (\mathbf{0}, (1, \mathbf{0}); 1)$, here $\mathbf{0}$ denote the infinite sequence with all components to be 0. Also in [H5], the notions of meromorphic functions on $\tilde{K}^c(1)$ and the meromorphic tangent space of $\tilde{K}^c(1)$ at $\tilde{Q} \in \tilde{K}^c(1)$ were introduced.

The following result is obvious:

Proposition 3.1 *The meromorphic tangent space $T_{\tilde{Q}}\tilde{K}^c(1)$ of $\tilde{K}^c(1)$ at any $\tilde{Q} \in \tilde{K}^c(1)$ is linearly isomorphic to the vector space of infinite linear combinations of the meromorphic tangent vectors $\frac{\partial}{\partial A_j^{(0)}}\big|_{\tilde{Q}}$, $\frac{\partial}{\partial A_j^{(1)}}\big|_{\tilde{Q}}$, $j > 0$, $\frac{\partial}{\partial a_0}\big|_{\tilde{Q}}$, and $\frac{\partial}{\partial C}\big|_{\tilde{Q}}$.* $\square$

A *meromorphic tangent field on* $\tilde{K}^c(1)$ is a tangent vector fields on $\tilde{K}^c(1)$ mapping meromorphic functions to meromorphic functions. Then we have:

Proposition 3.2 *A tangent vector field on $\tilde{K}^c(1)$ is meromorphic if and only if it is a an infinite linear combination of the tangent fields $\frac{\partial}{\partial A_j^{(0)}}$, $\frac{\partial}{\partial A_j^{(1)}}$, $j > 0$, $\frac{\partial}{\partial a_0}$ and $\frac{\partial}{\partial C}$ with meromorphic functions on $\tilde{K}^c(1)$ as coefficients.* $\square$

We denote the space of meromorphic tangent fields on $\tilde{K}^c(1)$ by $\Gamma(T\tilde{K}^c(1))$. This space has a subspace consists of meromorphic tangent fields which are finite linear combinations of the tangent fields $\frac{\partial}{\partial A_j^{(0)}}$, $\frac{\partial}{\partial A_j^{(1)}}$, $j > 0$, $\frac{\partial}{\partial a_0}$ and $\frac{\partial}{\partial C}$ with meromorphic functions on $\tilde{K}^c(1)$ as coefficients. We denote this subspace by $\Gamma(\hat{T}\tilde{K}^c(1))$.

In [H5], the first author found formal series $\Psi_j = \Psi_j(A, B, a_0)$, $j \in \mathbb{Z}$, in A_j, B_j and a_0 such that

$$e_{\mathcal{A}}(x)(\alpha_0)^{x\dfrac{d}{dx}} e_{\mathcal{B}}^{-1}(x^{-1}) = e_{\Psi^-}(x^{-1}) e_{\Psi^+}^{-1}(x)(\alpha_0)^{x\dfrac{d}{dx}} e^{-\Psi_0 x\dfrac{d}{dx}}, \tag{3.1}$$

where $\Psi^+ = \{\Psi_j(A, B, a_0)\}_{j>0}$, $\Psi^- = \{\Psi_{-j}(A, B, a_0)\}_{j>0}$,

$$e_A(x) = \exp\left(\sum_{j>0} A_j x^{j+1}\frac{d}{dx}\right)$$

for any sequence A and $e_A^{-1}(x)$ is the compositional inverse of $e_A(x)$.

For any $\tilde{P} \in \tilde{K}^c(1)$, we define the partial map $\ell_{\tilde{P}} : \tilde{K}^c(1) \to \tilde{K}^c(1)$ by $\ell_{\tilde{P}}(\tilde{Q}) = \tilde{P}\,{}_1\widetilde{\infty}_0^c\,\tilde{Q}$ for any $\tilde{Q} \in \tilde{K}^c(1)$ whenever the sewing makes sense. Recall from [H5] that the tangent space $\hat{T}_{\tilde{I}}\tilde{K}^c(1)$ has a basis

$$\mathcal{L}(j) = -\frac{\partial}{\partial A_{-j}^{(0)}}\bigg|_{\tilde{I}}$$

for $j < 0$,

$$\mathcal{L}(j) = -\frac{\partial}{\partial A_j^{(1)}}\Big|_{\tilde{I}}$$

for $j > 0$,

$$\mathcal{L}(0) = -\frac{\partial}{\partial a_0}\Big|_{\tilde{I}},$$

$$\mathcal{K} = C\frac{\partial}{\partial C}\Big|_{\tilde{I}} = \frac{\partial}{\partial C}\Big|_{\tilde{I}}$$

and a bracket operation $[\cdot,\cdot]$ coming from the sewing operation ${}_1\widetilde{\infty}_0^c$ and satisfying the Virasoro relations with central charge c:

$$[\mathcal{L}(m), \mathcal{L}(n)] = (m-n)\mathcal{L}(m+n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\mathcal{K}.$$

We define tangent vector fields $\mathbb{K}$ and $\mathbb{L}(j)$ for $j \in \mathbb{Z}$ by

$$\mathbb{K}\big|_{\tilde{P}} = (\ell_{\tilde{P}})_*(\mathcal{K})$$

and

$$\mathbb{L}(j)\big|_{\tilde{P}} = (\ell_{\tilde{P}})_*(\mathcal{L}(j))$$

for $j \in \mathbb{Z}$ and $\tilde{P} \in \tilde{K}^c(1)$, where $(\ell_{\tilde{P}})_*$ is the map from $T_{\tilde{I}}\tilde{K}^c(1)$ to $T_{\tilde{P}}\tilde{K}^c(1)$ induced from $\ell_{\tilde{P}}$. Observe that the identity element $\tilde{I}$ can be sewn to any element $\tilde{P} \in \tilde{K}^c(1)$, therefore the the vector fields $\mathbb{K}$ and $\mathbb{L}(j)$, $j \in \mathbb{Z}$, are well defined everywhere and are left invariant with respect to the sewing operation. Next we want to write down explicitly the vector fields defined above. To do this we need more notations and lemmas.

Let

$$\tilde{P} = (A^{(0)}, (a_0, A^{(1)}); C_{\tilde{P}}) \in \tilde{K}^c(1)$$

and

$$\tilde{Q} = (B^{(0)}, (b_0, B^{(1)}); C_{\tilde{Q}}) \in \tilde{K}^c(1)$$

such that $\tilde{P}{}_1\widetilde{\infty}_0^c\tilde{Q}$ exists. In [H5], it was shown that

$$\tilde{P}{}_1\widetilde{\infty}_0^c\tilde{Q} = (C^{(0)}, (c_0, C^{(1)}); C_{\tilde{P}_1\widetilde{\infty}_0^c\tilde{Q}})$$

where $C^{(0)} = C^{(0)}(\tilde{P}, \tilde{Q}) \in H$, $C^{(1)} = C^{(1)}(\tilde{P}, \tilde{Q}) \in H$ and $c_0 = c_0(\tilde{P}, \tilde{Q}) \in \mathbb{C}^\times$ are determined by

$$\begin{aligned}
\exp\left(\sum_{j>0} C_j^{(0)} x^{j+1} \frac{d}{dx}\right) x &= \exp\left(-\sum_{j>0} \Psi_{-j} x^{j+1} \frac{d}{dx}\right) \cdot \\
&\quad \cdot \exp\left(\sum_{j>0} A_j^{(0)} x^{j+1} \frac{d}{dx}\right) x, \\
c_0^{(1)} \exp\left(\sum_{j>0} C_j^{(1)} x^{j+1} \frac{d}{dx}\right) x &= \exp\left(-\sum_{j>0} \Psi_j x^{j+1} \frac{d}{dx}\right) \left(a_0 e^{-\Psi_0}\right)^{x\frac{d}{dx}} \cdot \\
&\quad \cdot \exp\left(\sum_{j>0} (B_j^{(1)} x^{j+1} \frac{d}{dx}\right) \left(b_0^{(1)}\right)^{x\frac{d}{dx}} x
\end{aligned}$$

and

$$C_{\tilde{P}_1 \widetilde{\infty}_0^c \tilde{Q}} = e^{c\Gamma(A^{(1)}, B^{(0)}, a_0^{(1)})} C_{\tilde{P}} C_{\tilde{Q}},$$

where $\Gamma = \Gamma(A^{(1)}, B^{(0)}, a_0^{(1)})$ is a convergent series in $a_0^{(1)}$, $A_j^{(1)}$, $B_j^{(0)}$, $j > 0$. Recall the compositions $\circ$ for $\mathbb{C}^\infty$ and for $\mathbb{C}^\times \times \mathbb{C}^\infty$ defined in [H5], pages 46 and 47. Using these compositions, we have

$$\begin{aligned}
C^{(0)}(\tilde{P}, \tilde{Q}) &= (-\Psi^-) \circ A^{(0)}, && (3.2) \\
(c_0^{(1)}(\tilde{P}, \tilde{Q}), C^{(1)}(\tilde{P}, \tilde{Q})) &= (a_0^{(1)} e^{-\Psi_0}, -\Psi^+) \circ (b_0^{(1)}, B^{(1)}), && (3.3) \\
c_0^{(1)}(\tilde{P}, \tilde{Q}) &= a_0^{(1)} b_0^{(1)} e^{-\Psi_0}, && (3.4) \\
C^{(1)}(\tilde{P}, \tilde{Q}) &= (-\Psi^+) \circ B^{(1)}(a_0^{(1)} e^{-\Psi_0}). && (3.5)
\end{aligned}$$

The following lemma is needed later:

Lemma 3.3 *Define the weights in the space of formal series in* $a_0^{(1)}, b_0^{(1)}$, $A_j^{(0)}, A_j^{(1)}, B_j^{(0)}, B_j^{(1)}$, $j > 0$, *by setting*

$$\begin{aligned}
\text{wt } A_j^{(0)} &= -j, \\
\text{wt } A_j^{(1)} &= j, \\
\text{wt } B_j^{(0)} &= -j, \\
\text{wt } B_j^{(1)} &= j
\end{aligned}$$

for $j \in \mathbb{Z}$,

$$\mathrm{wt}\ a_0^{(1)} = 0$$

and

$$\mathrm{wt}\ b_0^{(1)} = 0,$$

Then we have

$$\begin{aligned}
\mathrm{wt}\ \Gamma &= 0,\\
\mathrm{wt}\ \Psi_j &= j,\\
\mathrm{wt}\ C_j^{(0)} &= -j,\\
\mathrm{wt}\ C_j^{(1)} &= j
\end{aligned}$$

for $j > 0$.

Proof. By keeping track of the weights of all the series and operators in the relevant proofs in [H5], it is easy to see that the conclusion is true. $\square$

Proposition 3.4 *For any* $\tilde{P} = (A^{(0)}, (a_0, A^{(1)}); C_{\tilde{P}}) \in \tilde{K}^c(1)$, *we have*

$$\mathbb{K}\big|_{\tilde{P}} = C\frac{\partial}{\partial C}\bigg|_{\tilde{P}}, \tag{3.6}$$

$$\mathbb{L}(0)\big|_{\tilde{P}} = -a_0\frac{\partial}{\partial a_0}\bigg|_{\tilde{P}}, \tag{3.7}$$

$$\begin{aligned}
\mathbb{L}(j)\big|_{\tilde{P}} &= -\sum_{k>0}\frac{\partial((-\Psi^+)\circ B^{(1)}(a_0^{(1)}e^{-\Psi_0}))_k}{\partial B_j^{(1)}}\bigg|_{B^{(0)}=B^{(1)}=\mathbf{0}}\frac{\partial}{\partial A_k^{(1)}}\bigg|_{\tilde{P}}\\
&= -a_0^j\frac{\partial}{\partial A_j^{(1)}}\bigg|_{\tilde{P}}\\
&\quad -\sum_{k>j}\frac{\partial((-\Psi^+)\circ B^{(1)}(a_0^{(1)}e^{-\Psi_0}))_k}{\partial B_j^{(1)}}\bigg|_{B^{(0)}=B^{(1)}=\mathbf{0}}\frac{\partial}{\partial A_k^{(1)}}\bigg|_{\tilde{P}},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\mathbb{L}(-j)\big|_{\tilde{P}} &= \sum_{k>0}\frac{\partial\Psi_k}{\partial B_j^{(0)}}\bigg|_{B^{(0)}=\mathbf{0}}\frac{\partial}{\partial A_k^{(1)}}\bigg|_{\tilde{P}} + \frac{\partial\Psi_0}{\partial B_j^{(0)}}\bigg|_{B^{(0)}=\mathbf{0}}a_0\frac{\partial}{\partial a_0}\bigg|_{\tilde{P}}\\
&\quad -\sum_{k>0}\frac{\partial C_k^{(0)}}{\partial B_j^{(0)}}\bigg|_{B^{(0)}=\mathbf{0}}\frac{\partial}{\partial A_k^{(0)}}\bigg|_{\tilde{P}} + c\frac{\partial\Gamma}{\partial B_j^{(0)}}\bigg|_{B^{(0)}=\mathbf{0}}C\frac{\partial}{\partial C}\bigg|_{\tilde{P}}
\end{aligned} \tag{3.9}$$

for $j > 0$.

Proof. By the chain rule,

$$\begin{aligned}
\mathbb{L}(0) &= -\frac{\partial c_0^{(1)}}{\partial b_0}\Big|_{B^{(0)}=B^{(1)}=\mathbf{0},b_0=1}\frac{\partial}{\partial a_0}\Big|_{\tilde{P}} \\
&\quad -\sum_{k>0}\frac{\partial C_k^{(0)}(\tilde{P},\tilde{Q})}{\partial b_0}\Big|_{B^{(0)}=B^{(1)}=\mathbf{0},b_0=1}\frac{\partial}{\partial A_k^{(0)}}\Big|_{\tilde{P}} \\
&\quad -\sum_{k>0}\frac{\partial C_k^{(1)}(\tilde{P},\tilde{Q})}{\partial b_0}\Big|_{B^{(0)}=B^{(1)}=\mathbf{0},b_0=1}\frac{\partial}{\partial A_k^{(1)}}\Big|_{\tilde{P}} - \frac{\partial C_{\tilde{P}_1\widetilde{\infty}_0^c\tilde{Q}}}{\partial b_0}\frac{\partial}{\partial C}\Big|_{\tilde{P}}
\end{aligned}$$

$$\begin{aligned}
\mathbb{L}(j)\big|_{\tilde{P}} &= -\frac{\partial c_0^{(1)}}{\partial B_j^{(1)}}\Big|_{B^{(0)}=B^{(1)}=\mathbf{0},b_0=1}\frac{\partial}{\partial a_0}\Big|_{\tilde{P}} \\
&\quad -\sum_{k>0}\frac{\partial C_k^{(0)}(A,B)}{\partial B_j^{(1)}}\Big|_{B^{(0)}=B^{(1)}=\mathbf{0},b_0=1}\frac{\partial}{\partial A_k^{(0)}}\Big|_{\tilde{P}} \\
&\quad -\sum_{k>0}\frac{\partial C_k^{(1)}(A,B)}{\partial B_j^{(1)}}\Big|_{B^{(0)}=B^{(1)}=\mathbf{0},b_0=1}\frac{\partial}{\partial A_k^{(1)}}\Big|_{\tilde{P}} - \frac{\partial C_{\tilde{P}_1\widetilde{\infty}_0^c\tilde{Q}}}{\partial B_j^{(1)}}\frac{\partial}{\partial C}\Big|_{\tilde{P}}
\end{aligned}$$

$j > 0$, and

$$\begin{aligned}
\mathbb{L}(-j) &= -\frac{\partial c_0^{(1)}}{\partial B_j^{(0)}}\Big|_{B^{(0)}=B^{(1)}=\mathbf{0},b_0=1}\frac{\partial}{\partial a_0}\Big|_{\tilde{P}} \\
&\quad -\sum_{k>0}\frac{\partial C_k^{(0)}(A,B)}{\partial B_j^{(0)}}{}_{B^{(0)}=B^{(1)}=\mathbf{0},b_0=1}\frac{\partial}{\partial A_k^{(0)}}\Big|_{\tilde{P}} \\
&\quad -\sum_{k>0}\frac{\partial C_k^{(1)}(A,B)}{\partial B_j^{(0)}}\Big|_{B^{(0)}=B^{(1)}=\mathbf{0},b_0=1}\frac{\partial}{\partial A_k^{(1)}}\Big|_{\tilde{P}} \\
&\quad -\frac{\partial C_{\tilde{P}_1\widetilde{\infty}_0^c\tilde{Q}}}{\partial B_j^{(0)}}\Big|_{B^{(0)}=B^{(1)}=\mathbf{0},b_0=1}\frac{\partial}{\partial C}\Big|_{\tilde{P}}
\end{aligned}$$

$j > 0$. Using (3.2)–(3.5), Lemma 3.3 above and Proposition 2.2.5 in [H5], we obtain the formulas (3.7)–(3.9). □

Proposition 3.5 *The tangent vector fields $\mathbb{L}(j)$, $j \in \mathbb{Z}$, are meromorphic and their restrictions $\mathbb{L}(j)\big|_{\tilde{Q}}$, $j \in \mathbb{Z}$, at any point $\tilde{Q} \in T_{\tilde{Q}}\tilde{K}^c(1)$ together with $\frac{\partial}{\partial C}\big|_{\tilde{Q}}$, form a basis at $\tilde{Q}$.*

Proof. The first conclusion follows from arguments using weights for all relevant formal series. In fact, all coefficients, except $\frac{\partial C_k^{(0)}}{\partial B_j^{(0)}}$, in the explicit expressions (3.7)–(3.9) of $\mathbb{L}(j)$, $j \in \mathbb{Z}$, are weight-homogeneous polynomials in $a_0^{(1)}$, $A_j^{(1)}$, $j > 0$. The coefficient $\frac{\partial C_k^{(0)}}{\partial B_j^{(0)}}$ is a weight-homogeneous polynomial in $a_0^{(1)}$, $A_j^{(1)}$, $A_j^{(0)}$, $j > 0$.

To prove the second conclusion, note that we have the decomposition $\tilde{Q} = \tilde{Q}_1 {}_1\widetilde{\infty}_0^c \tilde{Q}_2$, where the local coordinates of $\tilde{Q}_1$ and $\tilde{Q}_2$ at 0 and ∞, respectively, are trivial. In the case that $C_{\tilde{Q}} \neq 0$, the maps

$$(\ell_{\tilde{Q}_2})_* : T_{\tilde{I}}\tilde{K}^c(1) \to T_{\tilde{Q}_2}\tilde{K}^c(1)$$

and

$$(\ell_{\tilde{Q}_1})_* : T_{\tilde{Q}_2}\tilde{K}^c(1) \to T_{\tilde{Q}}\tilde{K}^c(1)$$

are linear isomorphisms, since $\tilde{Q}_1$ and $\tilde{Q}_2$ are invertible in $\tilde{K}^c(1)$. So $(\ell_{\tilde{Q}})_* = (\ell_{\tilde{Q}_1})_* \circ (\ell_{\tilde{Q}_2})_*$ is also a linear isomorphism, proving the second conclusion in this case. In the case of $C_{\tilde{Q}} = 0$, since $\tilde{K}^c(1) = K(1) \times \mathbb{C}$, $\tilde{Q} \in K(1) \times \{0\}$. Let $\tilde{Q} = (Q; 0)$ where $Q \in K(1)$. Then in the decomposition $\tilde{Q} = \tilde{Q}_1 {}_1\widetilde{\infty}_0^c \tilde{Q}_2$, we can also choose $\tilde{Q}_1$ and $\tilde{Q}_2$ to be in $K(1) \times \{0\}$ such that $\tilde{Q}_1 = (Q_1; 0)$ and $\tilde{Q}_2 = (Q_2, 0)$. Since Q_1 and Q_2 are invertible, $(\ell_{\tilde{Q}_2})_*$ and $(\ell_{\tilde{Q}_1})_*$ induce linear isomorphisms from $T_{\tilde{I}}\tilde{K}^c(1)/\mathbb{C}\frac{\partial}{\partial C}\big|_{\tilde{I}}$ to $T_{\tilde{Q}_2}\tilde{K}^c(1)/\mathbb{C}\frac{\partial}{\partial C}\big|_{\tilde{Q}_2}$ and from $T_{\tilde{Q}_2}\tilde{K}^c(1)/\mathbb{C}\frac{\partial}{\partial C}\big|_{\tilde{Q}_2}$ to $T_{\tilde{Q}}\tilde{K}^c(1)/\mathbb{C}\frac{\partial}{\partial C}\big|_{\tilde{Q}}$, respectively. Thus the second conclusion is true in this case. $\square$

Let $\mathfrak{L}$ be the Virasoro algebra and let L_j, $j \in \mathbb{Z}$, and d, be the usual basis of $\mathfrak{L}$. Recall that in [H5], the space of meromorphic functions on $\tilde{K}^c(1)$ is denoted by $\tilde{D}_1^c$. The eigenspaces of $\mathbb{L}(0)$ as an operator on $\tilde{D}_1^c$ give a $\mathbb{Z}$-grading to $\tilde{D}_1^c$. Let $\tilde{D}_1^{c;1}$ be the space of meromorphic functions on $\tilde{K}^c(1)$ consisting of elements which are linear in the last coordinate C of $\tilde{Q} = (A^{(0)}, (a_0^{(1)}, A^{(1)}); C) \in \tilde{K}^c(1)$.

Proposition 3.6 *The restriction $\mathbb{L}(j)\big|_{\tilde{D}_1^{c;1}}$, $j \in \mathbb{Z}$, and $\mathbb{K}\big|_{\tilde{D}_1^{c;1}}$, of the meromorphic vector fields $\mathbb{L}(j)$, $j \in \mathbb{Z}$, and $\mathbb{K}$, to $\tilde{D}_1^{c;1}$ satisfy the Virasoro relations with central charge c:*

$$[\mathbb{L}(m)\big|_{\tilde{D}_1^{c;1}}, \mathbb{L}(n)\big|_{\tilde{D}_1^{c;1}}] \quad = \quad (m-n)\mathbb{L}(m+n)\big|_{\tilde{D}_1^{c;1}} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\mathbb{K}\big|_{\tilde{D}_1^{c;1}},$$

$$(3.10)$$

$$[\mathbb{L}(m)\big|_{\tilde{D}_1^{c;1}}, \mathbb{K}\big|_{\tilde{D}_1^{c;1}}] \;=\; 0. \tag{3.11}$$

In particular, the $\mathbb{Z}$-graded vector space $\tilde{D}_1^{c;1}$ equipped with the map $\pi : \mathfrak{L} \to$ End $\tilde{D}_1^{c;1}$ defined by $\pi(L_j) = \mathbb{L}(j)\big|_{\tilde{D}_1^{c;1}}$, $j \in \mathbb{Z}$ and $\pi(d) = c\mathbb{K}\big|_{\tilde{D}_1^{c;1}} = cI_{\tilde{D}_1^{c;1}}$ ($I_{\tilde{D}_1^{c;1}}$ being the identity map on $\tilde{D}_1^{c;1}$), is a module for the Virasoro algebra with central charge c.

Proof. The second equation (3.11) is obvious from the Proposition 3.4. We prove the first equation (3.10). Fix $\tilde{Q}_0 \in \tilde{K}^c(1)$. Let F be any meromorphic function on $\tilde{K}^c(1)$. Then for any $m > 0$, $n < 0$ and $\tilde{Q} \in \tilde{K}^c(1)$, by definition, we have

$$\begin{aligned}
(\mathbb{L}(i)F)(\tilde{Q}) &= -\frac{\partial}{\partial A_i^{(1)}}\Big|_{\tilde{P}_1=\tilde{I}} F(\tilde{Q}_1 \widetilde{\infty}_0^c \tilde{P}_1), \\
(\mathbb{L}(j)F)(\tilde{Q}) &= -\frac{\partial}{\partial B_j^{(0)}}\Big|_{\tilde{P}_2=\tilde{I}} F(\tilde{Q}_1 \widetilde{\infty}_0^c \tilde{P}_2),
\end{aligned}$$

where $\tilde{P}_1 = (A^{(0)}, (\alpha_0, A^{(1)}), C_1)$ $\tilde{P}_2 = (B^{(0)}, (\beta_0, B^{(1)}), C_2)$ are elements near $\tilde{I}$.

We have

$$\begin{aligned}
(\mathbb{L}(m)\mathbb{L}(n))\big|_{\tilde{Q}_0} F &= \mathbb{L}(m)\big|_{\tilde{Q}_0}(\mathbb{L}(n)F) \\
&= -\frac{\partial}{\partial A_m^{(1)}}\Big|_{\tilde{P}_1=\tilde{I}} (\mathbb{L}(n)F)(\tilde{Q}_0{}_1\widetilde{\infty}_0^c \tilde{P}_1) \\
&= \frac{\partial}{\partial A_m^{(1)}}\Big|_{\tilde{P}_1=\tilde{I}} \frac{\partial}{\partial B_n^{(0)}}\Big|_{\tilde{P}_2=\tilde{I}} F(\tilde{Q}_0{}_1\widetilde{\infty}_0^c \tilde{P}_1{}_1\widetilde{\infty}_0^c \tilde{P}_2) \\
&= \frac{\partial}{\partial A_m^{(1)}}\Big|_{\tilde{P}_1=\tilde{I}} \frac{\partial}{\partial B_n^{(0)}}\Big|_{\tilde{P}_2=\tilde{I}} ((\ell_{Q_0})^* F)(\tilde{P}_1{}_1\widetilde{\infty}_0^c \tilde{P}_2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\mathbb{L}(n)\mathbb{L}(m))\big|_{\tilde{Q}_0} F &= \mathbb{L}(n)\big|_{\tilde{Q}_0}(\mathbb{L}(m)F) \\
&= -\frac{\partial}{\partial A_n^{(0)}}\Big|_{\tilde{P}_1=\tilde{I}} (\mathbb{L}(m)F)(\tilde{Q}_0{}_1\widetilde{\infty}_0^c \tilde{P}_1) \\
&= \frac{\partial}{\partial A_n^{(0)}}\Big|_{\tilde{P}_1=\tilde{I}} \frac{\partial}{\partial B_m^{(1)}}\Big|_{\tilde{P}_2=\tilde{I}} F(\tilde{Q}_0{}_1\widetilde{\infty}_0^c \tilde{P}_1{}_1\widetilde{\infty}_0^c \tilde{P}_2)
\end{aligned}$$

$$= \frac{\partial}{\partial A_n^{(0)}}\Big|_{\tilde{P}_1=\tilde{I}} \frac{\partial}{\partial B_m^{(1)}}\Big|_{\tilde{P}_2=\tilde{I}} ((\ell_{\tilde{Q}_0})^* F)(\tilde{P}_1 {}_1\widetilde{\infty}_0^c \tilde{P}_2).$$

Thus, by Proposition 3.5.2 in [H5],

$$\begin{aligned}
&[\mathbb{L}(m), \mathbb{L}(n)]\big|_{\tilde{Q}_0} F \\
&\quad = \left(\frac{\partial}{\partial A_m^{(1)}}\Big|_{\tilde{P}_1=\tilde{I}} \frac{\partial}{\partial B_n^{(0)}}\Big|_{\tilde{P}_2=\tilde{I}} - \frac{\partial}{\partial A_n^{(0)}}\Big|_{\tilde{P}_1=\tilde{I}} \frac{\partial}{\partial B_m^{(1)}}\Big|_{\tilde{P}_2=\tilde{I}} \right) \cdot \\
&\qquad\qquad \cdot ((\ell_{\tilde{Q}_0})^* F)(\tilde{P}_{11}\widetilde{\infty}_0^c \tilde{P}_2) \\
&\quad = ((m-n)\mathcal{L}(m+n) + \frac{c}{12}(m^3-m)\delta_{m+n,0} C \frac{\partial}{\partial C})(\ell_{\tilde{Q}_0})^*(F) \\
&\quad = (\ell_{\tilde{Q}_0})_*((m-n)\mathcal{L}(m+n) + \frac{c}{12}(m^3-m)\delta_{m+n,0} C \frac{\partial}{\partial C}) F \\
&\quad = ((m-n)\mathbb{L}(m+n) + \frac{c}{12}(m^3-m)\delta_{m+n,0}\mathbb{K})\big|_{\tilde{Q}_0} F,
\end{aligned}$$

proving the Virasoro relation (3.10) for $m > 0, n < 0$. We can prove (3.10) for other m and n similarly. $\square$

3.2 Semi-infinite forms on $K(0)$

In this subsection we introduce and study semi-infinite forms on $K(0)$. We shall use structures on $K(1)$ to define semi-infinite forms on $K(0)$.

We need the embedding map $\mathfrak{E}$ from $K(0)$ to $K(1)$ defined by

$$\mathfrak{E}(A) = (A, (1, \mathbf{0})).$$

It has a left inverse $\mathfrak{D}$ defined by

$$\mathfrak{D}(A^{(0)}, (a_0^{(1)}, A^{(1)})) = (A^{(0)}, (a_0^{(1)}, A^{(1)})) {}_1\infty_0 \mathbf{0}.$$

These two maps induce $\tilde{\mathfrak{E}} : \tilde{K}^c(0) \to \tilde{K}^c(1)$ and $\tilde{\mathfrak{D}} : \tilde{K}^c(1) \to \tilde{K}^c(0)$.

For $\tilde{Q} \in \tilde{K}^c(1)$, we denote the subspace of $T_{\tilde{Q}}\tilde{K}^c(1)$ consisting of finite linear combinations of $\mathbb{L}(j)\big|_{\tilde{Q}}$, $j \in \mathbb{Z}$, $\frac{\partial}{\partial C}\big|_{\tilde{P}}$, by $\hat{T}_{\tilde{Q}}\tilde{K}^c(1)$. This space has a $\mathbb{Z}$-grading called *weight* defined by wt $\mathbb{L}(j)\big|_{\tilde{P}} = -j$, $j \in \mathbb{Z}$, wt $\mathbb{K}\big|_{\tilde{P}} = 0$. For $\tilde{P} \in \tilde{K}^c(0)$, we denote the subspace of $T_{\tilde{P}}\tilde{K}^c(0)$ consisting of finite linear combinations of $\tilde{\mathfrak{D}}_*(\mathbb{L}(j))\big|_{\tilde{P}}$, $j < -1$, $\tilde{\mathfrak{D}}_*(\frac{\partial}{\partial C})\big|_{\tilde{P}}$, by $\hat{T}_{\tilde{P}}\tilde{K}^c(0)$. Clearly,

$$\hat{T}_{\tilde{P}}\tilde{K}^c(0) = (\tilde{\mathfrak{D}}_*)\big|_{\tilde{\mathfrak{E}}(\tilde{P})}(\hat{T}_{\tilde{\mathfrak{E}}(\tilde{P})}\tilde{K}^c(1)).$$

The union of $\hat{T}_{\tilde{P}}\tilde{K}^c(0)$ for $\tilde{P} \in \tilde{K}^c(0)$ is a holomorphic vector bundle $\hat{T}\tilde{K}^c(0)$ over $\tilde{K}^c(0)$. Using the flat section ψ_0 of $\tilde{K}^c(0)$ constructed in [H5], we pull $\hat{T}\tilde{K}^c(0)$ back to a holomorphic vector bundle $\psi_0^*(\hat{T}\tilde{K}^c(0))$ over $K(0)$.

For $\tilde{Q} \in \tilde{K}^c(1)$, we denote the graded dual space of $\hat{T}_{\tilde{Q}}\tilde{K}^c(1)$ by $\hat{T}'_{\tilde{Q}}\tilde{K}^c(1)$. The union of $\hat{T}'_{\tilde{Q}}\tilde{K}^c(0)$ for $\tilde{Q} \in \tilde{K}^c(1)$ is a holomorphic vector bundle $\hat{T}'\tilde{K}^c(1)$ over $\tilde{K}^c(1)$. Let $\mathbb{L}'(j)$, $j \in \mathbb{Z}$, and $(\frac{\partial}{\partial C})'$ be the sections of $\hat{T}'\tilde{K}^c(1)$ such that for $\tilde{Q} \in \tilde{K}^c(1)$, $\mathbb{L}'(j)\big|_{\tilde{Q}}$, $j \in \mathbb{Z}$, and $(\frac{\partial}{\partial C})'\big|_{\tilde{Q}}$ form the dual basis of the basis $\mathbb{L}(j)\big|_{\tilde{Q}}$, $j \in \mathbb{Z}$, and $\frac{\partial}{\partial C}\big|_{\tilde{Q}}$. For $\tilde{P} \in \tilde{K}^c(0)$, we denote the subspace of the dual space of $T_{\tilde{P}}\tilde{K}^c(0)$ consisting of finite linear combinations of $\tilde{\mathfrak{E}}^*(\mathbb{L}'(j))\big|_{\tilde{P}}$, $j < -1$, and $\tilde{\mathfrak{E}}^*((\frac{\partial}{\partial C})')\big|_{\tilde{P}}$ by $\hat{T}'_{\tilde{P}}\tilde{K}^c(0)$. Clearly,

$$\hat{T}'_{\tilde{P}}\tilde{K}^c(0) = \tilde{\mathfrak{E}}^*\big|_{\tilde{\mathfrak{E}}(\tilde{P})}(\hat{T}'_{\tilde{\mathfrak{E}}(\tilde{P})}\tilde{K}^c(1)).$$

Note that the kernel of $\tilde{\mathfrak{E}}^*\big|_{\tilde{\mathfrak{E}}(\tilde{P})}$ is spanned by $\mathbb{L}'(j)$, $j \geq -1$. The union of kernels of $\tilde{\mathfrak{E}}^*\big|_{\tilde{\mathfrak{E}}(\tilde{P})}$ for $\tilde{P} \in \tilde{K}^c(0)$ forms a holomorphic vector bundle Ker $\tilde{\mathfrak{E}}^*$ over $\tilde{K}^c(0)$. Using the flat section ψ_0 of $\tilde{K}^c(0)$, we pull Ker $\tilde{\mathfrak{E}}^*$ back to a holomorphic vector bundle $\psi_0^*(\text{Ker } \tilde{\mathfrak{E}}^*)$ over $K(0)$.

Motivated by the semi-infinite cohomology of graded Lie algebras introduced by Feigin in [Fe] and developed by Frenkel-Garland-Zuckerman in [FGZ] and by Lian-Zuckerman [LZ1], we now consider the holomorphic vector bundle $(\wedge\psi_0^*(\text{Ker } \tilde{\mathfrak{E}}^*)) \wedge (\wedge\psi_0^*(\hat{T}\tilde{K}^c(0)))$ where $\wedge\psi_0^*(\text{Ker } \tilde{\mathfrak{E}}^*)$ and $\wedge\psi_0^*(\hat{T}\tilde{K}^c(0))$ are the wedge product bundles of $\psi_0^*(\text{Ker } \tilde{\mathfrak{E}}^*)$ and $\psi_0^*(\hat{T}\tilde{K}^c(0))$, respectively. For any $P \in K(0)$, the fiber of $(\wedge\psi_0^*(\text{Ker } \tilde{\mathfrak{E}}^*)) \wedge (\wedge\psi_0^*(\hat{T}\tilde{K}^c(0)))$ over P is a $\mathbb{Z} \times \mathbb{Z}$-graded vector spaces spanned by the canonical basis

$$\mathbb{L}'(i_1)\big|_{\tilde{\mathfrak{E}}(\psi_0(P))} \wedge \ldots \wedge \mathbb{L}'(i_m)\big|_{\tilde{\mathfrak{E}}(\psi_0(P))} \wedge \tilde{\mathfrak{D}}_*(\mathbb{L}(j_1))\big|_{\psi_0(P)} \wedge \ldots \wedge \tilde{\mathfrak{D}}_*(\mathbb{L}(j_n))\big|_{\psi_0(P)},$$

$m, n > 0$, $i_1, \ldots, i_m \geq -1$, $j_1, \ldots, j_n < -1$. The integer $m + n$ is called the *fermion number* or *ghost number* of these basis elements. If $m + n \in 2\mathbb{Z}$ ($\in 2\mathbb{Z} + 1$), these basis elements are said to be *even* (*odd*). The integers $-\sum_{k=1}^m i_k + \sum_{k=1}^n j_k$ are called *weights* or *degrees* of the canonical basis elements.

This bundle has holomorphic sections

$$P \mapsto \mathbb{L}'(i_1)\big|_{\tilde{\mathfrak{E}}(\psi_0(P))} \wedge \ldots \wedge \mathbb{L}'(i_m)\big|_{\tilde{\mathfrak{E}}(\psi_0(P))} \wedge \tilde{\mathfrak{D}}_*(\mathbb{L}(j_1))\big|_{\psi_0(P)} \wedge \ldots \wedge \tilde{\mathfrak{D}}_*(\mathbb{L}(j_n))\big|_{\psi_0(P)},$$

$m,n \geq 0,\ i_1,\ldots,i_m \geq -1,\ j_1,\ldots,j_n < -1$. We shall write these sections simply as

$$\psi_0^*(\mathbb{L}'(i_1)) \wedge \ldots \wedge \psi_0^*(\mathbb{L}'(i_m)) \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_1))) \wedge \ldots \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_n))).$$

We denote the space of these sections by $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$.

We also consider the graded dual bundle

$$((\wedge\psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*)) \wedge (\wedge\psi_0^*(\hat{T}\tilde{K}^c(0))))' = (\wedge\psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*))' \wedge (\wedge\psi_0^*(\hat{T}\tilde{K}^c(0)))'$$

of $(\wedge\psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*)) \wedge (\wedge\psi_0^*(\hat{T}\tilde{K}^c(0)))$. For any $P \in K(0)$, the fiber of

$$(\wedge\psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*))' \wedge (\wedge\psi_0^*(\hat{T}\tilde{K}^c(0)))'$$

is a $\mathbb{Z} \times \mathbb{Z}$-graded vector spaces spanned by the canonical basis

$$\mathbb{L}(i_1)\big|_{\tilde{\mathfrak{E}}(\psi_0(P))} \wedge \ldots \wedge \mathbb{L}(i_m)\big|_{\tilde{\mathfrak{E}}(\psi_0(P))} \wedge \tilde{\mathfrak{E}}^*(\mathbb{L}'(j_1))\big|_{\psi_0^*(P)} \wedge \ldots \wedge \tilde{\mathfrak{E}}^*(\mathbb{L}'(j_n))\big|_{\psi_0^*(P)},$$

$m,n \geq 0$, $m+n > 0$, $i_1,\ldots,i_m \geq -1$, $j_1,\ldots,j_n < -1$. The integer $m+n$ is called the *fermion number* or *ghost number* of these basis elements. If $m+n \in 2\mathbb{Z}(\in 2\mathbb{Z}+1)$, these basis elements are said to be *even* (*odd*). The integers $-\sum_{k=1}^m i_k + \sum_{k=1}^n j_k$ are called *weights* or *degrees* of the canonical basis elements.

This bundle has holomorphic sections

$$P \mapsto \mathbb{L}(i_1)\big|_{\tilde{\mathfrak{E}}(\psi_0(P))} \wedge \ldots \wedge \mathbb{L}(i_m)\big|_{\tilde{\mathfrak{E}}(\psi_0(P))} \wedge \tilde{\mathfrak{E}}^*(\mathbb{L}'(j_1))\big|_{\psi_0(P)} \wedge \ldots \wedge \tilde{\mathfrak{E}}^*(\mathbb{L}'(j_n))\big|_{\psi_0(P)},$$

$m,n > 0,\ i_1,\ldots,i_m \geq -1,\ j_1,\ldots,j_n < -1$. We shall write these sections simply as

$$\psi_0^*(\mathbb{L}(i_1)) \wedge \ldots \wedge \psi_0^*(\mathbb{L}(i_m)) \wedge \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'(j_1))) \wedge \ldots \wedge \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'(j_n))).$$

We denote the space of these sections by $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$. It is clear that $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ is naturally isomorphic to the graded dual of $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$. We shall identify $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ with the graded dual of $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$.

Note that both $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ and $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ are linearly isomorphic to the space $\wedge_\infty \mathfrak{W}$ of semi-infinite forms on the Witt algebra $\mathfrak{W}$.

A holomorphic section of

$$(\wedge\psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*)) \wedge (\wedge\psi_0^*(\hat{T}\tilde{K}^c(0)))$$

or

$$(\wedge\psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*))'\wedge(\wedge\psi_0^*(\hat{T}\tilde{K}^c(0)))'$$

is said to be *meromorphic* if it is a linear combination of sections of the form

$$P\mapsto f(P)\psi_0^*(\mathbb{L}'(i_1))\wedge\ldots\wedge\psi_0^*(\mathbb{L}'(i_m))\wedge\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_1)))\wedge\ldots\wedge\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_n)))$$

or

$$P\mapsto f(P)\psi_0^*(\mathbb{L}(i_1))\wedge\ldots\wedge\psi_0^*(\mathbb{L}(i_m))\wedge\psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'(j_1)))\wedge\ldots\wedge\psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'(j_n))),$$

respectively, where f is a meromorphic function on $K(0)$ and $m,n>0$, $i_1,\ldots,i_m\geq -1$, $j_1,\ldots,j_n<-1$. In other words, the spaces of meromorphic sections of

$$(\wedge\psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*))\wedge(\wedge\psi_0^*(\hat{T}\tilde{K}^c(0)))$$

and

$$(\wedge\psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*))'\wedge(\wedge\psi_0^*(\hat{T}\tilde{K}^c(0)))'$$

are linearly isomorphic to $D_0\otimes\wedge_\infty\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ and $D_0\otimes\wedge_\infty\psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$, respectively. Note that $D_0\otimes\wedge_\infty\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ is the semi-infinite analogue of the space of skew-symmetric holomorphic poly-vector fields on a complex manifold while $D_0\otimes\wedge_\infty\psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ is the semi-infinite analogue of the space of holomorphic forms on a complex manifold.

The semi-infinite analogue $D_0\otimes\wedge_\infty\psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ of the space of holomorphic forms on a complex manifold can also be constructed as the semi-infinite analogue on another submanifold of $K(1)$ of the space of skew-symmetric holomorphic poly-vector fields on a complex manifold. Since in the next subsection we shall need not only $\wedge_\infty\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ but also $\wedge_\infty\psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ and since this construction gives a conceptual explanation of why we need $\wedge_\infty\psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$, we give this construction in detail here.

Let $K_{\geq -1}(1)$ be the submanifold of $K(1)$ consisting of elements of the form $(A(1;z),(a_0^{(1)},A^{(1)}))$, $z\in\mathbb{C}$, $a_0\in\mathbb{C}^\times$ and $A^{(1)}\in H$ (recalling from [H5] that $A(1;z)$ is the sequence with the first component z and the other components 0). Then it is clear that $K_{\geq -1}(1)$ is closed under the sewing operation.

We need the embedding map $\mathfrak{I}$ from $K_{\geq -1}(1)$ to $K(1)$. It has a left inverse $\mathfrak{N}$ defined by

$$\mathfrak{N}(A^{(0)},(a_0^{(1)},A^{(1)}))=(A(1;A_1^{(0)}),(a_0^{(1)},A^{(1)})).$$

Let $\tilde{K}^c_{\geq -1}(1)$ be the restriction of the line bundle $\tilde{K}^c(1)$ to $K_{\geq -1}(1)$. Then the two maps above induce $\tilde{\mathfrak{I}}: \tilde{K}^c_{\geq -1}(1) \to \tilde{K}^c(1)$ and $\tilde{\mathfrak{R}}: \tilde{K}^c(1) \to \tilde{K}^c_{\geq -1}(1)$.

For $\tilde{P} \in \tilde{K}^c_{\geq -1}(1)$, we denote the subspace of $T_{\tilde{P}}\tilde{K}^c_{\geq -1}(1)$ consisting of finite linear combinations of $\tilde{\mathfrak{R}}_*(\mathbb{L}(j))\big|_{\tilde{P}}$, $j \geq -1$, $\tilde{\mathfrak{R}}_*(\frac{\partial}{\partial C})\big|_{\tilde{P}}$, by $\hat{T}_{\tilde{P}}\tilde{K}^c_{\geq -1}(1)$. Then

$$\hat{T}_{\tilde{P}}\tilde{K}^c_{\geq -1}(1) = (\tilde{\mathfrak{R}}_*)\big|_{\tilde{\mathfrak{I}}(\tilde{P})}(\hat{T}_{\tilde{\mathfrak{I}}(\tilde{P})}\tilde{K}^c(1)).$$

The union of $\hat{T}_{\tilde{P}}\tilde{K}^c_{\geq -1}(1)$ for $\tilde{P} \in \tilde{K}^c_{\geq -1}(1)$ is a holomorphic vector bundle $\hat{T}\tilde{K}^c_{\geq -1}(1)$ over $\tilde{K}^c_{\geq -1}(1)$. For simplicity, we also use the same notation ψ_1 to denote the flat section ψ_1 of $\tilde{K}^c(1)$ constructed in [H5] and its restriction to $K_{\geq -1}(1)$ of . Using ψ_1, we pull $\hat{T}\tilde{K}^c_{\geq -1}(1)$ back to a holomorphic vector bundle $\psi_1^*(\hat{T}\tilde{K}^c_{\geq -1}(1))$ over $K_{\geq -1}(1)$.

For $\tilde{P} \in \tilde{K}^c_{\geq -1}(1)$, we denote the subspace of the dual space of $T_{\tilde{P}}\tilde{K}^c_{\geq -1}(1)$ consisting of finite linear combinations of $\tilde{\mathfrak{I}}^*(\mathbb{L}'(j))\big|_{\tilde{P}}$, $j \geq -1$, and $\tilde{\mathfrak{I}}^*((\frac{\partial}{\partial C})')\big|_{\tilde{P}}$ by $\hat{T}'_{\tilde{P}}\tilde{K}^c_{\geq -1}(1)$. Then

$$\hat{T}'_{\tilde{P}}\tilde{K}^c_{\geq -1}(1) = \tilde{\mathfrak{I}}^*\big|_{\tilde{\mathfrak{I}}(\tilde{P})}(\hat{T}'_{\tilde{\mathfrak{I}}(\tilde{P})}\tilde{K}^c(1)).$$

Note that the kernel of $\tilde{\mathfrak{I}}^*\big|_{\tilde{\mathfrak{I}}(\tilde{P})}$ is spanned by $\mathbb{L}'(j)$, $j < -1$. The union of kernels of $\tilde{\mathfrak{I}}^*\big|_{\tilde{\mathfrak{I}}(\tilde{P})}$ for $\tilde{P} \in \tilde{K}^c_{\geq -1}(1)$ forms a holomorphic vector bundle $\mathrm{Ker}\ \tilde{\mathfrak{I}}^*$ over $\tilde{K}^c_{\geq -1}(1)$. Using the flat section ψ_1 of $\tilde{K}^c_{\geq -1}(1)$, we pull $\mathrm{Ker}\ \tilde{\mathfrak{I}}^*$ back to a holomorphic vector bundle $\psi_1^*(\mathrm{Ker}\ \tilde{\mathfrak{I}}^*)$ over $K_{\geq -1}(1)$.

We consider the holomorphic vector bundle

$$(\wedge\psi_1^*(\mathrm{Ker}\ \tilde{\mathfrak{I}}^*)) \wedge (\wedge\psi_1^*(\hat{T}\tilde{K}^c_{\geq -1}(1))$$

where $\wedge\psi_1^*(\mathrm{Ker}\ \tilde{\mathfrak{I}}^*)$ and $\wedge\psi_1^*(\hat{T}\tilde{K}^c_{\geq -1}(1))$ are the wedge product bundles of $\psi_1^*(\mathrm{Ker}\ \tilde{\mathfrak{I}}^*)$ and $\psi_1^*(\hat{T}\tilde{K}^c_{\geq -1}(1))$. For any $P \in K_{\geq -1}(1)$, the fiber of

$$(\wedge\psi_1^*(\mathrm{Ker}\ \tilde{\mathfrak{I}}^*)) \wedge (\wedge\psi_1^*(\hat{T}\tilde{K}^c_{\geq -1}(1))$$

over P is a $\mathbb{Z} \times \mathbb{Z}$-graded vector spaces spanned by the canonical basis

$$\mathbb{L}'(i_1)\big|_{\tilde{\mathfrak{I}}(\psi_1(P))} \wedge \ldots \wedge \mathbb{L}'(i_m)\big|_{\tilde{\mathfrak{I}}(\psi_1(P))} \wedge \tilde{\mathfrak{R}}_*(\mathbb{L}(j_1))\big|_{\psi_1(P)} \wedge \ldots \wedge \tilde{\mathfrak{R}}_*(\mathbb{L}(j_n))\big|_{\psi_1(P)},$$

$m, n > 0$, $i_1, \ldots, i_m < -1$, $j_1, \ldots, j_n \geq -1$.

This bundle has holomorphic sections

$$P \mapsto \mathbb{L}'(i_1)\big|_{\tilde{\mathfrak{I}}(\psi_1(P))}\wedge\ldots\wedge\mathbb{L}'(i_m)\big|_{\tilde{\mathfrak{I}}(\psi_1(P))}\wedge\tilde{\mathfrak{N}}_*(\mathbb{L}(j_1))\big|_{\psi_1(P)}\wedge\ldots\wedge\tilde{\mathfrak{N}}_*(\mathbb{L}(j_n))\big|_{\psi_1(P)},$$

$m, n > 0$, $i_1, \ldots, i_m < -1$, $j_1, \ldots, j_n \geq -1$. We shall denote the space of these holomorphic sections by $\wedge_\infty \psi_1^*(\tilde{\mathfrak{N}}_*(\mathbb{L}))$. It is clear that $\wedge_\infty \psi_1^*(\tilde{\mathfrak{N}}_*(\mathbb{L}))$ is naturally isomorphic to $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$. Note that these sections are the semi-infinite analogue of skew-symmetric holomorphic poly-vector fields. So we obtain the other construction of $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ we need.

Since the space D_0 of meromorphic functions on $K(0)$ is linearly isomorphic to $\tilde{D}_0^{c;1}$, the spaces $D_0 \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ and $D_0 \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ of meromorphic sections of

$$(\wedge \psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*)) \wedge (\wedge \psi_0^*(\hat{T}\tilde{K}^c(0)))$$

and

$$(\wedge \psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*))' \wedge (\wedge \psi_0^*(\hat{T}\tilde{K}^c(0)))',$$

respectively, are linearly isomorphic to the space $\tilde{D}_0^{c;1} \otimes \wedge_\infty \mathfrak{W}$. By Proposition 3.6, we know that $\tilde{D}_0^{c;1}$ is a module for the Virasoro algebra with central charge c. A space of the form $V \otimes \wedge_\infty \mathfrak{W}$ where V is a module for the Virasoro algebra has been studied in detail in the theory of the semi-infinite cohomology of the Virasoro algebra [Fe] [FGZ] [LZ1] [LZ2]. The space $\wedge_\infty \mathfrak{W}$ is in fact the space of semi-infinite forms for the Virasoro algebra relative to the center. Thus we can apply their theory in our case. In the remaining part of this subsection, we recall some basic facts we need in the case $V = \tilde{D}_1^{c;1}$. We first discuss $\tilde{D}_1^{c;1} \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$.

We define a module structure on $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ for the Virasoro algebra. We define linear operators $\varepsilon(\psi_0^*(\mathbb{L}'(j)))$, $\iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j))))$, $j \in \mathbb{Z}$, on $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ as follows: For any $m, n \geq 0$, $i_1, \ldots, i_m \geq -1$, $j_1, \ldots, j_n < -1$,

$$\begin{aligned}
\varepsilon(\psi_0^*(\mathbb{L}'(j)))(\psi_0^*(\mathbb{L}'(i_1)) \wedge \ldots \wedge \psi_0^*(\mathbb{L}'(i_m))&\\
\wedge\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_1)) \wedge \ldots \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_n)))&)\\
= \psi_0^*(\mathbb{L}'(j)) \wedge \psi_0^*(\mathbb{L}'(i_1)) \wedge \ldots \wedge \psi_0^*(\mathbb{L}'(i_m))&\\
\wedge\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_1)) \wedge \ldots \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_n))&
\end{aligned}$$

for $j \geq -1$,

$$\varepsilon(\psi_0^*(\mathbb{L}'(j)))(\psi_0^*(\mathbb{L}'(i_1)) \wedge \ldots \wedge \psi_0^*(\mathbb{L}'(i_m))$$

$$\wedge\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_1)) \wedge \ldots \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_n)))$$
$$= \sum_{k=1}^{n}(-1)^{m+k-1}\delta_{jj_k}\psi_0^*(\mathbb{L}'(i_1)) \wedge \ldots \wedge \psi_0^*(\mathbb{L}'(i_m)) \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_1)) \wedge \ldots$$
$$\wedge\psi_0^*(\widehat{\tilde{\mathfrak{D}}_*(\mathbb{L}(j_k)}) \wedge \ldots \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_n)))$$

for $j < -1$,

$$\iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j))))(\psi_0^*(\mathbb{L}'(i_1)) \wedge \ldots \wedge \psi_0^*(\mathbb{L}'(i_m))$$
$$\wedge\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_1))) \wedge \ldots \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_n))))$$
$$= \sum_{k=1}^{m}(-1)^{k-1}\delta_{ji_k}\psi_0^*(\mathbb{L}'(i_1)) \wedge \ldots \wedge \psi_0^*(\widehat{\mathbb{L}'(i_k})) \wedge \ldots \wedge \psi_0^*(\mathbb{L}'(i_m))$$
$$\wedge\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_1))) \wedge \ldots \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_n)))$$

for $j \geq -1$, and

$$\iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j))))(\psi_0^*(\mathbb{L}'(i_1)) \wedge \ldots \wedge \psi_0^*(\mathbb{L}'(i_m))$$
$$\wedge\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_1)) \wedge \ldots \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_n)))$$
$$= (-1)^m\psi_0^*(\mathbb{L}'(i_1)) \wedge \ldots \wedge \psi_0^*(\mathbb{L}'(i_m)) \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j)))$$
$$\wedge\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_1))) \wedge \ldots \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j_n)))$$

for $j < -1$.

Recall that from the definition, $\wedge_\infty\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ is $\mathbb{Z}\times\mathbb{Z}$-graded. This $\mathbb{Z}\times\mathbb{Z}$-grading (fermion numbers and weights) gives a $\mathbb{Z}\times\mathbb{Z}$-grading (fermion numbers and weights) on the space of linear operators on $\wedge_\infty\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$. Given any two homogeneous operators O_1 and O_2 on $\wedge_\infty\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$, we denote their fermion numbers by $|O_1|$ and $|O_2|$, respectively, and we define

$$[O_1, O_2] = O_1O_2 - (-1)^{|O_1||O_2|}O_2O_1.$$

Then we have:

Proposition 3.7 *The operators $\varepsilon(\psi_0^*(\mathbb{L}'(j)))$, $\iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j))))$, $j \in \mathbb{Z}$, are all odd operators. For any $j \in \mathbb{Z}$, the weight of $\varepsilon(\psi_0^*(\mathbb{L}'(j)))$ is $-j$ and the weight of $\iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j))))$ is j. These operators satisfy the following bracket formulas:*

$$\begin{aligned}
[\varepsilon(\psi_0^*(\mathbb{L}'(i))), \varepsilon(\psi_0^*(\mathbb{L}'(j)))] &= 0\\
[\iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(i)))), \iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j))))] &= 0\\
[\varepsilon(\psi_0^*(\mathbb{L}'(i))), \iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j))))] &= \delta_{ij}
\end{aligned}$$

for $i, j \in \mathbb{Z}$. $\square$

For $j \in \mathbb{Z}$, we define the operators

$$L_\wedge(j) = \sum_{i\in\mathbb{Z}} : \varepsilon(\psi_0^*(\mathbb{L}'(i)))\iota(\psi_0^*(\tilde{\mathfrak{D}}_*([\mathbb{L}(i), \mathbb{L}(j)]))) :$$

on $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$, where the normal ordering $: \cdot :$ is defined by

$$: \varepsilon(\psi_0^*(\mathbb{L}'(i)))\iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j)))) := \begin{cases} -\iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j))))\varepsilon(\psi_0^*(\mathbb{L}'(i))) & j < -1 \\ \varepsilon(\psi_0^*(\mathbb{L}'(i)))\iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j))) & j \geq -1. \end{cases}$$

Proposition 3.8 *The operators* $L_\wedge(j)$, $j \in \mathbb{Z}$, *satisfy the bracket relations*

$$[L_\wedge(i), L_\wedge(j)] = (i-j)L_\wedge(i+j) + \frac{-26}{12}(i^3 - i)\delta_{i+j,0}$$

for $i, j \in \mathbb{Z}$. $\square$

By this proposition, we see that $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ is a module of central charge -26 for the Virasoro algebra. Thus the space $\tilde{D}_0^{c;1} \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$, or equivalently, $D_0 \otimes \wedge_\infty \mathfrak{W}$ of meromorphic forms on $K(1)$ is a module of central charge $c - 26$ for the Virasoro algebra.

As in the case of skew-symmetric poly-vector fields on a finite-dimensional manifold, we have a differential δ on the space $\tilde{D}_0^{c;1} \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ of meromorphic forms on $K(1)$. In fact such a differential has been defined for $V \otimes \wedge_\infty \mathfrak{W}$, or equivalently, $V \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ for any module V in the category $\mathcal{O}$ for the Virasoro algebra. If the Virasoro operators on V are denoted by $L(j)$, $j \in \mathbb{Z}$, then

$$\begin{aligned} \delta &= \sum_{j\in\mathbb{Z}} L(j) \otimes \varepsilon(\psi_0^*(\mathbb{L}'(j))) \\ &\quad -\frac{1}{2} I_V \otimes \sum_{i,j\in\mathbb{Z}} : \iota(\psi_0^*(\tilde{\mathfrak{D}}_*([\mathbb{L}(i), \mathbb{L}(j)])))\varepsilon(\psi_0^*(\mathbb{L}'(i)))\varepsilon(\psi_0^*(\mathbb{L}'(j))) :, \end{aligned}$$

where I_V is the identity operator on V. In the case that $V = \tilde{D}_0^{c;1}$, we have

$$\begin{aligned} \delta &= \sum_{j\in\mathbb{Z}} \mathbb{L}(j) \otimes \varepsilon(\psi_0^*(\mathbb{L}'(j))) \\ &\quad -\frac{1}{2} I_{\tilde{D}_0^{c;1}} \otimes \sum_{i,j\in\mathbb{Z}} : \iota(\psi_0^*(\tilde{\mathfrak{D}}_*([\mathbb{L}(i), \mathbb{L}(j)])))\varepsilon(\psi_0^*(\mathbb{L}'(i)))\varepsilon(\psi_0^*(\mathbb{L}'(j))) : . \end{aligned}$$

Proposition 3.9 *When $c = 26$, $\delta^2 = 0$.* $\square$

We also have a *fermion number operator* or *ghost number operator*

$$U = \sum_{j \in \mathbb{Z}} : \varepsilon(\psi_0^*(\mathbb{L}'(j)))\iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j)))) : .$$

This operator gives a $\mathbb{Z}$-grading to $\tilde{D}_0^{c;1} \otimes \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ and this grading is the same as the first $\mathbb{Z}$-grading or the grading given by fermion numbers.

Since $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ is the graded dual of $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$, the adjoint operator d of δ defines a differential on $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ and we have:

Proposition 3.10 *When $c = 26$, $d^2 = 0$.* $\square$

The adjoint operator U' of U is called the *fermion number operator* or *ghost number operator* on $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$, and it gives the first $\mathbb{Z}$-grading or the grading given by fermion numbers.

Let $b(j) = \iota(\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(j))))$ and $c(j) = \varepsilon(\psi_0^*(\mathbb{L}'(-j)))$ for $j \in \mathbb{Z}$ and let $b(x) = \sum_{j \in \mathbb{Z}} b(j)x^{-j-2}$ and $c(x) = \sum_{j \in \mathbb{Z}} c(j)x^{-j+1}$. We define a vertex operator map

$$Y : \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L})) \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L})) \to \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))((x))$$

as follows: We use recurrence to define Y. First we define $Y(1, x)$ to be the identity map on $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$. Assume that u is of the form $c(i_1) \cdots c(i_m)1$, $m \geq 0$, $i_1 < \cdots < i_m \leq -1$, and $Y(u, x)$ is already defined. For $v = c(i)u$, we define

$$\begin{aligned} Y(v, x) &= \mathrm{Res}_{x_1}(x_1 - x)^{i-2}c(x_1)Y(u, x) \\ &\quad -(-1)^{|u|}\mathrm{Res}_{x_1}(-x + x_1)^{i-2}Y(u, x)c(x_1). \end{aligned}$$

Assume that u is of the form $b(j_1) \cdots b(j_n)c(i_1) \cdots c(i_m)1$, $m, n \geq 0$, $i_1 < \cdots < i_m \leq -1$, $j_1 < \cdots < j_n < -1$, and $Y(u, x)$ is already defined. For $v = b(i)u$, we define

$$\begin{aligned} Y(v, x) &= \mathrm{Res}_{x_1}(x_1 - x)^{i+1}b(x_1)Y(u, x) \\ &\quad -(-1)^{|u|}\mathrm{Res}_{x_1}(-x + x_1)^{i+1}Y(u, x)b(x_1). \end{aligned}$$

By definition, elements of the form $b(j_1) \cdots b_{j_n} c(i_1) \cdots c(i_m)1$, $m, n \geq 0$, $i_1 < \cdots < i_m \leq -1$, $j_1 < \cdots < j_n < -1$, form a basis of $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$. Thus the above procedure indeed defines a linear map

$$Y : \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L})) \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L})) \to \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))((x)).$$

Proposition 3.11 *The $\mathbb{Z} \times \mathbb{Z}$-graded vector space $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$, equipped with the vertex operator map Y defined above, the vacuum 1 and the Virasoro element*

$$\begin{aligned}\omega_\wedge &= 2c(0)b(-2)1 + c(1)b(-3)1 \\ &= 2\psi_0^*(\mathbb{L}'(0)) \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(-2))) + \psi_0^*(\mathbb{L}'(-1)) \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(-3)))\end{aligned}$$

is a $\mathbb{Z} \times \mathbb{Z}$-graded vertex operator algebra (satisfying the grading-restriction conditions). Moreover,

$$Y(\omega_\wedge, x) =: c(x)\frac{d}{dx}b(x) : +2 : \left(\frac{d}{dx}c(x)\right) b(x) : .$$

Proof. The proof of the first conclusion is similar to the proof in [H3] that the quotient $M_{c,0}/\langle L(-1)1\rangle$ of the Verma module $M_{c,0}$ of lowest weight 0 for the Virasoro algebra is a vertex operator algebra. We omit it here. The second conclusion is a direct calculation. $\square$

The vertex operator algebra $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ is sometimes called the *ghost vertex operator algebra*.

In Section 6.6 and 7.3 of [H5], it was shown that the vertex operator algebra $M_{c,0}/\langle L(-1)1\rangle$ is isomorphic to a geometrically constructed vertex operator algebra $M(c)$. We have:

Proposition 3.12 *The $\mathbb{Z}$-graded space $\tilde{D}_0^{c;1}$ has a structure of a module for the vertex operator algebra $M(c)$.*

Proof. From Chapter 6 of [H5], we know that $M(c)$ is isomorphic to the vector space spanned by the restrictions to $\psi_0(\mathbf{0})$ of differential operators (of all orders) on the space $\tilde{D}_0^c$ of meromorphic functions on $\tilde{K}^c(0)$. We define a vertex operator map

$$M(c) \otimes \tilde{D}_0^{c;1} \to \tilde{D}_0^{c;1}((x))$$

as follows: Let $v \in M(c)$ and $f \in \tilde{D}_0^{c;1}$. We think of v as the restriction to $\psi_0(\mathbf{0})$ of a derivative on $\tilde{D}_0^c$. For any $z \in \mathbb{C}^\times$, let ψ_2 be the flat section on $\tilde{K}^c(2)$ and $P(z)$ the element of $K(2)$ introduced in [H5]. Then for fixed $\tilde{Q}_2 \in \tilde{K}^c(0)$,

$$f((\psi_2(P(z))_1\widetilde{\infty}_0^{c/2}\tilde{Q}_1)_1\widetilde{\infty}_0^{c/2}\tilde{Q}_2)$$

as a function of $\tilde{Q}_1 \in \tilde{K}^c(0)$ is analytic. It is clear that v can always be extended so that it acts on analytic functions, not only on meromorphic functions. Thus v acts on this function of $\tilde{Q}_1$ and

$$v(f((\psi_2(P(z)))_1\widetilde{\infty}_0^{c/2}\tilde{Q}_1)_1\widetilde{\infty}_0^{c/2}\tilde{Q}_2))$$

as a function of $\tilde{Q}_2 \in \tilde{K}^c(0)$ is analytic. In addition, this function is also analytic in z. Thus we can expand it as a Laurent series in z. It is easy to see from the definition of the sewing operation that this series has a pole at $z = 0$ and the coefficients of this expansion are meromorphic functions of $\tilde{Q}_2 \in \tilde{K}^c(0)$. In addition, since f is proportional to C, so are all the coefficients of this expansion. Thus these coefficients give an element of $\tilde{D}_0^{c;1}((x))$. We define the image $Y(v, x)f$ of $v \otimes f$ under Y to be this element.

Since our vertex operator map is defined geometrically, it is now easy to use the geometry and the geometric proof that $M(c)$ is a vertex operator algebra in [H5] to prove that $\tilde{D}_0^{c;1}$ with the vertex operator map just defined is indeed a module for $M(c)$. $\square$

By Propositions 3.11 and 3.12, we have:

Corollary 3.13 *The $\mathbb{Z} \times \mathbb{Z}$-graded vector space $\tilde{D}_0^{c;1} \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ is a module for the tensor product vertex operator algebra $M(c) \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$.* $\square$

We shall denote the vertex operator algebra $M(c) \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ by G. We shall use the same notation to denote the vertex operator maps for $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$, for G and for the module $\tilde{D}_0^{c;1} \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$.

Corollary 3.14 *Let*

$$b = b(-2)1 = \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(-2)))$$

and

$$c = c(1)1 = \psi_0^*(\mathbb{L}'(-1)).$$

Then

$$\begin{aligned} b(x) &= Y(b, x), \\ c(x) &= Y(c, x). \quad \square \end{aligned}$$

Proposition 3.15 *Let*

$$
\begin{aligned}
q_G &= \mathbb{L}(-2)1 \otimes c + 1 \otimes (b(-2)c(1)c(0)1) \\
&= \mathbb{L}(-2)1 \otimes c + 1 \otimes \psi_0^*(\mathbb{L}'(0)) \wedge \psi_0^*(\mathbb{L}'(-1)) \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(-2))), \\
f_G &= 1 \otimes (c(1)b(-2)1) \\
&= 1 \otimes (\psi_0^*(\mathbb{L}'(-1)) \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(-2)))), \\
\omega_G &= (L(-2)1) \otimes 1 + 1 \otimes \omega_\wedge
\end{aligned}
$$

in G. Then ω_G is the Virasoro element of G and in $\tilde{D}_0^{c;1} \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$,

$$
\begin{aligned}
Y(q_G, x) &= \,:L(x)c(x): + :b(x)c(x)\frac{d}{dx}c(x):, \\
\delta &= \mathrm{Res}_x Y(q_G, x), \\
Y(f_G, x) &= \,:c(x)b(x):, \\
U &= \mathrm{Res}_x Y(f_G, x). \quad \square
\end{aligned}
$$

By definition, we know that the $\mathbb{Z} \times \mathbb{Z}$-graded vertex operator algebra $\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ is generated by b and c.

For $j \in \mathbb{Z}$, let $b'(-j)$ and $c'(-j)$ be the adjoint operators of $b(j)$ and $c(j)$, respectively. Since $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ is the graded dual of $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$, $b'(-j)$ and $c'(-j)$, $j \in \mathbb{Z}$, act on $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$, and we have:

Proposition 3.16 *The $\mathbb{Z} \times \mathbb{Z}$-graded vector space $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ is a module for the $\mathbb{Z} \times \mathbb{Z}$-graded vertex operator algebra $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$. The $\mathbb{Z} \times \mathbb{Z}$-graded vector space $\tilde{D}_0^{c;1} \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$ is a module for G. If we still use Y to denote the vertex operator map for the second module, then*

$$
\begin{aligned}
Y(b, x) &= \sum_{j \in \mathbb{Z}} b'(-j) x^{-x-2}, \\
Y(c, x) &= \sum_{j \in \mathbb{Z}} c'(-j) x^{-x+1}, \\
d &= \mathrm{Res}_x Y(q, x), \\
U' &= \mathrm{Res}_x Y(f, x). \qquad \square
\end{aligned}
$$

3.3 Semi-infinite forms on $K(n)$, $n \geq 0$

In this subsection, we introduce semi-infinite forms on $K(n)$ for $n \geq 0$.

We fix $n \geq 0$. From the definition of $K(n)$, it is easy to see that there is a canonical injective map $\mathfrak{K}_n : K(n) \to K(0) \times (K_{\geq -1}(1))^n$ and $\mathfrak{K}_n(K(n))$ is in fact an open subset of $K(0) \times (K_{\geq -1}(1))^n$. Over $K(0)$ and $K_{\geq -1}(1)$, we have the holomorphic bundles

$$(\wedge \psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*)) \wedge (\wedge \psi_0^*(\hat{T}\tilde{K}^c(0)))$$

and

$$(\wedge \psi_1^*(\mathrm{Ker}\ \tilde{\mathfrak{I}}^*)) \wedge (\wedge \psi_1^*(\hat{T}\tilde{K}^c_{\geq -1}(1)),$$

respectively. Thus we have the exterior product bundle (not the wedge product bundle)

$$(\wedge \psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*)) \wedge (\wedge \psi_0^*(\hat{T}\tilde{K}^c(0))) \boxtimes ((\wedge \psi_1^*(\mathrm{Ker}\ \tilde{\mathfrak{I}}^*)) \wedge (\wedge \psi_1^*(\hat{T}\tilde{K}^c_{\geq -1}(1)))^{\boxtimes n} \tag{3.12}$$

over $K(0) \times (K_{\geq -1}(1))^n$. The map $\mathfrak{K}_n$ pulls this holomorphic bundle back to a holomorphic bundle over $K(n)$. We denote this holomorphic bundle over $K(n)$ by $\mathcal{G}(n)$.

By the definition of exterior product bundle, we have linear injective maps from the space of sections of $(\wedge \psi_0^*(\mathrm{Ker}\ \tilde{\mathfrak{E}}^*)) \wedge (\wedge \psi_0^*(\hat{T}\tilde{K}^c(0)))$ and the spaces of sections of copies of $(\wedge \psi_1^*(\mathrm{Ker}\ \tilde{\mathfrak{I}}^*)) \wedge (\wedge \psi_1^*(\hat{T}\tilde{K}^c_{\geq -1}(1))$ to the space of sections of the bundle (3.12). Moreover, the images of these spaces of sections in the space of sections of (3.12) intersect with each other at 0. Let $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}^{(0)}))$ be a copy of $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ and for $i = 1, \ldots, n$, let $\wedge_\infty \psi_1^*(\tilde{\mathfrak{N}}_*(\mathbb{L}^{(i)}))$ be a copy of $\wedge_\infty \psi_1^*(\tilde{\mathfrak{N}}_*(\mathbb{L}))$. Then we have a linear injective map from

$$\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}^{(0)})) \wedge (\wedge_{i=1}^n \wedge_\infty \psi_1^*(\tilde{\mathfrak{N}}_*(\mathbb{L}^{(i)}))) \tag{3.13}$$

to the space of sections of (3.12). Since $\mathfrak{K}_n$ is injective and $\mathfrak{K}_n(K(n))$ is in fact an open subset of $K(0) \times (K_{\geq -1}(1))^n$, $\mathfrak{K}_n^*$ is an isomorphism from the space of holomorphic sections of (3.12) to the space of holomorphic sections of $\mathcal{G}(n)$. Thus we obtain a linear injective map from (3.13) to the space of holomorphic sections of $\mathcal{G}(n)$. We denote the image of this map by $\hat{\Gamma}(\mathcal{G}(n))$. Then $\hat{\Gamma}(\mathcal{G}(n))$ is isomorphic to (3.13). The tensor product space $D_n \otimes \hat{\Gamma}(\mathcal{G}(n))$ is the semi-infinite analogue on $K(n)$ of skew-symmetric holomorphic poly-vector fields on a complex manifold.

In the preceding subsection, we have shown that $\wedge_\infty \psi_1^*(\tilde{\mathfrak{N}}_*(\mathbb{L}))$ is naturally isomorphic to $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$. Thus if for $i = 1, \ldots, n$, we use $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*((\mathbb{L}^{(i)})'))$ to denote a copy $\wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*(\mathbb{L}'))$, then $\hat{\Gamma}(\mathcal{G}(n))$ is isomorphic to

$$\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}^{(0)})) \wedge (\wedge_{i=1}^n \wedge_\infty \psi_0^*(\tilde{\mathfrak{E}}^*((\mathbb{L}^{(i)})'))). \tag{3.14}$$

We shall identify $\hat{\Gamma}(\mathcal{G}(n))$ with this space.

The space $\hat{\Gamma}(\mathcal{G}(n))$ is $\mathbb{Z}\times\mathbb{Z}$-graded. We denote the graded dual of $\hat{\Gamma}(\mathcal{G}(n))$ by $\hat{\Omega}(K(n))$. The tensor product space $D_n \otimes \hat{\Omega}(K(n))$ is the semi-infinite analogue on $K(n)$ of the space of holomorphic forms on a complex manifold.

We now define differentials on the spaces $D_n\otimes\hat{\Gamma}(\mathcal{G}(n))$ and $D_n\otimes\hat{\Omega}(K(n))$. We discuss $D_n \otimes \hat{\Gamma}(\mathcal{G}(n))$ first. Since $\hat{\Gamma}(\mathcal{G}(n))$ is isomorphic to (3.14) and we have a differential δ for $\wedge_{\infty}\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ and a differential d for $\wedge_{\infty}\psi_0^*(\tilde{\mathfrak{E}}^*((\mathbb{L}^{(i)})'))$, we have a differential δ_n for $\hat{\Gamma}(\mathcal{G}(n))$. Since $\mathfrak{K}_n(K(n))$ is in fact an open subset of $K(0)\times(K_{\geq -1}(1))^n$, tangent fields on $K(0)\times(K_{\geq -1}(1))^n$ can be restricted to $\mathfrak{K}_n(K(n))$ to obtain tangent fields on $\mathfrak{K}_n(K(n))$ which can be further pulled back to tangent fields on $K(n)$. On $K(0)\times(K_{\geq -1}(1))^n$, we have the tangent fields $\mathbb{L}^{(0)}(j)$, $j<-1$, which are the push-forwards of the tangent fields $\mathbb{L}(j)$ on $K(0)$ and for $i=1,\dots,n$, we have the tangent fields $\mathbb{L}^{(i)}(j)$, $j\geq -1$, which are the push-forwards of the tangent fields $\mathbb{L}(j)$ on the i-th copy of $K_{\geq -1}(1)$. We shall use the same notations $\mathbb{L}^{(0)}(j)$, $j<-1$, and $\mathbb{L}^{(i)}(j)$, $j\geq -1$, $i=1,\dots,n$, to denote the pullbacks to $K(n)$ of their restrictions to $\mathfrak{K}_n(K(n))$. Recall from the preceding subsection that on $\wedge_{\infty}\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ and $\wedge_{\infty}\psi_0^*(\tilde{\mathfrak{E}}^*((\mathbb{L})'))$, we have the operators $c(j)$ and $c'(-j)$, respectively, $j\in\mathbb{Z}$. We denote $c(j)$, $j\in\mathbb{Z}$, on $\wedge_{\infty}\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}^{(0)}))$ by $c^{(0)}(j)$ and $c'(-j)$, $j\in\mathbb{Z}$, on $\wedge_{\infty}\psi_0^*(\tilde{\mathfrak{E}}^*((\mathbb{L}^{(i)})'))$ for $i=1,\dots,n$ by $(c')^{(i)}(-j)$. The operators $c^{(0)}(j)$, $j<-1$, and for $i=1,\dots,n$, the operators $(c')^{(i)}(j)$, $j\geq -1$, act on $\hat{\Gamma}(\mathcal{G}(n))$ naturally, and we shall use the same notation to denote their actions on $\hat{\Gamma}(\mathcal{G}(n))$.

We define a differential on $D_n\otimes\hat{\Gamma}(\mathcal{G}(n))$, still denoted δ_n, by

$$\delta_n = \sum_{j<-1}\mathbb{L}^{(0)}(j)\otimes c^{(0)}(-j) + \sum_{i=1}^{n}\sum_{j\geq -1}\mathbb{L}^{(i)}(j)\otimes (c')^{(i)}(j) + I_{D_n}\otimes\delta_n.$$

From this definition and Propositions 3.9 and 3.10, we conclude:

Proposition 3.17 *When $c=26$, $\delta_n^2=0$.* $\square$

We define a differential d_n on $D_n\otimes\hat{\Omega}(K(n))$ using δ_n in the obvious way. Then we have:

Proposition 3.18 *When $c=26$, $d_n^2=0$.* $\square$

3.4 A partial operad for semi-infinite forms

In this subsection, we construct a partial operad $\mathfrak{G}$ from semi-infinite forms introduced in the preceding two subsections. We also construct a partial suboperad $\mathcal{T}_G$ of $\mathfrak{G}$ which is the motivation of the construction of the (strong) topological vertex partial operad in Subsection 4.2.

In [H5], $M(c)$ is constructed as a space of differential operators (of all orders) on $\tilde{K}^c(0)$ restricted to $\psi_0(\mathbf{0})$. Using the tangent fields $\tilde{\mathfrak{D}}_*(\mathbb{L}(j))$, $j < -1$, we see that $M(c)$ is isomorphic to a space of differential operators (without restricting to $\psi_0(\mathbf{0})$). The union of the vector spaces of restrictions of elements of this space to Q for all $Q \in \tilde{K}^c(0)$ is a holomorphic bundle over $\tilde{K}^c(0)$ and $M(c)$ is a space of holomorphic sections of this holomorphic bundle. This holomorphic bundle is pulled back to a holomorphic bundle $\mathcal{O}(0)$ over $K(0)$ and $M(c)$ is in fact isomorphic to a space of holomorphic sections of this pullback bundle $\mathcal{O}(0)$. From now on we shall identify $M(c)$ with this space of holomorphic sections.

We now consider the space of differential operators (of all orders) on $\tilde{K}^c_{\geq -1}(1)$ generated by $\tilde{\mathfrak{N}}_*(\mathbb{L}(j))$, $j \geq -1$. The union of the vector spaces of restrictions of elements of this space to Q for all $Q \in \tilde{K}^c_{\geq -1}(1)$ is a holomorphic bundle over $\tilde{K}^c_{\geq -1}(1)$ and this space is a space of holomorphic sections of this bundle. This bundle and this space of holomorphic sections are pulled back by ψ_1^* to a holomorphic bundle $\mathcal{O}_{\geq -1}(1)$ over $K_{\geq -1}(1)$ and a space $M_{\geq -1}(c)$ of holomorphic sections of $\mathcal{O}_{\geq -1}(1)$, respectively. Note that $M(c)$ and $M_{\geq -1}(c)$ are in fact isomorphic to the universal enveloping algebras of the Lie subalgebras of the Virasoro algebra spanned by $L(j)$ for $j < -1$ and by $L(j)$ for $j \geq -1$, respectively. Since $M(c)$ is a module for the Virasoro algebra, both $M(c)$ and $M_{\geq -1}(c)$ act on $M(c)$ naturally. These actions can also be interpreted geometrically as follows: The sewing operation gives the map ${}_1\infty_0 : K_{\geq -1}(1) \times K(0) \to K(0)$. Note that elements of $M_{\geq -1}(c) \otimes M(c)$ can be viewed as operators on functions on $K_{\geq -1}(1) \times K(0)$. The map ${}_1\infty_0$ pushes the restriction to $(I, \mathbf{0}) \in K_{\geq -1}(1) \times K(0)$ of an operator in $M_{\geq -1}(c) \otimes M(c)$ to an operator on the space of functions holomorphic at $\mathbf{0} \in K(0)$. Using the left multiplication in $K(0)$ defined by the embedding from $K(0)$ to $K(1)$ and the sewing operation, we see that this push-forward operator is the restriction to $\mathbf{0}$ of an operator in $M(c)$. Thus we obtain a map from $M_{\geq -1}(c) \otimes M(c)$ to $M(c)$. This map is the action of $M_{\geq -1}(c)$ on $M(c)$. Similarly, the action of $M(c)$ on $M(c)$ has a geometric meaning. For the details of the geometric construction of the action of the Virasoro algebra on $M(c)$, see [H5].

Both $M(c)$ and $M_{\geq -1}(c)$ are $\mathbb{Z}$-graded (by weights). Note that the homogeneous subspaces of $M(c)$ of weights less than 0 are all 0 and homogeneous subspaces of $M_{\geq -1}(c)$ of weights larger than 1 are all 0. Also note that the homogeneous subspaces of $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ of weights less than 0 are all 0 and homogeneous subspaces of $\wedge_\infty \psi_1^*(\tilde{\mathfrak{N}}_*(\mathbb{L}))$ of weights larger than 1 are all 0.

From the definition, we see that the space $\wedge_\infty \psi_1^*(\tilde{\mathfrak{N}}_*(\mathbb{L}))$ is isomorphic to the space spanned by operators on $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ of the form

$$b(i_1)\cdots b(i_m)c(-j_1)\cdots c(-j_n),$$

$m, n > 0$, $i_1, \dots, i_m \geq -1$, $j_1, \dots, j_n < -1$. We shall identify $\wedge_\infty \psi_1^*(\tilde{\mathfrak{N}}_*(\mathbb{L}))$ with this space of operators. The space $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ is isomorphic to the space spanned by operators on itself of the form

$$c(-i_1)\cdots c(-i_m)b(j_1)\cdots b(j_n),$$

$m, n > 0$, $i_1, \dots, i_m \geq -1$, $j_1, \dots, j_n < -1$. We shall identify $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ with this space of operators.

Consider the tensor products

$$T_\wedge = M(c) \otimes \wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$$

and

$$T_\wedge^{\geq -1} = M_{\geq -1}(c) \otimes \wedge_\infty \psi_1^*(\tilde{\mathfrak{N}}_*(\mathbb{L})).$$

Then $T_\wedge$ is a $\mathbb{Z} \times \mathbb{Z}$-graded vertex operator algebra and both $T_\wedge$ and $T_\wedge^{\geq -1}$ can be identified with spaces of operators on $T_\wedge$ spanned by homogeneous vectors. The homogeneous subspaces of $T_\wedge$ of weights less than 0 are all 0 and homogeneous subspaces of $T_\wedge^{\geq -1}$ of weights larger than 1 are all 0. We have the algebraic completions $\overline{T}_\wedge$ and $\overline{T}_\wedge^{\geq -1}$ of $T_\wedge$ and $T_\wedge^{\geq -1}$, respectively. When $c = 26$, the central charge of $T_\wedge$ is 0. From now on, we shall take $c = 26$.

Since $M(c)$ and $\wedge_\infty \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ ($M_{\geq -1}(c)$ and $\wedge_\infty \psi_1^*(\tilde{\mathfrak{N}}_*(\mathbb{L}))$) are spaces of holomorphic sections of holomorphic vector bundles over $K(0)$ ($K_{\geq -1}(1)$), the unions of the vector spaces of restrictions of elements of $T_\wedge$ and $T_\wedge^{\geq -1}$ to Q for all $Q \in \tilde{K}^c_{\geq -1}(1)$ form holomorphic bundles $\mathfrak{G}(0)$ and $\mathfrak{G}_{\geq -1}(1)$ over $K(0)$ and $K_{\geq -1}(1)$, respectively, and $T_\wedge$ and $T_\wedge^{\geq -1}$ are spaces of holomorphic sections of $\mathfrak{G}(0)$ and $\mathfrak{G}_{\geq -1}(1)$, respectively.

In [H6] and [H7], a locally convex topological completion $H^V \subset \overline{V}$ of a vertex operator algebra V is constructed such that the maps in $\mathrm{Hom}(V^{\otimes n}, \overline{V})$ corresponding to elements of $K_{\mathfrak{H}_1}(n)$ can in fact be extended to maps in $\mathrm{Hom}((H^V)^{\otimes n}, H^V)$. In particular, when the central charge of V is 0, H^V has a structure of an algebra over the (non-partial) suboperad $K_{\mathfrak{H}_1}$ of K. These results generalize to $\mathbb{Z}\times\mathbb{Z}$-graded vertex operator algebras in an obvious way.

Applying this result to the vertex operator algebra $T_\wedge$ of central charge 0, we obtain a locally convex completion $H^{T_\wedge} \subset \overline{T}_\wedge$ such that $H^{T_\wedge}$ has a structure of an algebra over $K_{\mathfrak{H}_1}$. Elements of $\mathrm{Hom}((H^{T_\wedge})^{\otimes n}, H^{T_\wedge})$ give elements of $\mathrm{Hom}(T_\wedge^{\otimes n}, \overline{T}_\wedge)$. Thus we have a map from $\mathrm{Hom}((H^{T_\wedge})^{\otimes n}, H^{T_\wedge})$ to $\mathrm{Hom}(T_\wedge^{\otimes n}, \overline{T}_\wedge)$. In fact this map is linear and injective. From now on, we shall view $\mathrm{Hom}((H^{T_\wedge})^{\otimes n}, H^{T_\wedge})$ as a subspace of $\mathrm{Hom}(T_\wedge^{\otimes n}, \overline{T}_\wedge)$.

Note that since elements of $T_\wedge$ and $T_\wedge^{\geq -1}$ have been identified with spaces of operators on $T_\wedge$, elements of the algebraic completions $\overline{T}_\wedge$ and $\overline{T}_\wedge^{\geq -1}$ can be identified with elements of $\mathrm{Hom}(T_\wedge, \overline{T}_\wedge)$. Let $\widetilde{T}_\wedge$ ($\widetilde{T}_\wedge^{\geq -1}$) be the subspaces of $\overline{T}_\wedge$ ($\overline{T}_\wedge^{\geq -1}$) consisting of elements $u \in \overline{T}_\wedge$ ($u \in \widetilde{T}_\wedge^{\geq -1}$) satisfying the following property: There exists a positive number a such that $a^{L(0)}ua^{-L(0)}$ as an element of $\mathrm{Hom}(T_\wedge, \overline{T}_\wedge)$ is in fact in the subspace $\mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge})$.

Using the same construction of $\mathfrak{G}(0)$ and $\mathfrak{G}_{\geq -1}(1)$ in terms of $T_\wedge$ and $T_\wedge^{\geq -1}$, respectively, we construct holomorphic vector bundles $\widetilde{\mathfrak{G}}(0)$ and $\widetilde{\mathfrak{G}}_{\geq -1}(1)$ over $K(0)$ and $K_{\geq -1}(1)$, respectively, such that $\widetilde{T}_\wedge$ and $\widetilde{T}_\wedge^{\geq -1}$ are spaces of holomorphic sections of $\widetilde{\mathfrak{G}}(0)$ and $\widetilde{\mathfrak{G}}_{\geq -1}(1)$, respectively.

We have the exterior product bundle

$$\widetilde{\mathfrak{G}}(0) \boxtimes (\widetilde{\mathfrak{G}}_{\geq -1}(1))^{\boxtimes n}$$

over $K(0) \times (K(1))^n$. The space $\widetilde{T}_\wedge$ can be embedded into the space of holomorphic sections of this bundle and there are n embeddings of $\widetilde{T}_\wedge^{\geq -1}$ into the space of holomorphic sections of this bundle. The images of these embeddings can be pulled back to subspaces of the space of holomorphic sections of the holomorphic bundle

$$\mathfrak{K}_n^*(\widetilde{\mathfrak{G}}(0) \boxtimes (\widetilde{\mathfrak{G}}_{\geq -1}(1))^{\boxtimes n}).$$

We shall denote these pullbacks by $\widetilde{T}_\wedge^{(0)}$ and by $\widetilde{T}_\wedge^{(i)}$, $i = 1, \dots, n$. Let

$$\widetilde{\mathcal{T}}_\wedge(n) = \mathfrak{K}_n^*(\widetilde{T}_\wedge^{(0)} \otimes \otimes_{i=1}^n \widetilde{T}_\wedge^{(i)}).$$

Then this subspace of the space of holomorphic sections of

$$\mathfrak{K}_n^*(\widetilde{\mathfrak{G}}(0) \boxtimes (\widetilde{\mathfrak{G}}_{\geq -1}(1))^{\boxtimes n})$$

gives a holomorphic bundle $\widetilde{\mathfrak{G}}(n)$ over $K(n)$ for each $n \geq 0$.

We now construct a partial operad structure on $\widetilde{\mathfrak{G}} = \{\widetilde{\mathfrak{G}}(n)\}_{n\geq 0}$. We first define the composition maps for $\widetilde{\mathfrak{G}}$.

We need some notations. Let

$$Q = (z_1, \dots, z_{n-1}; A^{(0)}, (a_0^{(1)}, A^{(1)}), \dots, (a_0^{(n)}, A^{(n)})) \in K(n).$$

We shall use $z_1(Q), \dots, z_n(Q)$, $A^{(0)}(Q), \dots, A^{(n)}(Q)$ and $a_0^{(1)}(Q), \dots, a_0^{(n)}(Q)$ to denote $z_1, \dots, z_n$, $A^{(0)}, \dots, A^{(n)}$ and $a_0^{(1)}, \dots, a_0^{(n)}$, respectively. Also recall the notations

$$\begin{aligned} e^{L^+(A)} &= \exp\left(\sum_{j>0} A_j L(j)\right), \\ e^{L^-(A)} &= \exp\left(\sum_{j>0} A_j L(-j)\right) \end{aligned}$$

in [H5] for any sequence $A \in \mathbb{C}^\infty$.

Let $n_1 > 0$, $n_2 \geq 0$ and $1 \leq i \leq n_1$. Let $(Q_1, \alpha_1) \in \widetilde{\mathfrak{G}}(n_1)$ and $(Q_2, \alpha_2) \in \widetilde{\mathfrak{G}}(n_2)$ where $Q_1 \in K(n_1)$, $Q_2 \in K(n_2)$ and α_1 and α_2 are elements of the fibers at Q_1 and Q_2, respectively, of $\widetilde{\mathfrak{G}}(n_1)$ and $\widetilde{\mathfrak{G}}(n_2)$. We would like to define $(Q_1, \alpha_1) \circ_i (Q_2, \alpha_2)$ when $Q_1\, {}_i\infty_0\, Q_2$ exists and an additional condition (see below) on α_1 and α_2 holds.

By the definition of $\widetilde{\mathfrak{G}}(n_1)$ and $\widetilde{\mathfrak{G}}(n_2)$, we need only consider those α_1 and α_2 which are the restrictions to Q_1 and Q_2, respectively, of the sections of the forms

$$\mathfrak{K}_{n_1}^*(u^{(0)} \otimes u^{(1)} \otimes \cdots \otimes u^{(n_1)})$$

and

$$\mathfrak{K}_{n_2}^*(v^{(0)} \otimes v^{(1)} \otimes \cdots \otimes v^{(n_2)}),$$

respectively, where $u^{(0)}, v^{(0)} \in \widetilde{T}_\wedge$, and $u^{(k)}, v^{(l)} \in \widetilde{T}_\wedge^{\geq -1}$ for $k = 1, \dots, n_1$ and $l = 1, \dots, n_2$.

From the definition of sewing operation, we know that it is possible to find positive numbers $r_0, \dots, r_{n_1}, s_0, \dots, s_{n_2}$ such that

$$\begin{aligned}(\cdots(((\mathbf{0},(r_0,\mathbf{0}))\,{}_1\infty_0 Q_1)\,{}_1\infty_0(\mathbf{0},(r_1,\mathbf{0})))\,{}_1\infty_0\cdots)\,{}_1\infty_0(\mathbf{0},(r_{n_1},\mathbf{0})))\\ =(\cdots(((A^{(0)},(a_0^{(0)},\mathbf{0}))\,{}_1\infty_0\hat{Q}_1)\,{}_1\infty_0(\mathbf{0},(a_0^{(1)},A^{(1)})))\,{}_1\infty_0\cdots)\\ {}_1\infty_0(\mathbf{0},(a_0^{(n_1)},A^{(n_1)})))\in K_{\mathfrak{H}_1}(n_1)\end{aligned}$$

and

$$\begin{aligned}(\cdots(((\mathbf{0},(s_0,\mathbf{0}))\,{}_1\infty_0 Q_2)\,{}_1\infty_0(\mathbf{0},(s_1,\mathbf{0})))\,{}_1\infty_0\cdots)\,{}_1\infty_0(\mathbf{0},(s_{n_2},\mathbf{0})))\\ =(\cdots(((B^{(0)},(b_0^{(0)},\mathbf{0}))\,{}_1\infty_0\hat{Q}_2)\,{}_1\infty_0(\mathbf{0},(b_0^{(1)},B^{(1)})))\,{}_1\infty_0\cdots)\,{}_1\\ \infty_0(\mathbf{0},(b_0^{(n_2)},B^{(n_2)})))\in K_{\mathfrak{H}_1}(n_2),\end{aligned}$$

where

$$(A^{(0)},(a_0^{(0)},\mathbf{0})),(\mathbf{0},(a_0^{(k)},A^{(k)})),(B^{(0)},(b_0^{(0)},\mathbf{0})),\quad(\mathbf{0},(b_0^{(l)},B^{(l)}))\in K_{\mathfrak{H}_1}(1),$$

for $k = 1, \dots, n_1$, $l = 1, \dots, n_2$, and

$$\begin{aligned}\hat{Q}_1 &= (\xi_1,\dots,\xi_{n_1};\mathbf{0},(\mathbf{0},(c_0^{(1)},\mathbf{0})),\dots,(\mathbf{0},(c_0^{(n_1)},\mathbf{0})))\in K_{\mathfrak{H}_1}(n_1),\\ \hat{Q}_2 &= (\eta_1,\dots,\eta_{n_1};\mathbf{0},(\mathbf{0},(d_0^{(1)},\mathbf{0})),\dots,(\mathbf{0},(d_0^{(n_2)},\mathbf{0})))\in K_{\mathfrak{H}_1}(n_2).\end{aligned}$$

If we can find such positive numbers such that in addition to the property above, $s_0 = r_i^{-1}$ and

$$\begin{aligned}r_i^{L(0)}u^{(i)}s_0^{L(0)}e^{-L^-(B^{(0)})}(b_0^{(0)})^{-L(0)}v^{(0)}\\ =r_i^{L(0)}u^{(i)}r_i^{-L(0)}e^{-L^-(B^{(0)})}(b_0^{(0)})^{-L(0)}v^{(0)}\in\mathrm{Hom}(T_\wedge,\overline{T}_\wedge)\end{aligned}$$

is in $\mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge})$, then we say that $(Q_1, \alpha_1) \circ_i (Q_2, \alpha_2)$ *exists*. It is clear that when $(Q_1, \alpha_1) \circ_i (Q_2, \alpha_2)$ exists, $Q_1\,{}_i\infty_0 Q_2$ exists.

We assume now that $(Q_1, \alpha_1) \circ_i (Q_2, \alpha_2)$ exists. Let

$$\nu_n : K(n) \to \mathrm{Hom}(T_\wedge^{\otimes n}, \overline{T}_\wedge)$$

for $n \geq 0$ be the maps defining the geometric vertex operator algebra structure of central charge 0 on $T_\wedge$. Consider the expression

$$(e^{-L^-(A^{(0)}(Q_1))}u^{(0)}e^{L^-(A^{(0)}(Q_1))})\circ\nu_{n_1}(Q_1)$$

$$
\begin{aligned}
&\circ(u^{(1)} \otimes \cdots \otimes (u^{(i)} \circ ((e^{-L^-(A^{(0)}(Q_2))} v^{(0)} e^{L^-(A^{(0)}(Q_2))})\\
&\circ \nu_{n_2}(Q_2) \circ (v^{(1)} \otimes \cdots \otimes v^{(n_2)})) \otimes \cdots \otimes u^{(n_1)}))\\
= {}&(r_0^{L(0)} e^{-L^-(A^{(0)})} (a_0^{(0)})^{-L(0)} u^{(0)}) \circ \nu_{n_1}(\hat{Q}_1) \circ ((e^{-L^+(A^{(1)})} (a_0^{(1)})^{-L(0)} r_1^{L(0)} u^{(1)})\\
&\otimes \cdots \otimes ((e^{-L^+(A^{(i)})} (a_0^{(i)})^{-L(0)} r_i^{L(0)} u^{(i)}) \circ ((s_0^{L(0)} e^{-L^-(B^{(0)})} (b_0^{(0)})^{-L(0)} v^{(0)})\\
&\circ \nu_{n_2}(\hat{Q}_2) \circ ((e^{-L^+(B^{(1)})} (b_0^{(1)})^{-L(0)} s_1^{L(0)} v^{(1)}) \otimes \cdots\\
&\otimes (e^{-L^+(B^{(n_2)})} (b_0^{(n_2)})^{-L(0)} s_{n_2}^{L(0)} v^{(n_2)}))) \otimes \cdots\\
&\otimes (e^{-L^+(A^{(n_1)})} (a_0^{(n_1)})^{-L(0)} r_{n_1}^{-L(0)} u^{(n_1)}))) \qquad (3.15)
\end{aligned}
$$

where $\circ$ means the composition.

We first show that (3.15) makes sense and can in fact be viewed as an element of

$$\mathrm{Hom}(T_\wedge^{\otimes(n_1+n_2-1)}, \overline{T}_\wedge).$$

Since $\hat{Q}_1 \in K_{\mathfrak{H}_1}(n_1)$, $\hat{Q}_2 \in K_{\mathfrak{H}_1}(n_2)$ and $(\mathbf{0}, (a_0^{(i)}, A^{(i)})), ((B^{(0)}, (b_0^{(0)}, \mathbf{0})) \in K_{\mathfrak{H}_1}(1)$, $\nu_{n_1}(\hat{Q}_1)$, $\nu_{n_2}(\hat{Q}_2)$ and $e^{-L^+(A^{(i)}} (a_0^{(i)})^{-L(0)}$ are in $\mathrm{Hom}((H^{T_\wedge})^{\otimes n_1}, H^{T_\wedge})$, $\mathrm{Hom}((H^{T_\wedge})^{\otimes n_2}, H^{T_\wedge})$ and $\mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge})$, respectively. By assumption, we also know that

$$r_i^{L(0)} u^{(i)} s_0^{L(0)} e^{-L^-(B^{(0)})} (b_0^{(0)})^{-L(0)} v^{(0)}$$

is in $\mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge})$. Thus

$$
\begin{aligned}
&\nu_{n_1}(\hat{Q}_1) \circ (1 \otimes \cdots\\
&\qquad \otimes ((e^{-L^+(A^{(i)}} (a_0^{(i)})^{-L(0)} r_i^{L(0)} u^{(i)} s_0^{-L(0)} e^{-L^-(B^{(0)}} (b_0^{(0)})^{-L(0)} v^{(0)})\\
&\qquad \circ \nu_{n_2}(\hat{Q}_2))) \otimes \cdots \otimes 1)
\end{aligned}
$$

is in fact in $\mathrm{Hom}((H^{T_\wedge})^{\otimes(n_1+n_2-1)}, H^{T_\wedge})$. Since $r_0^{-L(0)} e^{-L^-(A^{(0)})} (a_0^{(0)})^{-L(0)} u^{(0)}$ can be viewed as a map in $\mathrm{Hom}(\overline{T}_\wedge, \overline{T}_\wedge)$ and since $e^{-L^+(A^{(k)})} (a_0^{(k)})^{-L(0)} r_k^{L(0)} u^{(k)}$, $k = 1, \dots, n_1$, and $e^{-L^+(B^{(l)})} (b_0^{(l)})^{-L(0)} s_l^{L(0)} v^{(l)}$, $l = 1, \dots, n_2$, are maps in $\mathrm{Hom}(T_\wedge, T_\wedge)$, (3.15) can be viewed as an element of $\mathrm{Hom}(T_\wedge^{\otimes(n_1+n_2-1)}, \overline{T}_\wedge)$.

Next we rewrite (3.15) in a different form. Since both $e^{-L^+(A^{(i)})} (a_0^{(i)})^{-L(0)}$ and

$$r_i^{L(0)} u^{(i)} s_0^{-L(0)} e^{-L^-(B^{(0)})} (b_0^{(0)})^{-L(0)} v^{(0)}$$

are in $\mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge})$, their product is also in $\mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge})$. But this product can be rewritten as

$$
\begin{aligned}
&e^{-L^+(A^{(i)})} (a_0^{(i)})^{-L(0)} e^{-L^-(B^{(0)})} (b_0^{(0)})^{-L(0)} \cdot\\
&\qquad \cdot (b_0^{(0)})^{L(0)} e^{L^-(B^{(0)})} r_i^{L(0)} u^{(i)} s_0^{-L(0)} e^{-L^-(B^{(0)})} (b_0^{(0)})^{-L(0)} v^{(0)}. \qquad (3.16)
\end{aligned}
$$

Since as elements of $\mathrm{Hom}(T_\wedge, \overline{T}_\wedge)$,

$$e^{-L^+(A^{(i)})}(a_0^{(i)})^{-L(0)} = \nu_1((\mathbf{0}, (a_0^{(i)}, A^{(i)})))$$

and

$$e^{-L^-(B^{(0)})}(b_0^{(0)})^{-L(0)} = \nu_1((B^{(0)}, (b_0^{(0)}, \mathbf{0})))$$

and since

$$\nu_1((\mathbf{0}, (a_0^{(i)}, A^{(i)}))), \nu_1((B^{(0)}, (b_0^{(0)}, \mathbf{0}))) \in \mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge}),$$

we have

$$\begin{aligned} &e^{-L^+(A^{(i)})}(a_0^{(i)})^{-L(0)} e^{-L^-(B^{(0)})}(b_0^{(0)})^{-L(0)} \\ &\quad = \nu_1((\mathbf{0}, (a_0^{(i)}, A^{(i)})))\nu_1((B^{(0)}, (b_0^{(0)}, \mathbf{0}))). \end{aligned} \tag{3.17}$$

But

$$\nu_1((\mathbf{0}, (a_0^{(i)}, A^{(i)})))\nu_1((B^{(0)}, (b_0^{(0)}, \mathbf{0})))$$

is in fact in $\mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge})$. By the sewing axiom for geometric vertex operator algebras (see Chapter 5 of [H5]), the right-hand side of (3.17) is equal to

$$\nu_1((\mathbf{0}, (a_0^{(i)}, A^{(i)}))_1\infty_0(B^{(0)}, (b_0^{(0)}, \mathbf{0}))). \tag{3.18}$$

Let $(E^{(0)}, (e_0^{(0)}, \mathbf{0}))$ and $(\mathbf{0}, (e_0^{(1)}, E^{(1)}))$ be elements of $K_{\mathfrak{H}_1}$ such that

$$(E^{(0)}, (e_0^{(1)}, \mathbf{0}))_1\infty_0(\mathbf{0}, (e_0^{(1)}, E^{(1)})) = (\mathbf{0}, (a_0^{(i)}, A^{(i)}))_1\infty_0(B^{(0)}, (b_0^{(0)}, \mathbf{0})).$$

Then (3.18) is equal to

$$\begin{aligned} &\nu_1((E^{(0)}, (e_0^{(0)}, \mathbf{0}))_1\infty_0(\mathbf{0}, (e_0^{(1)}, E^{(1)})) \\ &\quad = e^{-L^-(E^{(0)})}(e_0^{(0)})^{-L(0)} e^{-L^+(E^{(1)})}(e_0^{(1)})^{-L(0)}. \end{aligned} \tag{3.19}$$

Since

$$(E^{(0)}, (e_0^{(0)}, \mathbf{0})), (\mathbf{0}, (e_0^{(1)}, E^{(1)})) \in K_{\mathfrak{H}_1},$$

we have

$$e^{-L^-(E^{(0)})}(e_0^{(0)})^{-L(0)}, e^{-L^+(E^{(1)})}(e_0^{(1)})^{-L(0)} \in \mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge}).$$

By the calculations (3.17)–(3.19), (3.16) is equal to

$$\begin{aligned}
&e^{-L^-(E^{(0)})}(e_0^{(0)})^{-L(0)}e^{-L^+(E^{(1)})}(e_0^{(1)})^{-L(0)}\cdot\\
&\qquad\cdot e^{L^-(B^{(0)})}r_i^{L(0)}u^{(i)}s_0^{-L(0)}e^{-L^-(B^{(0)})}(b_0^{(0)})^{-L(0)}v^{(0)}\\
&\quad= e^{-L^-(E^{(0)})}(e_0^{(0)})^{-L(0)}\Big(e^{-L^+(E^{(1)})}(e_0^{(1)})^{-L(0)}e^{L^-(B^{(0)})}r_i^{L(0)}u^{(i)}s_0^{-L(0)}\cdot\\
&\qquad\cdot e^{-L^-(B^{(0)})}(b_0^{(0)})^{-L(0)}v^{(0)}e^{-L^+(E^{(1)})}(e_0^{(1)})^{-L(0)}\Big)(e_0^{(1)})^{L(0)}e^{L^+(E^{(1)})}.
\end{aligned} \tag{3.20}$$

Since

$$e^{-L^-(E^{(0)})}(e_0^{(0)})^{-L(0)}, e^{-L^+(E^{(1)})}(e_0^{(1)})^{-L(0)} \in \mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge})$$

and (3.16) are also in $\mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge})$ and since both $e^{-L^-(E^{(0)})}(e_0^{(0)})^{-L(0)}$, $e^{-L^+(E^{(1)})}(e_0^{(1)})^{-L(0)}$ are invertible,

$$\begin{aligned}
&e^{-L^+(E^{(1)})}(e_0^{(1)})^{-L(0)}e^{L^-(B^{(0)})}r_i^{L(0)}u^{(i)}s_0^{-L(0)}\\
&\qquad\cdot e^{-L^-(B^{(0)})}(b_0^{(0)})^{-L(0)}v^{(0)}e^{-L^+(E^{(1)})}(e_0^{(1)})^{-L(0)}
\end{aligned} \tag{3.21}$$

is also in $\mathrm{Hom}(H^{T_\wedge}, H^{T_\wedge})$. Moreover, we can in fact express (3.21) as a finite linear combination of elements of the form w^-w^+ where w^- and w^+ are elements of $\widetilde{T}_\wedge$ and $\widetilde{T}_\wedge^{\geq -1}$, respectively; here we view $\widetilde{T}_\wedge$ and $\widetilde{T}_\wedge^{\geq -1}$ as subspaces of $\mathrm{Hom}(T_\wedge, \overline{T}_\wedge)$. Thus by the calculations (3.17)–(3.20), (3.16) is a finite linear combination of elements of the form

$$e^{-L^-(E^{(0)})}(e_0^{(0)})^{-L(0)}w^-w^+(e_0^{(1)})^{L(0)}e^{L^+(E^{(1)})}.$$

Using the calculations and discussions above, the Jacobi identity and the study of exponentials of infinite sums of Virasoro operators in [H5], we see that (3.15) can be written as a finite linear combination of elements of $\mathrm{Hom}(T_\wedge^{\otimes(n_1+n_2-1)}, \overline{T}_\wedge)$ of the form

$$\begin{aligned}
&(e^{-L^-(A^{(0)}(Q_1\ {}_i\infty_0 Q_2))}w^{(0)}e^{-L^-(A^{(0)}(Q_1\ {}_i\infty_0 Q_2))})\circ\\
&\qquad \nu_{n_1+n_2-1}(Q_1\ {}_i\infty_0 Q_2)\circ(w^{(1)}\otimes\cdots\otimes w^{(n_1+n_2-1)}),
\end{aligned} \tag{3.22}$$

where $w^{(0)}$ is an element of $\mathrm{Hom}(T_\wedge, \overline{T}_\wedge)$ induced from an element (denoted by the same notations $w^{(0)}$) of $\widetilde{T}_\wedge$, and $w^{(k)}$, $k = 1, \dots, n_1+n_2-1$, are elements of $\mathrm{Hom}(T_\wedge, \overline{T}_\wedge)$ induced from elements (denoted by the same notations $w^{(k)}$) of $\widetilde{T}_\wedge^{\geq -1}$.

The element (3.22) gives an element

$$\mathfrak{K}^*_{n_1+n_2-1}(w^{(0)} \otimes w^{(1)} \otimes \cdots \otimes w^{(n_1+n_2-1)}) \tag{3.23}$$

of $\widetilde{\mathcal{T}}_\wedge(n_1+n_2-1)$. Taking the same linear combination as what we obtained for (3.15) but replacing maps of the form (3.22) by the corresponding elements (3.23) of $\widetilde{\mathcal{T}}_\wedge(n_1+n_2-1)$, we obtain an element of $\widetilde{\mathcal{T}}_\wedge(n_1+n_2-1)$, that is, a section of $\mathfrak{G}(n_1+n_2-1)$. Let $\alpha_1{}_i *_0 \alpha_2$ be the value of this section at $Q_1\ {}_i\infty_0 Q_2$. It is clear from the construction above that $\alpha_1{}_i *_0 \alpha_2$ is uniquely determined by (Q_1, α_1) and (Q_2, α_2). We define

$$(Q_1, \alpha_1) \circ_i (Q_2, \alpha_2) = (Q_1\ {}_i\infty_0 Q_2, \alpha_1{}_i *_0 \alpha_2).$$

This finishes the definition of the composition maps.

The bundle $\mathfrak{G}(n)$ has a constant section 1. When $n = 1$, the value of 1 at the identity I of $K(1)$ gives an element $I_{\mathfrak{G}} = (I, 1\big|_I)$. For $n \geq 0$, there is an obvious action of S_n on $\mathfrak{G}(n)$. From the definition, the following result is obvious:

Proposition 3.19 *The sequence $\mathfrak{G}$ equipped with the composition maps defined above, the identity $I_{\mathfrak{G}}$ and the obvious actions of S_n is an analytic partial operad.* $\square$

We consider the vertex operator subalgebra T_G of $T_\wedge$ generated by b, ω_G, f_G and q_G. We consider the elements of $\widetilde{\mathcal{T}}_\wedge(1)$ of the form $\mathfrak{K}^*_{n_1}(\sum_{j=1}^k u_j^{(0)} \otimes u_j^{(1)})$, where $u_j^{(0)} \in \widetilde{T}_\wedge \cap \mathrm{Hom}(T_G, \overline{T}_G)$ and $u_j^{(1)} \in \widetilde{T}_\wedge^{\geq -1} \cap \mathrm{Hom}(T_G, \overline{T}_G)$, $j = 1, \ldots, k$. These elements give a holomorphic subbundle $\mathcal{T}_G(1)$ of $\mathfrak{G}(1)$ and T_G gives a holomorphic subbundle $\mathcal{T}_G(0)$ of $\mathfrak{G}(0)$. The value at $P(1)$ of the section 1 of $\mathfrak{G}(2)$ above gives an element $(P(1), 1\big|_{P(1)})$ of $\mathfrak{G}(2)$. Let $\mathcal{T}_G$ be the partial suboperad of $\mathfrak{G}$ generated by $\mathcal{T}_G(0)$, $\mathcal{T}_G(1)$ and $(P(1), 1\big|_{P(1)})$. This partial operad is still not the (strong) topological partial operad we need but it motivates our construction of the (strong) topological partial operad in Subsection 4.2.

4 (Strong) topological vertex partial operads and the proofs of the main results

4.1 Properties of topological vertex operator algebras

In this subsection, we prove some results on topological vertex operator algebras needed in the construction of (strong) topological partial operads and the proofs of the main theorem.

First we need several lemmas:

Lemma 4.1 *Let V be a $\mathbb{Z}$-graded vector space. Consider the space* $\mathrm{End}\, V$ *spanned by homogeneous operators on V. Then* $\mathrm{End}\, V$ *is a $\mathbb{Z}$-graded associative algebra. If we use $|A|$ to denote the grading of a homogeneous element A of* $\mathrm{End}\, V$ *and define a bracket $[\cdot, \cdot]$ on* $\mathrm{End}\, V$ *by*

$$[A, B] = AB - (-1)^{|A||B|}BA,$$

then $\mathrm{End}\, V$ *equipped with the graded associative algebra structure and the bracket $[\cdot, \cdot]$ is a graded Poisson algebra. In other words, in addition to the axioms for a graded associative algebra, the following conditions are satisfied:*

$$[A, B] = (-1)^{|A||B|}[B, A] \tag{4.1}$$

$$[A, [B, C]] = [[A, B], C] + (-1)^{|A||B|}[B, [A, C]] \tag{4.2}$$

$$[A, BC] = [A, B]C + (-1)^{|A||B|}B[A, C]. \quad \square \tag{4.3}$$

Lemma 4.2 *Let V be a topological vertex operator algebra. Then for $i, j \in \mathbb{Z}$,*

$$\begin{aligned} [g(i), L(j)] &= [L(i), g(j)] \\ &= (i - j)g(i + j). \end{aligned} \tag{4.4}$$

Proof. Since $L(n)g = 0$ for $n > 0$ and $L(0)g = 2g$, the vertex operator $Y(g, x)$ of g is a primary field of weight 2. So for $i, j \in \mathbb{Z}$, we have

$$\begin{aligned} [L(i), g(j)] &= [L(i), \mathrm{Res}_x x^{j+1} Y(g, x)] \\ &= \mathrm{Res}_x x^{j+1}[L(i), Y(g, x)] \\ &= \mathrm{Res}_x x^{j+1}\left(x^{i+1}\frac{d}{dx} + 2(i + 1)x^i\right)Y(g, x) \\ &= (i - j)g(i + j). \end{aligned} \tag{4.5}$$

By (4.1) and (4.5), we obtain

$$\begin{aligned}
[g(i), L(j)] &= -[L(j), g(i)] \\
&= -(j-i)g(j+i) \\
&= (i-j)g(i+j). \quad \square
\end{aligned}$$

Proposition 4.3 *Let V be a topological vertex operator algebra. Then for any $n \in \mathbb{Z}$ and $v \in V$,*

$$[Q, [Q, v_n]] = 0, \tag{4.6}$$

where $v_n = \mathrm{Res}_x x^n Y(v, x)$.

Proof. For any $u \in V$, by the commutator formula for V,

$$[q_0, Y(u, x)] = Y(q_0 u, x).$$

Thus

$$\begin{aligned}
[Q, [Q, v_n]] &= [q_0, [q_0, v_n]] \\
&= [q_0, [q_0, \mathrm{Res}_x x^n Y(v, x)]] \\
&= \mathrm{Res}_x x^n [q_0, [q_0, Y(v, x)]] \\
&= \mathrm{Res}_x x^n [q_0, Y(q_0 v, x)] \\
&= \mathrm{Res}_x x^n Y(q_0^2 v, x) \\
&= \mathrm{Res}_x x^n Y(Q^2 v, x) \\
&= 0. \quad \square
\end{aligned}$$

Proposition 4.4 *The central charge of a topological vertex operator algebra V is* 0*.*

Proof. Let the central charge of V be c. Then for any $i \in \mathbb{Z}$,

$$[L(i), L(-i)] = 2iL(0) + \frac{c}{12}(i^3 - i). \tag{4.7}$$

On the other hand, using $L(-i) = [Q, g(-i)]$ and the Jacobi identity (4.2),

$$\begin{aligned}
[L(i), L(-i)] &= [L(i), [Q, g(-i)]] \\
&= [[L(i), Q], g(-i)] + [Q, [L(i), g(-i)]], \qquad (4.8)
\end{aligned}$$

By (4.4), $[L(i), g(-i)] = 2ig(0)$. Thus

$$[Q, [L(i), g(-i)]] = [Q, 2ig(0)] = 2iL(0).$$

So we see that the right-hand side of (4.8) is equal to $2iL(0)$. Comparing it with the right-hand side of (4.7), we obtain $c = 0$. $\square$

Proposition 4.5 *Let V be a topological vertex operator algebra, then*

1. *For any $i, j \in \mathbb{Z}$, $[Q, [g(i), g(j)]] = 0$.*
2. *If $g(0)^2 = 0$, then we have $[g(i), g(j)] = 0$ for any $i, j \in \mathbb{Z}$. In particular, we have $g(i)^2 = 0$ for any $i \in \mathbb{Z}$.*

Proof. The first conclusion follows from the following straightforward calculation using $[Q, g(i)] = L(i)$, $i \in \mathbb{Z}$, Lemmas 4.1 and 4.2:

$$\begin{aligned}
[Q, [g(i), g(j)]] &= [[Q, g(i)], g(j)] - [g(i), [Q, g(j)]] \\
&= [L(i), g(j)] - [g(i), L(j)] \\
&= (i-j)g(i+j) - (i-j)g(i+j) \\
&= 0.
\end{aligned} \tag{4.9}$$

To prove the second conclusion, first we claim that $[g(i), g(0)] = 0$ for any $i \in \mathbb{Z}$. For $i = 0$, this follows trivially from the condition $g(0)^2 = 0$. So we assume $i \neq 0$. Then by (4.3) and (4.4),

$$\begin{aligned}
0 &= [L(i), g(0)^2] \\
&= [L(i), g(0)]g(0) + g(0)[L(i), g(0)] \\
&= i(g(i)g(0) + g(0)g(i)) \\
&= i[g(i), g(0)]
\end{aligned} \tag{4.10}$$

So we also have $[g(0), g(i)] = 0$ for $i \neq 0$. Finally we assume $i, j \neq 0$. Using $[g(i), g(0)] = 0$ for $i \in \mathbb{Z}$ which we have just proved, the formulas (4.4) and (4.2), we have

$$[g(i), g(j)] = \frac{1}{j}[g(i), [L(j), g(0)]]$$

$$
\begin{aligned}
&= \frac{1}{j}[[g(i), L(j)], g(0)] + \frac{1}{j}[L(j), [g(i), g(0)]] \\
&= \frac{i-j}{j}[g(i+j), g(0)] + \frac{1}{j}[L(j), [g(i), g(0)]] \\
&= 0. \quad \square \qquad (4.11)
\end{aligned}
$$

By this proposition, for any strong topological vertex operator algebra, the operators $g(i)$, $i \in \mathbb{Z}$, are anti-commutative.

4.2 Construction of (strong) topological vertex partial operads

We construct topological vertex partial operads $\mathcal{T}^k$ of type k and strong topological partial operad $\bar{\mathcal{T}}^k$ of type k for $k < 0$ in this subsection.

To construct (strong) topological vertex partial operads, we need to construct a grading-restricted (strong) topological vertex operator algebra T^k ($\bar{T}^k$) of type k for each $k < 0$ such that it is *universal* in the sense that given any locally-grading-restricted (strong) topological vertex operator algebra V of type k, there is a unique homomorphism from T^k ($\bar{T}^k$) to V.

We first construct a $\mathbb{Z} \times \mathbb{Z}$-graded vertex algebra U. Consider the set

$$S = \{\omega_G, b, f_G, q_G\} \subset T_G.$$

Vertex operators of elements of this set have the component operators $L(n)$, $b(n), f_G(n), q_G(n)$, $n \in \mathbb{Z}$. Clearly, these operators are linearly independent. Let U_S be the vector space spanned by the operators $L(n), b(n)$ and $f_G(n), q_G(n)$ and let $T(U_S)$ be the tensor algebra over U_S. Then as a vector space, $T(U_S)$ has a basis consisting 1 and elements of the form $u_1(n_1) \otimes \ldots \otimes u_l(n_l)$ for $l \geq 0$, $u_1, \ldots, u_l \in S$ and $n_1, \ldots, n_l \in \mathbb{Z}$. Let J be the ideal of $T(U_S)$ generated by $L(n)$, $b(n)$, $n \geq -1$, and $f_G(n)$, $q_G(n)$, $n \geq 0$, and let $U = T(U_S)/J$. We shall use $[u_1(n_1) \otimes \ldots \otimes u_l(n_l)]$ to denote the coset in U containing $u_1(n_1) \otimes \ldots \otimes u_l(n_l)$. We shall denote $[\omega_G(-2)]$, $[b(-2)]$, $[f_G(-1)]$ and $[q_G(-1)]$ in U by ω_U, g_U, f_U and q_U, respectively.

We define the weight of the element $[u_1(n_1) \otimes \cdots \otimes u_l(n_l)]$ to be $-n_1 - \cdots - n_l$. We define the fermion number of the same element to be the number of b's in $(u_1, \ldots, u_l)$ minus the number of q_G's in $(u_1, \ldots, u_l)$. We use the notation $|v|$ to denote the fermion number of an element $v \in U$. Thus U becomes a $\mathbb{Z} \times \mathbb{Z}$-graded vector space.

For any $u \in S$ and $n \in \mathbb{Z}$, we define an action of $u(n)$ on U as follows: We define $u(n)[1] = [u(n)]$ and

$$u(n)([u_1(n_1) \otimes \ldots \otimes u_l(n_l)]) = [u(n) \otimes u_1(n_1) \otimes \ldots \otimes u_l(n_l)]$$

for elements of U of the form $[u_1(n_1) \otimes \ldots \otimes u_l(n_l)]$, $l > 0$. We have

$$[u_1(n_1) \otimes \ldots \otimes u_l(n_l)] = u_1(n_1) \ldots u_l(n_l)[1].$$

We define a vertex operator map $Y : U \to (\text{End } U)[[x, x^{-1}]]$ as follows: We define $Y([1], x) = I_U$, the identity operator on U. We also define

$$\begin{aligned}
Y(\omega_U, x) &= \sum_{n \in \mathbb{Z}} L(n) x^{-n-2} \\
Y(g_U, x) &= \sum_{n \in \mathbb{Z}} b(n) x^{-n-2} \\
Y(f_U, x) &= \sum_{n \in \mathbb{Z}} f_G(n) x^{-n-1} \\
Y(q_U, x) &= \sum_{n \in \mathbb{Z}} q_G(n) x^{-n-1}.
\end{aligned}$$

Let $v = u(m)w$. We define

$$\begin{aligned}
Y(v, x) &= \text{Res}_{x_1}(x_1 - x)^{m+1} Y(u, x_1) Y(w, x) \\
&\quad -(-1)^{|u||w|} \text{Res}_{x_1}(-x + x_1)^{m+1} Y(w, x) Y(u, x_1)
\end{aligned}$$

when $u = \omega_U, g_U$, and define

$$\begin{aligned}
Y(v, x) &= \text{Res}_{x_1}(x_1 - x)^{m} Y(u, x_1) Y(w, x) \\
&\quad -(-1)^{|u||w|} \text{Res}_{x_1}(-x + x_1)^{m} Y(w, x) Y(u, x_1)
\end{aligned}$$

when $u = f_U, q_U$. By recursion, we obtain a vertex operator map $Y : U \to (\text{End } U)[[x, x^{-1}]]$.

Proposition 4.6 *The $\mathbb{Z} \times \mathbb{Z}$-graded vector space U equipped with the vertex operator map Y defined above, the vacuum $[1]$ and the Virasoro element ω_U is a $\mathbb{Z} \times \mathbb{Z}$-graded vertex operator algebra without grading restrictions. Given any locally-grading-restricted topological vertex operator algebra V, there exists a unique homomorphism of $\mathbb{Z} \times \mathbb{Z}$-graded vertex operator algebra without grading restrictions from U to V such that the images of g_U, f_U and q_U are g, f and q, respectively.*

Proof. The proof is the same as the proof that $M(c)$ is a vertex operator algebra in [H3]. $\square$

The $\mathbb{Z}\times\mathbb{Z}$-graded vertex algebra U is not a topological vertex operator algebra because the additional axioms for g_U, f_U and q_U are not satisfied. For $k<0$, the grading-restricted topological vertex operator algebra T^k of type k is constructed from U as follows: Let R^k be the ideal of U generated by elements of the forms $[u_1(n_1)\otimes\cdots\otimes u_l(n_l)]$ for $l>0$, $u_1,\dots,u_l\in S$, $n_1,\dots,n_l\in\mathbb{Z}$ satisfying $-n_1-\cdots-n_l<k$, $f_U(0)v-|v|v$, $L(n)q_U$, $q_U(0)^2v$, $L(n)g_U$, and $q_U(0)g_U-\omega_U$ for $v\in U$, $n>0$. Let $T^k=U/R^k$. Let g_{T^k}, f_{T^k} and q_{T^k} be the cosets in T^k containing g_U, f_U and q_U, respectively. The following result is obvious from Proposition 4.6 and the construction of T^k:

Proposition 4.7 *For any $k<0$, T^k together with g_{T^k}, f_{T^k} and q_{T^k} is a grading-restricted topological vertex operator algebra of type k. It satisfies the following universal property: Given any locally-grading-restricted topological vertex operator algebra V of type k, there exists a unique homomorphism of locally-grading-restricted topological vertex operator algebras from T^k to V.* $\square$

For $k<0$, using T^k, we construct a grading-restricted strong topological vertex operator algebra $\bar{T}^k$ of type k as follows: Let O^k be the ideal of T^k generated by elements of the form $g_{T^k}^2(0)v$ for $v\in T^k$. We define $\bar{T}=T^k/O^k$. Let $g_{\bar{T}^k}$, $f_{\bar{T}^k}$ and $q_{\bar{T}^k}$ be the cosets in T^k containing g_{T^k}, f_{T^k} and q_{T^k}, respectively. The following result is obvious:

Proposition 4.8 *For any $k<0$, $\bar{T}^k$ together with $g_{\bar{T}^k}$, $f_{\bar{T}^k}$ and $q_{\bar{T}^k}$ is a grading-restricted strong topological vertex operator algebra of type k. It satisfies the following universal property: Given any locally-grading-restricted strong topological vertex operator algebra V satisfying of type k, there exists a unique homomorphism of locally-grading-restricted topological vertex operator algebras from U to V.* $\square$

Motivated by the partial operad $\mathcal{T}_G$, we now construct the topological vertex partial operad $\mathcal{T}^k$ of type k for $k<0$ using the topological vertex operator algebra T^k of type k and the sphere partial operad K as follows: Since components of vertex operators of elements of $U_S\subset T_G$ can be identified with holomorphic sections of the holomorphic bundle $\mathcal{T}_G(1)$ over $K(1)$, there exists a holomorphic bundle over $K(1)$ such that components of vertex operators of elements of U span a space of holomorphic sections of this holomorphic bundle over $K(1)$ and fibers over $Q \in K(1)$ of this holomorphic bundle are spanned by the values of the sections in this space at Q. Note that the space $Y(U)$ of components of vertex operators of elements of U is a $\mathbb{Z}$-graded vector space. Its algebraic completion $\overline{Y(U)}$ is a subspace of $\mathrm{Hom}(U, \overline{U})$. Let H^U be the locally convex topological completion of U constructed in [H6] and [H7] (see Subsection 3.4) and let $\widetilde{Y(U)}$ be the subspace of $\overline{Y(U)}$ consisting of elements $u \in \overline{Y(U)}$ satisfying the following property: There exists a positive number a such that $a^{L(0)}ua^{-L(0)} \in \mathrm{Hom}(U, \overline{U})$ is in fact in $\mathrm{Hom}(H^U, H^U) \subset \mathrm{Hom}(U, \overline{U})$. Then there exists a holomorphic vector bundle $\tilde{\mathcal{U}}(1)$ over $K(1)$ such that elements of $\widetilde{Y(U)}$ span a space of holomorphic sections of $\tilde{\mathcal{U}}(1)$ and fibers of $\tilde{\mathcal{U}}(1)$ are spanned by the values of the sections in this space. Using $\mathfrak{E} : K(0) \to K(1)$, we pull $\tilde{\mathcal{U}}(1)$ back to a holomorphic bundle $\tilde{\mathcal{U}}(0)$ over $K(0)$. Also we have a holomorphic bundle $\tilde{\mathcal{U}}(1)\big|_{K_{\geq -1}(1)}$ over $K_{\geq -1}(1)$. It is clear that $\tilde{\mathcal{U}}(1)$ is canonically isomorphic to the pullback of the exterior product bundle $\tilde{\mathcal{U}}(0) \boxtimes \tilde{\mathcal{U}}(1)\big|_{K_{\geq -1}(1)}$ by the map $\mathfrak{K}_1 : K(1) \to K(0) \times K_{\geq -1}(1)$. For $n \geq 2$, using the canonical injective map $\mathfrak{K}_n : K(n) \to K(0) \times (K_{\geq -1}(1))^n$, we pull the exterior product bundle $\tilde{\mathcal{U}}(0) \boxtimes (\tilde{\mathcal{U}}(1)\big|_{K_{\geq -1}(1)})^{\boxtimes n}$ over $K(0) \times (K_{\geq -1}(1))^n$ to obtain a holomorphic bundle $\tilde{\mathcal{U}}(n)$ over $K(n)$. We obtain a sequence $\tilde{\mathcal{U}} = \{\tilde{\mathcal{U}}(n)\}_{n \geq 0}$ of holomorphic vector bundles.

Let $\widetilde{Y(R)}^k$ be the intersection of $\widetilde{Y(U)}$ and the algebraic completion $\overline{R}^k$ of R^k. Since R^k is a $\mathbb{Z} \times \mathbb{Z}$-graded vertex operator subalgebra without grading-restrictions of U and since fibers of $\tilde{\mathcal{U}}(1)$ are spanned by the values of the sections in $\widetilde{Y(U)}$ which is viewed as a space of holomorphic sections of $\tilde{\mathcal{U}}(1)$, $\widetilde{Y(R)}^k$ is isomorphic to a space of holomorphic sections of a holomorphic subbundle $\tilde{\mathcal{R}}(1)$ of $\tilde{\mathcal{U}}(1)$ such that the fibers of $\tilde{\mathcal{R}}(1)$ are spanned by the values of the sections in $\widetilde{Y(R)}^k$. The pullback of $\tilde{\mathcal{R}}(1)$ by $\mathfrak{E}$ is a holomorphic subbundle $\tilde{\mathcal{R}}^k(0)$ of $\tilde{\mathcal{U}}(0)$. Using the same construction as the one for $\tilde{\mathcal{U}}(n)$, $n \geq 2$, we construct holomorphic subbundles $\tilde{\mathcal{R}}^k(n)$, $n \geq 2$, of $\tilde{\mathcal{U}}(n)$. Let $\mathcal{T}^k(n) = \tilde{\mathcal{U}}(n)/\tilde{\mathcal{R}}^k(n)$ for $n \geq 0$.

Let $Y(T^k)$ be the space of components of vertex operators of elements of the topological vertex operator algebra T^k and $\overline{Y(T^k)}$ its algebraic completion. Let H^{T^k} be the locally convex completion of T^k and $\widetilde{Y(T^k)}$ the subspace of $\overline{Y(T^k)}$ consisting of elements $u \in \overline{Y(T^k)}$ satisfying the following property: There exists a positive number a such that $a^{L(0)}ua^{-L(0)} \in \mathrm{Hom}(T^k, \overline{T}^k)$ is in fact in $\mathrm{Hom}(H^{T^k}, H^{T^k}) \subset \mathrm{Hom}(T^k, \overline{T}^k)$. Then by definition, we see that $\overline{Y(T^k)}$ is isomorphic to a generating subspace of holomorphic sections of $\mathcal{T}^k(1)$. The holomorphic bundles $\mathcal{T}^k(n)$ can also be constructed from $\mathcal{T}^k(1)$ using the same construction as the one for $\tilde{\mathcal{U}}(n)$, $n \geq 2$. That is, we have holomorphic bundles $\mathcal{T}^k(0)$ and $\mathcal{T}^k(1)|_{K_{\geq -1}(1)}$ over $K(0)$ and $K_{\geq -1}(1)$, respectively, such that

$$\mathcal{T}^k(n) = \mathfrak{K}_n^*(\mathcal{T}^k(0) \boxtimes (\mathcal{T}^k(1)|_{K_{\geq -1}(1)})^{\boxtimes n}.$$

Thus we see that the sequence $\mathcal{T}^k = \{\mathcal{T}^k(n)\}_{n \geq 0}$ of holomorphic bundles can be constructed from the topological vertex operator algebra T^k of type k. Since T^k is a grading-restricted topological vertex operator algebra, the same method as the one in the construction of the composition maps for the partial operad $\mathfrak{G}$ (see Subsection 3.4) gives composition maps for $\mathcal{T}^k$. The symmetric group S_n acts on $\mathcal{T}^k(n)$ in the obvious way for $n \geq 0$. The constant term of the vertex operator of the vacuum of T^k is a holomorphic section of $\mathcal{T}^k(1)$. We denote the value of this section at $I \in K(1)$ by $I_{\mathcal{T}^k}$.

Similarly, using the strong topological vertex operator algebra $\bar{T}^k$ of type k, we can construct a sequence $\bar{\mathcal{T}}^k = \{\bar{\mathcal{T}}^k(n)\}_{n \geq 0}$ of holomorphic bundles together with composition maps, an identity $I_{\bar{\mathcal{T}}^k}$ and actions of S_n from $\mathcal{T}^k$. We have:

Proposition 4.9 *The sequence $\mathcal{T}^k$ equipped with the composition maps, the identity $I_{\mathcal{T}^k}$ and the actions of S_n induced from $\mathcal{T}^k$ is an analytic partial operad. The sequence $\bar{\mathcal{T}}^k$ equipped with the composition maps, the identity $I_{\bar{\mathcal{T}}^k}$ and the actions of S_n is also an analytic partial operad.* $\square$

For $k < 0$, the partial operad $\mathcal{T}^k$ is called the *topological vertex partial operad of type k* and the partial operad $\bar{\mathcal{T}}^k$ is called the *strong topological vertex partial operad of type k*. Clearly these partial operads are $\mathbb{C}^\times$-rescalable (see [HL1], [HL2] and Appendix C of [H5] for the definition). By definition, for any $k < 0$, there is a canonical morphism of $\mathbb{C}^\times$-rescalable analytic partial operads from $\mathcal{T}^k$ to $\bar{\mathcal{T}}^k$.

We now define the notion of (weakly) meromorphic $\mathbb{Z} \times \mathbb{Z}$-graded algebra over $\mathcal{T}^k$ and $\bar{\mathcal{T}}^k$, $k < 0$. First, recall from [H4] that for a $\mathbb{Z} \times \mathbb{Z}$-graded space

V (graded by fermion numbers and by weights) and a $\mathbb{Z}\times\mathbb{Z}$-graded subspace W, we can define the graded endomorphism partial pseudo-operad $\mathcal{H}_V$ to be the sequence $\{\mathrm{Hom}(V^{\otimes n}, \overline{V})\}_{n\in\geq 0}$ as in [HL1], [HL2] and Appendix C of [H5], except that the left actions of the symmetric groups are defined such that for any $f \in \mathrm{Hom}(V^{\otimes n}, \overline{V})$ and $\sigma_{i,i+1} \in S_n$ which is the transposition permuting i and $i+1$,

$$(\sigma_{i,i+1}(f))(v_1, \dots, v_i, v_{i+1}, \dots, v_n) = (-1)^{|v_i||v_{i+1}|} f(v_1, \dots, v_{i+1}, v_i, \dots, v_n) \tag{4.12}$$

for any $v_1, \dots, v_n \in V$ with homogeneous fermion numbers. Using this graded endomorphism partial pseudo-operad, we can define the notions of $\mathbb{Z}\times\mathbb{Z}$-*graded algebra* over any $\mathbb{C}^\times$-rescalable partial operads in the obvious way.

Let $\mathbf{1}_{T^k}$ be the vacuum of T^k and $\mathbf{1}_{T^k}(0) = \mathrm{Res}_x x^{-1} Y(\mathbf{1}_{T^k}, x) \in \widetilde{Y(T^k)}$. Then the holomorphic sections $\mathfrak{K}_n^*(\mathfrak{E}^*(\mathbf{1}_{T^k}(0)) \boxtimes (\mathbf{1}_{T^k}(0)|_{K_{\geq -1}(1)})^{\boxtimes n})$ of $\mathcal{T}^k(n)$, $n \geq 0$, give a morphism of partial operads from K to $\mathcal{T}^k$. Composing with the morphisms of partial operads from $\mathcal{T}^k$ to $\bar{\mathcal{T}}^k$, we obtain a morphism of partial operads from K to $\bar{\mathcal{T}}^k$. In particular, an $\mathbb{Z}\times\mathbb{Z}$-graded algebra over $\mathcal{T}^k$ or $\bar{\mathcal{T}}^k$ is also an $\mathbb{Z}\times\mathbb{Z}$-graded algebra over K, or equivalently, an algebra over $\tilde{K}^0$.

Recall the following definition of meromorphic algebra over $\tilde{K}^c$ in [H5]:

Definition 4.10 A *meromorphic algebra over* $\tilde{K}^c$ is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ equipped with a morphism ν of partial operads from $\tilde{K}^c$ to $\mathcal{H}_V = \{\mathrm{Hom}(V^{\otimes n}, \overline{V})$ satisfying the following axioms:

1. The grading-restriction axiom: $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $V_{(n)} = 0$ for n sufficiently small.

2. For any $n \in \mathbb{N}$, ν_n is linear on any fiber of $\tilde{K}^c(n)$.

3. For any positive integer n, $v' \in V'$, $v_1, \dots, v_n \in V$, the function

$$Q \to \langle v', \nu_n(\psi_n(Q)))(v_1 \otimes \cdots \otimes v_n)\rangle$$

on $K(n)$ is meromorphic (in the sense of Section 3.1) and if z_i and z_j are the i-th and j-th punctures of $Q \in K(n)$ respectively, then for any v_i and v_j in V there exists a positive integer $N(v_i, v_j)$ such that for any

$v' \in V'$, $v_k \in V$, $k \neq i, j$, the order of the pole $z_i = z_j$ (we use the convention $z_n = 0$) of

$$\langle v', \nu_n(\psi_n(Q)))(v_1 \otimes \cdots \otimes v_n)\rangle$$

is less than $N(v_i, v_j)$.

We also need:

Definition 4.11 A *weakly meromorphic algebra over* $\tilde{K}^c$ is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ equipped with a morphism ν of partial operads from $\tilde{K}^c$ to $\mathcal{H}_V = \{\mathrm{Hom}(V^{\otimes n}, \overline{V})$ satisfying all the axioms for meromorphic algebras over $\tilde{K}^c$ except the grading-restriction axioms. The notions of *(weakly) meromorphic ($\mathbb{Z}\times\mathbb{Z}$-graded) algebra over* $\tilde{K}$ and *(weakly) meromorphic* $\mathbb{Z}\times\mathbb{Z}$-*graded algebra over* $\tilde{K}^c$ are defined similarly.

We have the following notions:

Definition 4.12 For $k < 0$, a $\mathbb{Z}\times\mathbb{Z}$-graded algebra V over the topological vertex partial operad $\mathcal{T}^k$ of type k (or over the strong topological vertex partial operad $\bar{\mathcal{T}}^k$ of type k) is called *(weakly) meromorphic* if it is (weakly) meromorphic as a $\mathbb{Z}\times\mathbb{Z}$-graded algebra over $\tilde{K}^0$ and if the map from V to $\overline{V}$ corresponding to $((\mathbf{0}, (1, \mathbf{0})), f_{\mathcal{T}^k}(0)) \in \mathcal{T}^k(1)$ (or $((\mathbf{0}, (1, \mathbf{0})), f_{\bar{\mathcal{T}}^k}(0)) \in \bar{\mathcal{T}}^k(1)$) defines the fermion grading on V.

4.3 Proof of Theorem 2.10

We prove the main theorem, Theorem 2.10, of the present paper in this subsection using the main theorem of [H5] and its generalization to locally-grading-restricted conformal vertex algebra (that is, vertex operator algebras without grading-restrictions but satisfying the local grading-restriction conditions, see Remark 2.3) given in [H7].

The case of grading-restricted (strong) topological vertex operator algebras of type $k < 0$ and the case of locally-grading-restricted strong topological vertex operator algebras of type k is an easy consequence of the case of locally-grading-restricted topological vertex operator algebras of type k. We only prove the case of locally-grading-restricted topological vertex operator algebras of type k.

Let V be a locally-grading-restricted topological vertex operator algebra of type k. Then by Proposition 4.4, the central charge of V is 0. In [H7], the main theorem of [H5] is generalized to locally-grading-restricted conformal vertex algebras, that is, vertex operator algebras without grading-restrictions but satisfying the local grading-restriction conditions (see Remark 2.3). This generalization states that the category of locally-grading-restricted conformal vertex algebras of central charge c is isomorphic to the category of weakly meromorphic algebras over $\tilde{K}^c$. See [H7] for details. By this generalization, V gives a weakly meromorphic $\mathbb{Z}\times\mathbb{Z}$-graded algebra over $\tilde{K}^0$. From the definitions of the partial operad structure on $\tilde{K}^0$ and the definition of a weakly meromorphic $\mathbb{Z}\times\mathbb{Z}$-graded algebra over $\tilde{K}^0$, we see that V in fact gives a $\mathbb{Z}\times\mathbb{Z}$-graded algebra over K.

Let H^V be the locally convex topological completion of V constructed in [H6] and [H7]. By Proposition 4.7, there is a homomorphism of locally-grading-restricted topological vertex operator algebras from T^k to V. In particular, elements of $\widetilde{Y(T^k)}$ can be viewed as elements of $\mathrm{Hom}(V,\overline{V})$. Recall that $\mathcal{T}^k(1)$ is canonically isomorphic to

$$\mathfrak{K}_1^*(\mathcal{T}^k(0)\boxtimes\mathcal{T}^k(1)|_{K_{\geq -1}(1)}).$$

So we need only consider elements of the subbundles $\mathfrak{K}_1^*(\mathcal{T}^k(0)\boxtimes\mathbf{1}_{T^k}(0)|_{K_{\geq -1}(1)})$ and $\mathfrak{K}_1^*(\mathfrak{E}^*(\mathbf{1}_{T^k}(0))\boxtimes\mathcal{T}^k(1)|_{K_{\geq -1}(1)})$ of $\mathcal{T}^k(1)$. Any element of $\mathcal{T}^k(1)$ is of the form $\hat{Q}=(Q,u\big|_Q)$, where $u\in\widetilde{Y(T^k)}$. (Recall that the fibers of $\mathcal{T}^k(1)$ are spanned by the values of the elements of $\widetilde{Y(T^k)}$.) Let $\nu_n: K(n)\to \mathrm{Hom}(V^{\otimes n},\overline{V})$, $n\geq 0$, be the maps defining the structure of a $\mathbb{Z}\times\mathbb{Z}$-graded algebra over K. We define

$$\nu_1^{\mathcal{T}^k}(\hat{Q})=e^{-L^-(A^{(0)}(Q))}ue^{L^-(A^{(0)}(Q))}\nu_1(Q)$$

when $\hat{Q}\in\mathfrak{K}_1^*(\mathcal{T}^k(0)\boxtimes\mathbf{1}_{T^k}(0)|_{K_{\geq -1}(1)})$, and we define

$$\nu_1^{\mathcal{T}^k}(\hat{Q})=\nu_1(Q)u$$

when $\hat{Q}\in\mathfrak{K}_1^*(\mathfrak{E}^*(\mathbf{1}_{T^k}(0))\boxtimes\mathcal{T}^k(1)|_{K_{\geq -1}(1)})$. It is easy to see that when $\hat{Q}\in\mathfrak{K}_1^*(\mathcal{T}^k(0)\boxtimes\mathbf{1}_{T^k}(0)|_{K_{\geq -1}(1)})$, u as an element of $\mathrm{Hom}(H^V,H^V)\subset\mathrm{Hom}(V,\overline{V})$ is in fact an infinite sum of homogeneous elements of $\mathrm{Hom}(V,V)$ of weights larger than an integer depending on u, and when

$$\hat{Q}\in\mathfrak{K}_1^*(\mathfrak{E}^*(\mathbf{1}_{T^k}(0))\boxtimes\mathcal{T}^k(1)|_{K_{\geq -1}(1)}),$$

u as an element of $\mathrm{Hom}(H^V, H^V) \subset \mathrm{Hom}(V, \overline{V})$ is in fact in $\mathrm{Hom}(V, V)$. Thus in both cases, $\nu_1^{\mathcal{T}^k}(\hat{Q})$ is indeed in $\mathrm{Hom}(V, \overline{V})$.

We also define

$$\nu_2^{\mathcal{T}^k}(\mathfrak{K}_2^*(\mathfrak{E}^*(\mathbf{1}_{T^k}(0)) \boxtimes \mathbf{1}_{T^k}(0) \boxtimes \mathbf{1}_{T^k}(0)\big|_{P(z)}) = \nu_2(P(z)).$$

Note that $\mathcal{T}^k(0)$ is constructed from $\mathcal{T}^k(1)$ and $\mathcal{T}^k$ is generated by elements of $\mathcal{T}^k(0)$, $\hat{Q}$ and

$$\mathfrak{K}_2^*(\mathfrak{E}^*(\mathbf{1}_{T^k}(0)) \boxtimes \mathbf{1}_{T^k}(0) \boxtimes \mathbf{1}_{T^k}(0)\big|_{P(z)}.$$

Also note that the definition of the composition maps for $\mathcal{T}^k$ depends only on the vertex operator algebra structure on T^k. Thus we can extend $\nu_1^{\mathcal{T}^k}$ and $\nu_2^{\mathcal{T}^k}$ defined on these particular elements of $\mathcal{T}^k$ to a morphism $\nu^{\mathcal{T}^k}$ of partial pseudo-operads from $\mathcal{T}^k$ to the endomorphism partial pseudo-operad $\mathcal{H}_V = \{\mathrm{Hom}(V^{\otimes n}, \overline{V})\}_{n\geq 0}$ of V. So V gives a $\mathbb{Z}\times\mathbb{Z}$-graded algebra over $\mathcal{T}^k$. Since as a $\mathbb{Z}\times\mathbb{Z}$-graded algebra over $\tilde{K}^0$, it is meromorphic and since the map from V to $\overline{V}$ corresponding to $(\mathbf{0}, (1, \mathbf{0}), f_{T^k}(0)) \in \mathcal{T}^k(1)$ is the fermion grading operator f_0 defining the fermion grading on V, the $\mathbb{Z}\times\mathbb{Z}$-graded algebra over $\mathcal{T}^k$ constructed above is weakly meromorphic. We define a functor from the category of locally-grading-restricted topological vertex operator algebras of type k to the category of weakly meromorphic $\mathbb{Z}\times\mathbb{Z}$-graded algebras over $\mathcal{T}^K$ by assigning to a locally-grading-restricted topological vertex operator algebra V of type k the weakly meromorphic $\mathbb{Z}\times\mathbb{Z}$-graded algebra over $\mathcal{T}^k$ constructed above from V.

Conversely, given a weakly meromorphic $\mathbb{Z}\times\mathbb{Z}$-graded algebra V over $\mathcal{T}^k$, we obtain a locally-grading-restricted topological vertex operator algebra as follows: Since by definition V is a weakly meromorphic $\mathbb{Z}\times\mathbb{Z}$-graded algebra over $\tilde{K}^0$, V has a structure of locally-grading-restricted $\mathbb{Z}\times\mathbb{Z}$-graded conformal vertex algebra of central charge 0 by the generalization of the main theorem of [H5] discussed above and in [H7]. Let $\nu_n^{\mathcal{T}^k}$, $n \geq 0$, be the maps from $\mathcal{T}^k(n)$ to $\mathrm{Hom}(V^{\otimes n}, \overline{V})$ defining the structure of a weakly meromorphic $\mathbb{Z}\times\mathbb{Z}$-graded algebra over $\mathcal{T}^k$. Let

$$\begin{aligned} g &= \nu_0^{\mathcal{T}^k}((\mathbf{0}, \mathfrak{E}^*(g_{T^k}(-2))\big|_{\mathbf{0}})), \\ f &= \nu_0^{\mathcal{T}^k}((\mathbf{0}, \mathfrak{E}^*(f_{T^k}(-1))\big|_{\mathbf{0}})), \\ q &= \nu_0^{\mathcal{T}^k}((\mathbf{0}, \mathfrak{E}^*(q_{T^k}(-1))\big|_{\mathbf{0}})). \end{aligned}$$

Then it is clear that V together with the elements g, f and q is a locally-grading-restricted topological vertex operator algebra. From the construction of V, we also see that there is a canonical homomorphism of locally-grading-restricted topological vertex operator algebras from T_k to V. So V is of type k. Thus we obtain a functor from the category of weakly meromorphic $\mathbb{Z} \times \mathbb{Z}$-graded algebras over $\mathcal{T}^k$ to the category of locally-grading-restricted topological vertex operator algebras of type k.

The two functors constructed above are clearly inverse to each other. Thus we see that the two categories are isomorphic.

4.4 Proof of Theorem 2.8

In this subsection, we prove Theorem 2.8 using Theorem 2.10.

Let V be a locally-grading-restricted strong topological vertex operator algebra. First we know that there exists $k < 0$ such that V is of type k. By Theorem 2.10, V has a structure of a weakly meromorphic $\mathbb{Z} \times \mathbb{Z}$-graded algebra over $\bar{\mathcal{T}}^k$. In particular, V has a structure of a weakly meromorphic $\mathbb{Z} \times \mathbb{Z}$-graded algebra over K. Let H^V be the locally convex topological completion of V constructed in [H6] and [H7]. Then it has been shown in [H6] and [H7] that the maps in $\mathrm{Hom}(V^{\otimes n}, \overline{V})$ corresponding to elements of $K_{\mathfrak{H}_1}(n)$ is in fact in $\mathrm{Hom}(H^{\otimes n}, H)$. Moreover, H^V has a structure of an algebra over the operad (not partial) $K_{\mathfrak{H}_1}$.

For $n \geq 0$, let $\bar{\mathcal{T}}^k_{\mathfrak{H}_1}(n)$ be the holomorphic subbundle over $K_{\mathfrak{H}_1}(n)$ of the holomorphic bundle $\bar{\mathcal{T}}^k(n)$ consisting elements of the form (Q, α) where $Q \in K_{\mathfrak{H}_1}(n)$ and α is the restriction to Q of a linear combination of sections

$$\mathfrak{K}^*_{n_1}(u^{(0)} \otimes u^{(1)} \otimes \cdots \otimes u^{(n_1)})$$

for $u^{(0)} \in \mathfrak{C}^*(\widetilde{Y(\bar{\mathcal{T}}^k)} \cap \mathrm{Hom}(H^{\bar{\mathcal{T}}^k}, H^{\bar{\mathcal{T}}^k}))$ satisfying

$$e^{-L^-(A^{(0)}(Q))} u^{(0)} e^{L^-(A^{(0)}(Q))} \in \mathrm{Hom}(H^{\bar{\mathcal{T}}^k}, H^{\bar{\mathcal{T}}^k})$$

and $u^{(k)} \in (\widetilde{Y(\bar{\mathcal{T}}^k)} \cap \mathrm{Hom}(H^{\bar{\mathcal{T}}^k}, H^{\bar{\mathcal{T}}^k}))|_{K_{\geq -1}(1)}$, $k = 1, \dots, n$. Then it is clear that $\bar{\mathcal{T}}^k_{\mathfrak{H}_1} = \{\bar{\mathcal{T}}^k_{\mathfrak{H}_1}(n)\}_{n \geq 0}$ is a suboperad (not partial) of $\bar{\mathcal{T}}^k$. From the definition of $\bar{\mathcal{T}}^k$ in Subsection 4.2 and the construction of the structure of a weakly meromorphic $\mathbb{Z} \times \mathbb{Z}$-graded algebra over $\bar{\mathcal{T}}^k$ on V in the preceding subsection, we see that the maps in $\mathrm{Hom}(V^{\otimes n}, \overline{V})$ corresponding to elements of $\bar{\mathcal{T}}^k_{\mathfrak{H}_1}(n)$

are actually in $\mathrm{Hom}((H^V)^{\otimes n}, H^V)$. Thus we see that H^V has a structure of an algebra over the operad (not partial) $\bar{\mathcal{T}}^k_{\mathfrak{H}_1}$.

On the other hand, from the construction of $\bar{\mathcal{T}}^k$ in Subsection 4.2, there is a morphism of operads from $\wedge TK_{\mathfrak{H}_1}$ to $\bar{\mathcal{T}}^k_{\mathfrak{H}_1}$ constructed as follows: Consider the vertex operator subalgebra T_b of T_G generated by b and ω_G. Using T_b and the same construction as the one used to construct $\mathcal{T}_G$ at the end of Subsection 3.3, we obtain a partial suboperad $\mathcal{T}_b$ of $\mathcal{T}_G$. Let $\wedge TK(n)$, $n \geq 0$, be the direct sums of all wedge powers of the tangent bundles of the moduli spaces $K(n)$. Then it is clear that $\wedge TK = \{\wedge TK(n)\}_{n\geq 0}$ is a partial operad. For any $n \geq 0$, the direct sums of all wedge powers of the holomorphic tangent bundles of the moduli space $K(n)$ is a subbundle of $\wedge TK(n)$. In fact it is clear that this subbundle is isomorphic to the bundle $\mathcal{T}_b(n)$ and there is a bundle map from $\wedge TK(n)$ to $\mathcal{T}_b(n)$ such on each fiber, this bundle map is the projection from the fiber of $\wedge TK(n)$ to the fiber of $\mathcal{T}_b(n)$. Moreover, by the definition of the composition maps for $\mathcal{T}_b$, we see that $\mathcal{T}_b$ is in fact a partial suboperad of $\wedge TK$ and the bundle maps from $\wedge TK(n)$ to $\mathcal{T}_b(n)$ give a morphism of partial operads.

Clearly the restriction of $\wedge TK$ to $K_{\mathfrak{H}_1}(n)$ is the operad $\wedge TK_{\mathfrak{H}_1}$. Since $\bar{T}$ is a strong topological vertex operator algebra, there is a homomorphism of vertex operator algebras from T_b to $\bar{T}$. This homomorphism induces a morphism of partial operad from $\mathcal{T}_b$ to $\bar{\mathcal{T}}^k$. Composing this morphism with the morphism from $\wedge TK$ to $\mathcal{T}_b$, we obtain a morphism from $\wedge TK$ to $\bar{\mathcal{T}}^k$. This morphism gives a morphism from $\wedge TK_{\mathfrak{H}_1}$ to $\bar{\mathcal{T}}^k_{\mathfrak{H}_1}$. Composing the morphism from $\wedge TK_{\mathfrak{H}_1}$ to $\bar{\mathcal{T}}^k_{\mathfrak{H}_1}$ and the morphism from $\bar{\mathcal{T}}^k_{\mathfrak{H}_1}$ to $\{\mathrm{Hom}((H^V)^{\otimes n}, H^V)\}_{n\geq 0}$, we obtain a morphism from $\wedge TK_{\mathfrak{H}_1}$ to $\mathcal{E}_{H^V} = \{\mathrm{Hom}((H^V)^{\otimes n}, H^V)\}_{n\geq 0}$.

The differential Q on H^V is the natural extension to H^V of the operator $Q = q_0 = \mathrm{Res}_x Y(q, x)$ on V. The proof of (2.2) is the same as the proof of the same formula for topological vertex algebras in [H4] (cf. also [KSV]). In fact, since in our case $b(j)$, $j \in \mathbb{Z}$, are anti-commutative with each other, the proof of (2.2) is simpler than the corresponding proof in [H4].

5 Appendix: Examples of locally-grading-restricted strong topological vertex operator algebras

5.1 Locally-grading-restricted conformal vertex algebra of central charge 26 tensored with the ghost vertex operator algebra

These examples are substructures of string backgrounds in string theory. They were discussed extensively by Lian and Zuckerman. See for example [LZ2] and the references there.

Let V be a locally-grading-restricted conformal vertex algebra of central charge 26, that is, a vertex operator algebra of without grading restrictions of central charge 26 satisfying local grading-restriction conditions (see Remark 2.3). Then $V \otimes \wedge_{\infty}\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ is a locally-grading-restricted $\mathbb{Z} \times \mathbb{Z}$-graded conformal vertex algebra of central charge 0. Let

$$
\begin{aligned}
g &= \mathbf{1}_V \otimes b,\\
q &= L_V(-2)\mathbf{1} \otimes c(1)1 + 1 \otimes (b(-2)c(1)c(0)1)\\
&= L_V(-2)\mathbf{1} \otimes c + \mathbf{1} \otimes (\psi_0^*(\mathbb{L}'(0)) \wedge \psi_0^*(\mathbb{L}'(-1)) \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(-2)))),\\
f_G &= \mathbf{1} \otimes (c(1)b(-2)1)\\
&= \mathbf{1} \otimes (\psi_0^*(\mathbb{L}'(-1)) \wedge \psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}(-2))))
\end{aligned}
$$

be three elements of $V \otimes \wedge_{\infty}\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$. Then $V \otimes \wedge_{\infty}\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$ together with these three elements is a locally-grading-restricted strong topological vertex operator algebra.

The above construction begin with a locally-grading-restricted conformal vertex algebra of central charge 26. Such an algebra can be constructed from a locally-grading-restricted conformal vertex algebra of arbitrary central charge c this algebra with a familiar locally-grading-restricted conformal vertex algebra of central charge $26 - c$. One example of such familiar locally-grading-restricted conformal vertex algebras is the one coming from the so-called Liouville models. We can also use other familiar examples associated to the Virasoro algebra, affine Lie algebras and lattices.

One important example of this class of locally-grading-restricted strong topological vertex operator algebra is the tensor product of the moonshine module vertex operator algebra constructed by Frenkel, Lepowsky and Meurman [FLM1] [FLM2], the locally-grading-restricted conformal vertex algebra constructed from a rank 2 Lorentz lattice and the ghost vertex operator algebra $\wedge_{\infty}\psi_0^*(\tilde{\mathfrak{D}}_*(\mathbb{L}))$. This example was studied in detail by Lian and Zuckerman in [LZ3]. Since the lattice used is Lorentzian, this example does not satisfy the grading-restriction conditions. So it is not a topological vertex operator algebra in the sense of [H4] and Definition 2.2. The tensor product of the first two algebras was used by Borcherds in his famous proof of the Monstrous Moonshine Conjecture in [B2]. The Monster Lie algebra constructed in [B2] can also be constructed using the Gerstenhaber algebra structure on the cohomology of this example.

5.2 Twisted $N = 2$ superconformal vertex operator superalgebras

In [EY], Eguchi and Yang constructed explicitly topological conformal field theories in a physical sense by twisting $N = 2$ superconformal field theories. A good exposition can be found in, for example, [W] by Warner. Here we formulate the theory in terms of the mathematical language of vertex operator algebras. These topological vertex operator algebras are closely related to the mirror symmetry.

Definition 5.1 An $N = 2$ *superconformal vertex operator superalgebra* is a vertex operator superalgebra $(V, Y, \mathbf{1}, \omega)$ together with odd elements τ^+, τ^- and even h satisfying the following axioms: Let

$$\begin{aligned} Y(\tau^+, x) &= \sum_{n\in\mathbb{Z}} G^+(n+\frac{1}{2})x^{-n-2},\\ Y(\tau^-, x) &= \sum_{n\in\mathbb{Z}} G^-(n-\frac{1}{2})x^{-n-1},\\ Y(h, x) &= \sum_{n\in\mathbb{Z}} J(n)x^{-n-1}. \end{aligned}$$

Then V is a direct sum of eigenspaces of $J(0)$ with integral eigenvalues which modulo $2\mathbb{Z}$ give the $\mathbb{Z}_2$ grading for the vertex operator superalgebra structure, and the following $N = 2$ Neveu-Schwarz relations hold: For $m, n \in \mathbb{Z}$,

$$\begin{aligned} [L(m), L(n)] &= (m-n)L(m+n) + \frac{c}{12}(m^3-m)\delta_{m+n,0},\\ [J(m), J(n)] &= \frac{c}{3}m\delta_{m+n,0},\\ [L(m), J(n)] &= -nJ_{m+n},\\ [L(m), G^\pm_{n\pm\frac{1}{2}}] &= \left(\frac{m}{2} - (n \pm \frac{1}{2})\right) G^\pm_{m+n\pm\frac{1}{2}}, \end{aligned}$$

$$
\begin{aligned}
[J(m), G^{\pm}_{n\pm\frac{1}{2}}] &= \pm G^{\pm}_{m+n\pm\frac{1}{2}},\\
[G^{+}_{m+\frac{1}{2}}, G^{-}_{n-\frac{1}{2}}] &= 2L(m+n) + (m-n+1)J(m+n) + \frac{c}{3}(m^2+m)\delta_{m+n,0},\\
[G^{\pm}_{m\pm\frac{1}{2}}, G^{\pm}_{n\pm\frac{1}{2}}] &= 0
\end{aligned}
$$

where $L(m)$, $m \in \mathbb{Z}$, are the Virasoro operators on V and c is the central charge of V.

We shall denote the $N = 2$ superconformal vertex operator superalgebra defined above by $(V, Y, \mathbf{1}, \omega, \tau^+, \tau^-, h)$ or simply by V.

Let $(V, Y, \mathbf{1}, \omega, \tau^+, \tau^-, h)$ be an $N = 2$ superconformal vertex operator superalgebra. We define

$$
\begin{aligned}
\omega_T &= \omega + \frac{1}{2}J(-2)\mathbf{1},\\
f &= h\\
q &= \tau^+,\\
g &= \tau^-.
\end{aligned}
$$

An easy calculation gives the following:

Proposition 5.2 *Let $(V, Y, \mathbf{1}, \omega, \tau^+, \tau^-, h)$ be an $N = 2$ superconformal vertex operator superalgebra. Then $(V, Y, \mathbf{1}, \omega_T, f, q, g)$ is a grading-restricted strong topological vertex operator algebra.* $\square$

References

[A] F. Akman, On some generalizations of Batalin-Vilkovisky algebras, *J. Pure Appl. Alg.* **120** (1997), 105–141.

[B1] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068–3071.

[B2] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, *Invent. Math.* **109** (1992), 405–444.

[C] F. R. Cohen, The homology of $\mathcal{C}_{n+1}$-spaces, $n \geq 0$, in: *The homology of iterated loop spaces,* Lecture Notes in Math., **533**, Springer, Berlin, 1976, 207–351.

[DL] C. Dong and J. Lepowsky, *Generalized Vertex Algebras and Relative Vertex Operators*, Progress in Math., **112**, Birkhäuser Boston, 1993.

[EY] T. Eguchi and S.-K. Yang, $N = 2$ superconformal models as topological field theories. *Modern Phys. Lett.* **A5** (1990), 1693–1701.

[Fe] B. L. Feigin, Semi-infinite homology of Lie, Kac-Moody and Virasoro algebras, *Uspekhi Mat. Nauk* **39** (1984), 195–196.

[Fr] I. B. Frenkel, talk presented at the Institute for Advanced Study, 1988; and private communications.

[FGZ] I. B. Frenkel, H. Garland, and G. J. Zuckerman, Semi-infinite cohomology and string theory. *Proc. Nat. Acad. Sci. U.S.A.* **83** (1986), 8442–8446.

[FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, 1993.

[FLM1] I. B. Frenkel, J. Lepowsky and A. Meurman, A natural representation of the Fischer-Griess monster with the modular function J as character, *Proc. Natl. Acad. Sci. USA* **81** (1984), 3256–3260.

[FLM2] I. B. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator algebras and the Monster*, Pure and Appl. Math., **134**, Academic Press, New York, 1988.

[G] E. Getzler, Batalin-Vilkovisky algebras and two-dimensional topological field theory, *Comm. Math. Phys.* **159** (1994), 265-285.

[GJ] E. Getzler and J. D. S. Jones, Operads, homotopy algebra and iterated integrals for double loop spaces, `hep-th/9403055`, to appear.

[H1] Y.-Z. Huang, *On the geometric interpretation of vertex operator algebras*, Ph.D thesis, Rutgers University, 1990.

[H2] Y.-Z. Huang, Geometric interpretation of vertex operator algebras, *Proc. Natl. Acad. Sci. USA* **88** (1991), 9964–9968.

[H3] Y.-Z. Huang, Vertex operator algebras and conformal field theory, *Intl. Jour. of Mod. Phys* **A7** (1992), 2109–2151.

[H4] Y.-Z. Huang, Operadic formulation of topological vertex algebras and Gerstenhaber or Batalin-Vilkovisky algebras, *Comm. in Math. Phys.* **164** (1994), 105–144.

[H5] Y.-Z. Huang, Two-dimensional conformal geometry and vertex operator algebras, *Progress in Mathematics*, Vol. 148, 1997, Birkhäuser, Boston.

[H6] Y.-Z. Huang, A functional-analytic theory of vertex (operator) algebras, I, 31 pages, *Comm. in Math. Phys.*, `math.QA/9808022`, to appear.

[H7] Y.-Z. Huang, A functional-analytic theory of vertex (operator) algebras, II, in preparation.

[HL1] Y.-Z. Huang and J. Lepowsky, Vertex operator algebras and operads, *The Gelfand Mathematical Seminar, 1990–1992*, ed. L. Corwin, I. Gelfand and J. Lepowsky, Birkhäuser Boston, 1993, 145–161.

[HL2] Y.-Z. Huang and J. Lepowsky, Operadic formulation of the notion of vertex operator algebra, in: *Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups, Proc. Joint Summer Research Conference, Mount Holyoke, 1992*, ed. P. Sally, M. Flato, J. Lepowsky, N. Reshetikhin and G. Zuckerman, Contemporary Math., Vol. 175, Amer. Math. Soc., Providence, 1994, 131-148.

[KSV] T. Kimura, J. D. Stasheff and A. A. Voronov, On operad structures of moduli spaces and string theory, *Comm. Math. Phys.* **171** (1995), 1–25.

[KVZ] T. Kimura, A. A. Voronov, and G. J. Zuckerman, Homotopy Gerstenhaber algebras and topological field theory, in: *Operads: Proceedings of Renaissance Conferences*, ed. J.-L. Loday, J. Stasheff, and A. A. Voronov, Contemporary Math., **202**, Amer. Math. Soc., Providence, 1997, 305–333.

[LZ1] B. H. Lian and G. J. Zuckerman, BRST cohomology and highest weight vectors. I, *Comm. Math. Phys.* **135** (1991), 547–580.

[LZ2] B. H. Lian and G. J. Zuckerman, New perspectives on the BRST-algebraic structure of string theory, *Comm. Math. Phys.* **154** (1993), 613–646.

[LZ3] B. H. Lian and G. J. Zuckerman, Moonshine cohomology, in: *Moonshine and vertex operator algebras*, RIMS Kokyuroku 904, RIMS, Kyoto, Japan, 1995, 87–115.

[S] G. Segal, Topology from the point of view of Q. F. T., Lectures at Yale University, March, 1993.

[V] A. A. Voronov, Homotopy Gerstenhaber algebras, in preparation.

[W] N.P. Warner, $N = 2$ supersymmetric integrable models and topological field theories, lectures given at the Summer School on High Energy Physics and Cosmology, Trieste, Italy, June 15th – July 3rd, 1992, `hep-th/9301088`, to appear.

Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019

E-mail address: yzhuang@math.rutgers.edu

Department of Mathematics, University of Chicago, Chicago, IL 60637

E-mail address: zhao@math.uchicago.edu