Floppy modes (FMs) play a fundamental role in the mechanics and design of spring networks [1–7], particle packings and jamming [8–10], and mechanical metamaterials [11–18]. Both generic, disordered systems [1, 2] and periodic lattices [19–23] are well understood. A more complex situation arises when both order and disorder are present, such as in randomly diluted lattices. For example, randomly diluted triangular spring networks featuring perfectly aligned bonds which spawn excess floppy modes (EFMs) that disappear under generic perturbations [1 2 24]. Such EFMs escape generic counting methodologies [1, 2] but are important for mechanical metamaterials [15–25] [30].

Here we consider the floppy modes of collections of rigid quads connected by flexible hinges — a geometry which underlies a range of metamaterials [11–18] [31 32]. Each quad has three degrees of freedom (DOF), and in a fully connected lattice each quad has four connections in the bulk, and less near the boundary. A simple counting argument shows that \( M \times N \) lattices of generic quads are over-constrained when \( M \geq 3 \) and \( N \geq 3 \); however, symmetric systems of linked squares always exhibit an EFM where the squares can counter-rotate, irrespective of system size [1] [Fig. 1(a)]. What happens when such lattices are diluted, is not clear: How are the FMs in generic and symmetric systems related? What are the statistics and structure of the EFMs? Can we leverage symmetries to design systems with EFMs.

System and Methods.— We consider diluted \( N \times N \) lattices of quads connected by springs of unit stiffness and zero rest length — unless noted otherwise, we use open boundary conditions [Fig. 1(b)]. We first fix a connection topology by removing quads (or removing links - see below), and then systematically compare the FMs for systems consisting of squares of side length 1 to stress-free generic systems obtained by randomly displacing the corners of linked squares with magnitude \( \epsilon = 0.1 \) [33]. We then, for each connection topology, determine the number of floppy modes of generic (\( n_g \)) and symmetric (\( n_s \)) systems, excluding the three global translational/rotational modes (for details, see S.I.).

Random Dilution.— We first study \( n_g \) and \( n_s \) for randomly diluted topologies with \( (1 - \rho)N^2 \) removed quads, and focus on the ensemble averaged number of EFMs, \( \langle \Delta \rangle := \langle n_s - n_g \rangle \) [Fig. 2(a)]. We find that for large filling fractions, \( n_s = 1 \) (counter rotating EFM) and \( n_g = 0 \), so that \( \langle \Delta \rangle \) is 1; consistent with the fully filled case discussed above. In contrast, for small filling fractions, we find that \( \langle \Delta \rangle \to 0 \) — the dilute limit corresponds to disconnected quads, for which \( n_s = n_g \). Strikingly, \( \langle \Delta \rangle \) exhibits a maximum \( \langle \Delta \rangle^* \gg 1 \) at intermediate filling fractions, evidencing the emergence of multiple EFMs. Intuitively, this can be grasped by considering a topology consisting of \( m \) disconnected clusters consisting of fully filled \( 3 \times 3 \) blocks. Each symmetric cluster has an internal hinging EFM, yielding \( n_s = 4m - 3 \), \( n_g = 3m - 3 \) and \( \Delta = m \). Clearly, much more intricate topologies also arise; nevertheless, we will show that the internal hinging motions in clusters of squares are crucial to understand EFMs.
Scaling. — The number of EFMs, \( \langle \Delta \rangle \), follows mean-field scaling. First, we find that its maximum, \( \langle \Delta \rangle^* \), grows linearly with \( N^2 \), which implies that the peak density of EFMs, \( \beta := \langle \Delta \rangle^*/N^2 \), is a constant [inset Fig. 2(a)]. Second, we can collapse our data for random dilution as \( \langle \Delta \rangle/N^2 = f(\rho) \), and find that the peak location approaches a constant as \( \rho^* = \rho_0 + \alpha/N \) with \( \rho_0 \approx 0.69 \), \( \alpha \approx 1.00 \) [Fig. 2(b)]. Third, we found that the distribution of \( \Delta \) at fixed \( \rho \) is Gaussian. Finally, we have also studied random bond removal, i.e. removal of individual links, and find a very similar scaling collapse and peak location (see S.I.). All this strongly suggests that EFMs occur with a constant density and finite correlation length.

Peak density. — We have studied the variation of the peak EFM density, \( \beta \) as function of dilution protocol. First, while for for random quad dilution \( \beta \approx 0.01 \), for random bond removal we find \( \beta \approx 0.007 \) (see S.I.). Second, we have explored the design of specific topologies that maximize \( \beta \), and have designed geometries where \( \beta \) can be as large as 0.06 (see S.I).

Clusters and Counting. — We now construct a counting argument to determine the number of EFMs for a given geometry. We focus our attention on \( n_s \), as \( n_g \) can be exactly determined by, e.g., the pebble-game \( \{1,2\} \). To estimate \( n_s \), we partition the system into clusters, where each cluster has an internal (hinging) EFM and three translational/rotational floppy modes. We then consider the connectors between such clusters to estimate \( n_s \). We will show that, despite several simplifying assumptions, counting clusters and connectors is sufficient to closely capture the numerically observed features of \( n_s \) and \( \Delta \).

As a first step, we partition a given topology in clusters and "remaining quads". We start by identifying each 4-block, defined as four quads connected in a loop — each 4-block corresponds to a node on a dual grid [Fig. 2(a)]. Two 4-blocks are defined to be adjacent if they correspond to adjacent nodes on the dual grid (not along diagonals), and we define clusters as groups of adjacent 4-blocks \([4]\). This procedure yields that each quad belongs to either zero, one, or two clusters; quads that do not belong to any cluster are either a type-1 connector (connected by two links to a cluster), or a remaining quad. Connections between clusters come in three variants [Fig. 3(b)]. First, a type-1 connection consists of a single quad and introduces one constraint. Second, a type-2 connection occurs when two clusters are adjacent and are connected though a bond which introduces two constraints. Third, a type-3 connection occurs when two clusters share a quad and introduces three constraints.

The identification of clusters, connectors and remaining quads is illustrated in Fig. 3(c). We note that this partition often yields a significant number of loosely connected remaining quads which, as we will verify below, do not significantly contribute to \( \Delta \), as they almost equally contribute to \( n_s \) and \( n_g \). We therefore remove these quads and consider pruned systems that solely consist of clusters and connectors \([5, 6]\); we refer to the quantities in the pruned systems by accents (e.g., \( \Delta' = n_s' - n_g' \)) [Fig. 3(d)].

To estimate \( n_s \), let us first consider the DOF of two connected clusters \( i \) and \( j \). We define the connection number \( C_{ij} := n_{ij}^1 + 2n_{ij}^2 + 3n_{ij}^3 \), where \( n_{ij}^k \) is the number of type-\( k \) connectors between clusters \( i \) and \( j \). Without connections \( (C_{ij} = 0) \), such a system has six rotational/translational and two internal hinging DOF. When \( C_{ij} \leq 4 \), the number of DOF is expected to be
8 − C_{ij}, but when C_{ij} ≥ 5, the system will always feature four degrees of freedom: three rotational/translational and one internal hinging DOF — symmetries thus cause degeneracies between strongly connected clusters, which must lead to extra states of self stress rather than constraining their DOF. Hence, when C_{ij} ≥ 4, these two clusters can be seen as one larger cluster, with three rotational/translational and one internal hinging DOF.

We now introduce a sequence of increasingly accurate predictions \( \Delta_0, \Delta_1, \ldots, \Delta_\infty \) for \( \Delta' \) of the pruned system. We count the number of clusters \( N_c \), and define \( \Delta_0 := 4N_c - \sum C_{ij}/2 - 3 - n'_2 \). We then merge pairs of clusters for which \( C_{ij} ≥ 4 \), determine \( N_c^1, C_{ij}^1 \), and define \( \Delta_1 := N_c^1 - \sum C_{ij}^1/2 - 3 - n'_2 \). As cluster merging may yield new pairs of clusters \( p \) and \( q \) for which \( C_{pq} ≥ 4 \), this procedure must be iterated, leading to \( \Delta_2, \Delta_3, \ldots \). Eventually, no cluster merging is possible when all \( C_{ij}^n < 4 \) [Fig. 4(e)], and the final result for \( \Delta_n \) is denoted as \( \Delta_\infty \).

Results.— We now show that the predictions \( \Delta_0 \) increasingly accurately capture \( \Delta' \), which in turn is close to \( \Delta \). Consider the results for \( \Delta_0, \Delta_1, \Delta_2, \Delta_\infty \) and \( \Delta' \) shown in Fig. 4(a). While \( \Delta_0 \) qualitatively captures the peak in the number of excess FMs, it fails to accurately describe the high density regimes. Cluster merging significantly suppress these discrepancies, and \( \Delta_\infty \) is within a few percent of \( \Delta' \) [Fig. 4(a)]. Moreover, for large \( N \), \( \Delta_\infty \) displays a scaling collapse closely matching that of \( \Delta \) [Fig. 4(b)]. Finally, when plotted together, \( \Delta_\infty, \Delta' \) and \( \Delta \) are very close [Fig. 4(c)]. We have checked that this agreement is similarly good for other system sizes.

We briefly discuss the minor differences between \( \Delta, \Delta' \) and \( \Delta_\infty \). First, our counting method yields a strict lower bound on \( n'_2 \) and thus on \( \Delta' \); \( \Delta_\infty ≤ \Delta' \). Cases where \( \Delta_\infty < \Delta' \) are rare, and occur when degeneracies between the constraints yield additional states of self stress and concomitant EFMs [see Fig. 4(d) and SI]. Second, \( \Delta' \) does not provide a strict bound on \( \Delta \), as (rare) cases with both \( \Delta < \Delta' \) and \( \Delta' < \Delta \) can be constructed (see SI). Finally, we have observed that \( \langle \Delta \rangle < \langle \Delta' \rangle \). We conclude that EFMs arise due to local hinging motions in weakly coupled clusters, and that their multiplicity can be accurately estimated by our cluster-merging methodology.

Multi-branch planar mechanisms.— We now leverage the symmetries in systems of hinging squares to design geometries with one continuous degree of freedom but multiple discrete branches of motion. Specifically, the deformations of such systems can be characterized by one continuous degree of freedom, and one discrete degree of freedom, where the latter labels the specific branch of motion. Multi-branch mechanism occur in, e.g., Origami [25, 60], but we are not aware of any 2D mechanisms that allow multi-branch behavior without self-intersections. Here we show that one can connect clusters of hinging squares such that their magnitude of hinging is coupled, while their sign of the hinging motion can be chosen independently.

To understand the design of such multi-branched mechanism, we first consider the hinging motion of a single cluster. We define the opening angle \( \psi := (\theta − π/2)/2 \), color the squares according to their alternating rotating motion, and note that we can distinguish four link types, associated with connections at the north, east, south or west tip of a dark square [Fig. 5(a)]. The crucial observation is that the distance between two links is invariant under opening angle inversion \( (\psi \leftrightarrow -\psi) \) if and only if both links are of equal type [Fig. 5(a)]. Specifically, if we consider links that are vertically aligned, such as \( A, B \) and \( C \) in Fig. 5(a), we find that \( AB \) is symmetric in \( \psi \), while \( AC \) is not [Fig. 5(b)].

We obtain multi-branched mechanisms by connecting clusters by type-2 connectors located at hinges \( A \) and \( B \), whose distance is symmetric in the opening angle. We define \( \psi_1, \psi_2, \ldots \) such that for a global hinging mode all opening angles are equal. The symmetry of \( AB(\psi) \) then implies that, e.g., the clusters 1 and 2 can be in a "homogeneous" state where \( \psi_1 = \psi_2 \), and in second state, where \( \psi_1 = -\psi_2 \); we note that configurations \( \{\psi_i\} \) and \( \{-\psi_i\} \) are related by the global
FIG. 5. (color online) (a) A cluster with positive opening angle $\psi := (\pi - \pi/2)/2$. The colored dots indicate the four types of hinges. (b) The distance $AB$ (full) is symmetric in $\psi$ as $A$ and $B$ correspond to the same hinge type ($AB = 2\sqrt{2}\cos(\psi)$) while $AC$ (dashed) is not invariant under $\psi \leftrightarrow -\psi$, as $A$ and $C$ correspond to different hinge types ($AC = AB + 2\sin(\psi + \pi/4)$). (c) Clusters connected by type-2 connectors at points of the same hinge type can either have equal or opposite $\psi$, the four branches of motion are characterized by the freely assignable signs of $\psi_2$ and $\psi_3$. (d) Compound cluster structure consisting of four identical unit cells that exhibits three branches of motion - see text and S.I.

More complex situations where the number of branches is different from a power of two arise when clusters are connected in more complex topologies. We designed a cluster which features the single and double-bump edge structures along both its vertical and horizontal edges, and connect it into a $2 \times 2$ lattice [Fig. 5(d)]. As the (horizontal) connections between cluster 1 and 2, respectively 3 and 4, are of a different link type than the (vertical) connections between cluster 1 and 3 respectively 2 and 4, the sign of $\psi_1$ has to be consistent along either rows or columns. Fixing $\psi_1 > 0$, we obtain two consistent row configurations with $\{s_i\} = \{0,0,0,0\}$ and $\{0,0,1,1\}$; similarly, consistent column configurations are $\{s_i\} = \{0,0,0,0\}$ and $\{0,1,0,1\}$. It is easy to show that these are all allowed possibilities, and as we double count the homogeneous configuration, this example constitutes a three-branch mechanism. We note that the table of allowed branches $s_2, s_3$ is equivalent to the truth table of a NAND gate. Extending this design to $m \times n$ tilings, we obtain $2^{m-1}$ different column arrangements, $2^{n-1}$ different row arrangements yielding $2^{m-1} + 2^{n-1} - 1$ branches. For small $m \leq 3$ and $n \leq 3$ this already allows to make mechanisms with 1, 2, 3, 4, 5, 7, 8, 9, 11 and 15 branches. It is an intriguing open question whether we can design compounds with any integer number of branches, or any desired truth table for a number of branch identifiers $s_i$.

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[33] This perturbation is sufficiently large so that in generic systems the eigenvalues of EFMs become large and numerically clearly separated from those of FMs (see SI).
[34] We note that there are other isolated groups of quads with $n_s = 4, n_g = 3$, such as a loop of thickness two — these are rare and excluding these from our cluster definition is a minor approximation.
[35] The only remaining quads that easily can be detected and understood to have the potential to change $\Delta$, are self-connectors — isolated quads that are connected to the same cluster. These never modify $n_s$, but they may rigidify an otherwise hinging generic group of quads, and we have therefore kept these self-connectors in our pruned systems - for an example, see SI.