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Simultaneous decompositions of two states

Armin Uhlmann, Leipzig [1]

Dedicated to Roman S. Ingarden

Abstract

Simultaneous decompositions of a pair of states into pure ones are examined. There are privileged decompositions which are distinguished from all the other ones.

Presently we witness that *quantum information theory* is becoming an interdisciplinary, quickly growing field of research. In its history Roman S. Ingarden is playing a significant role, both by his own research and by posing stimulating questions and problems [1]. It is about 40 years ago that I met Roman the first time, and he was already thinking about the role of information in quantum physics and, in particular, whether one can found the concept of probability onto that of information [2].
I feel honored by the possibility to dedicate to him the following paper.

1 Decomposing one density operator

A density operator, representing a state, is a positive operator with trace one. However, it is convenient for the following considerations not to insist

[1]postal address: Institut für Theoretische Physik, Universität Leipzig, Augustusplatz 10, D-04109, Leipzig, Germany. email: armin.uhlmann@itp.uni-leipzig.de

in normalization.
We shall assume, mainly for technical simplicity, a finite dimensional Hilbert space, $\mathcal{H}$, the dimension of which is denoted by $\dim \mathcal{H} = d$. Thus, mathematically, we are just dealing with positive operators (and with the Null operator) of a finite dimensional Hilbert space.

Let τ be a positive operator on our Hilbert space. Its decreasingly ordered eigenvalues are denoted by $\lambda_1, \lambda_2, \ldots$, i. e.

$$\mathrm{spec}(\tau) = \{\lambda_1 \geq \lambda_2 \geq \ldots\}$$

By a *decomposition of* τ I denote every set of vectors $|\chi_j\rangle$ such that

$$\tau = \sum |\chi_j\rangle\langle\chi_j| \tag{1}$$

As I showed in [3]

$$\sum_{j=1}^{m} \lambda_j \geq \sum_{j=1}^{m} \langle\chi_j|\chi_j\rangle \tag{2}$$

is valid for all $1 \leq m \leq \dim \mathcal{H}$. Moreover, equality is reached if and only if $|\chi_j\rangle$ is an eigenvector for λ_j of τ for all $j = 1, \ldots, m$.
The motivation for asking questions of that kind has been the problem whether the von Neumann entropy of a density operator is already fixed by its position as a point in the convex set of all density operators. The result just quoted gives, if written with normalized vectors, an affirmative answer. Indeed, my aim was to define on every (compact) convex set a function which just gives the von Neumann entropy if applied to state spaces of a quantum system. Up to day I do not know whether the construction is of any use for other convex sets than quantum state spaces.

In [4] M. A. Nielsen proved the reversed statement: If p_j are positive numbers which are majorized by $\mathrm{spec}(\tau)$, then there exists a decomposition (1) such that $p_j = \langle\chi_j|\chi_j\rangle$.

The results mentioned above will be slightly extended to the case that there are two decompositions of one and the same τ. Thus let

$$\tau = \sum |\chi'_j\rangle\langle\chi'_j| \tag{3}$$

be a further decomposition of τ. Adding (1) and (3) we get a decomposition of 2τ and

$$\sum_{j=1}^{m} (\langle\chi_j|\chi_j\rangle + \langle\chi'_j|\chi'_j\rangle) \geq 2 \sum_{j=1}^{m} |\langle\chi_j|\chi'_j\rangle|$$

Equality takes place iff $|\chi_j\rangle$ differs from $|\chi'_j\rangle$ by a phase factor only. Because the eigenvalues of 2τ are just $2\lambda_j$ we get

Proposition 1: *Let (1) and (3) be decompositions of τ and $\lambda_1 \geq \lambda_2 \geq \ldots$ the decreasingly ordered eigenvalues of τ, and $1 \leq m \leq d$. Then*

$$\sum_{j=1}^{m} \lambda_j \geq \sum_{j=1}^{m} |\langle \chi_j | \chi'_j \rangle| \tag{4}$$

Equality holds if and only if for $1 \leq j \leq m$

$$\tau|\chi_j\rangle = \lambda_j|\chi_j\rangle, \quad |\chi'_j\rangle = \epsilon_j|\chi_j\rangle \tag{5}$$

with unimodular numbers ϵ_j.

2 Decomposing two density operators

Let us now consider a pair, ρ and ω, of positive operators.
Definition: *$F_m^+(\rho, \omega)$ denotes the sum of the m largest eigenvalues of*

$$(\sqrt{\rho}\omega\sqrt{\rho})^{1/2} \tag{6}$$

The definition works well for $1 \leq m \leq d$. It is sometimes convenient to extend it by $F_m^+ = F_d^+$ if $m \geq d$ and to set $F_m^+ = 0$ for $m = 0$.
Remark that F_d^+ is the square root of the transition probability [6]. The square root of the transition probability is called *fidelity* and is denoted by $F(\rho, \omega)$ in the present paper. Notice, however, that Jozsa, who showed its use in quantum information theory [5], identified the general transition probability with his fidelity concept (and not with its square root).
A further remark is the following: In [7] I considered another quantity: The k-fidelity, F_k, which is the sum of *all but the first* k eigenvalues of (6). These partial fidelities are jointly concave (and super-additive) in its arguments for $k = 0, 1, \ldots$ Obviously,

$$F_m^+(\rho, \omega) = F(\rho, \omega) - F_m(\rho, \omega)$$

In contrast to the partial fidelities, the quantity (8) seems to be neither concave nor convex if m is smaller than $\dim \mathcal{H}$.

Finally, let us rewrite (4) of proposition 1 as

$$F(\tau,\tau) \geq \sum_{j=1}^{m} |\langle \chi_j | \chi_j' \rangle|, \tag{7}$$

Remember that equality in (7) can be reached by eigenvector decompositions of τ with decreasingly ordered eigenvalues.

Theorem 1: *Let be* $1 \leq m \leq d$. *It is*

$$F_m^+(\rho,\omega) = \max \sum_{j=1}^{m} |\langle \psi_j | \varphi_j \rangle| \tag{8}$$

where the maximum is to perform over all possible decompositions

$$\rho = \sum |\psi_j\rangle\langle\psi_j|, \quad \omega = \sum |\varphi_j\rangle\langle\varphi_j|. \tag{9}$$

If the length of a decomposition is less than $\dim \mathcal{H}$, *or if the length of the two compositions (9) are different, one adds some zero vectors to get decompositions of equal and large enough length.*

The proof of the theorem starts by stating the invariance of the eigenvalues of (6) with respect of a transformation

$$\{\rho,\omega\} \;\Rightarrow\; \{\rho,\omega\}^X := \{X\rho X^*, (X^{-1})^*\omega X^{-1}\} \tag{10}$$

for any invertible operator X, see [7]. ((In the present paper the Hermitian adjoint of an operator A is denoted by A^* and *not* by $A^\dagger$.)) Hence the sum of the m largest eigenvalues of (6) cannot be changed by such a transformation. On the other hand, if we simultaneously transform decompositions (9) according to

$$|\psi_j\rangle \to X|\psi_j\rangle, \quad |\varphi_j\rangle \to (X^{-1})^*|\varphi\rangle \tag{11}$$

then the right hand side of (8) remains unchanged. Therefore, if the assertion of the theorem is true for a pair of density operators $\{\rho,\omega\}$, it is true for every pair $\{\rho,\omega\}^X$.

Let now ρ and ω be invertible (i. e. faithful). If we then can choose X such that

$$\{\rho,\omega\}^X := \{\tau,\tau\} \tag{12}$$

with a certain τ yet to be determined, we are done: For the pair $\{\tau, \tau\}$ the theorem is equivalent to proposition 1. But

$$X\omega X^* = (X^{-1})^* \rho X^{-1} := \tau \tag{13}$$

is valid if X^*X is the geometric mean [8] of ρ and ω^{-1}, i.e.

$$X^*X = \omega^{-1/2}(\omega^{1/2}\rho\omega^{1/2})^{1/2}\omega^{-1/2} \tag{14}$$

Hence, the theorem is true for invertible ρ and ω.
Indeed, the proof covers the case of any pair ρ, ω, with equal supports: To see it we only have to replace $\mathcal{H}$ by the supporting Hilbert subspace because neither to F_m^+ nor to the decompositions there is a non-zero contribution from the null spaces (i.e. the kernels) of ρ and ω.

We now prove that the right hand side of (8) never exceeds F_m^+. Denote by P_0, Q_0 the projection operators onto the null spaces of ρ and ω. We choose decompositions of P_0 and Q_0 with vectors $|\psi_i'\rangle$ and $|\varphi_i'\rangle$ respectively. We complement arbitrarily chosen decompositions (9) to those of $\rho' = \rho + c_1 P_0$ and $\omega' = \omega + c_2 Q_0$ with $c_j > 0$. For ρ' this is done by

$$\rho' = \rho + c_1 P_0 = \sum |\psi_j\rangle\langle\psi_j| + c_1 \sum |\psi_j'\rangle\langle\psi_j'|$$

and similarly we proceed with ω'. Because ρ' and ω' are invertible, we already can apply theorem 1 to them. Because $F_m^+(\rho', \omega')$ is approaching $F_m^+(\rho, \omega)$ if $c_j \to 0$ we are done.

What remains to show is the following: There are decompositions (9) such that $\sum |\langle\psi_i, \varphi_i\rangle|$ is equal to F_m^+, whatsoever the support properties of ρ and ω may be. To get this we first assert:
Let Q be the projection operator onto the supporting space of ω. For all decompositions (9) we get

$$\langle Q\psi_i|\varphi\rangle = \langle\psi_i|Q|\varphi\rangle = \langle\psi_i|\varphi\rangle$$

because every vector of a decomposition of ω must be an eigenvector of Q. That is, every one of the sums in question for ρ, ω gives one for $Q\rho Q, \omega$ yielding the same value. On the other hand, if we start with decompositions of $Q\rho Q, \omega$, we can add terms orthogonal to ω into the decomposition of ρ to get a decompositions of ρ, ω without changing the value of the sum. Below

we shall show the equality of $F_m^+(\rho,\omega)$ with $F_m^+(Q\rho Q,\omega)$, and, all together, we obtain: If and only if theorem 1 is true for the pair $Q\rho Q,\omega$ it is true for the pair ρ,ω. Now we can proceed as following: If the supports of $Q\rho Q$ and ω are equal, we are done. If not, we consider the projection operator P_1 onto the support of $Q\rho Q$, yielding the same statement for the pairs ρ,ω, $Q\rho Q,\omega$, and $Q\rho Q, P_1\omega P_1$. Either the last pair is of equal support, and we are done, or continue the same game with the projection operator Q_1 onto the support of $P_1\omega P_1$. This procedure must terminate after a finite number of steps yielding a pair with equal supports. The obvious reason: In every necessary step, the rank of one member of the pair under consideration is diminished, and we are in finite dimensions.
The proof of theorem 1 is done after showing the equality of $F_m^+(\rho,\omega)$ with $F_m^+(Q\rho Q,\omega)$ if Q is the support projection of ω. This assertion is a particular case with $X=Q$ of the equation

$$F_m^+(\rho, X^*\omega X) = F_m^+(X\rho X^*,\omega) \tag{15}$$

For invertible X the transformation (10) does not change the eigenvalues of (6). By the replacement $\omega \to X^*\omega X$ we thus get (15) for invertible X. But F_m^+ is continuous in its arguments, and (15) is valid for all X.

Let us underline the main point in constructing decompositions (9) satisfying

$$F_m^+(\rho,\omega) = \sum_{j=1}^{m} \langle\psi_j|\varphi_j\rangle, \quad m=1,2,\ldots \tag{16}$$

We have to solve (13) so that X and τ are at our disposal. From the spectral decomposition of τ,

$$\tau = \sum |\chi_j\rangle\langle\chi_j|, \quad \langle\chi_j|\chi_k\rangle = \lambda_j\delta_{jk} \tag{17}$$

we get an optimal decomposition satisfying (16) by

$$|\psi_j\rangle = X^{-1}|\chi_j\rangle, \quad |\varphi_j\rangle = X^*|\chi_j\rangle \tag{18}$$

Such a choice fulfills the bi-orthogonal relations

$$\langle\psi_k|\varphi_j\rangle = \langle\psi_j|\varphi_k\rangle = \lambda_j\delta_{jk} \tag{19}$$

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