Solvable Models of Supercooled Liquids at the Avoided Mode-Coupling-Theory Transition

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Mode-Coupling Theory (MCT) provides an accurate quantitative description of many supercooled liquids models in the early stages of dynamical slowing down. In realistic non-mean-field models the description become incorrect close to the MCT singularity that is avoided and transformed into a crossover and the main open problem is to amend MCT and extend the description quantitatively to the crossover region and beyond. We consider a family of models in three dimensions obtained by the application of the $M$-layer construction to Fredrickson-Andersen Kinetically-Constrained-Models. We argue that deviations from mean-field MCT predictions in these models are described by the Stochastic-Beta-Relaxation (SBR) equations and compute the free parameters of the theory. By means of numerical simulations we show that SBR predictions are indeed accurate, providing the first instance of a model that can be described quantitatively at the glass crossover without any fitting parameter.

I. INTRODUCTION

Mode-Coupling Theory captures many features of the physics of liquids upon supercooling, notably two-step relaxation and stretched exponential decay [1]. On the other hand it predicts a dynamical arrest transition, characterized by power-law divergences, at a temperature where one observes instead a crossover from power-law to exponential increase of the relaxation time. The fundamental problem of fixing somehow the theory and develop a theory of supercooled liquid valid down to the glass temperature is thus open. It is generically believed that the spurious transition predicted by MCT is destroyed by fluctuations and the technical problem is how to take into account them.

There have been many unsatisfactory attempts to incorporate fluctuations at the microscopic level, while recently considerable progress has been made starting from the assumption that fluctuations are described by an effective theory. More precisely the assumption should be valid in the $\beta$ regime where the correlation remains close to a plateau value. Once this assumption is accepted the choice of the effective theory is dictated by the symmetries of the problem in much the same way as one derives the Landau effective theory for ferromagnetism and the dynamical effective theory corresponding to MCT can indeed be identified [2]. Nevertheless in general studying effective theories beyond the mean-field approximation is very difficult in general, but it turns out that the effective theory associated to MCT is equivalent to an intuitive dynamical model called Stochastic-Beta-Relaxation that can be thoroughly studied beyond the mean-field approximation. These results are promising because the SBR seems to cure the drawbacks of MCT, displaying in particular the power-law to exponential crossover and dynamical heterogeneities [3–5] without spoiling the successes of MCT, in particular two-step relaxations and stretched exponentials.

In order to determine if an actual supercooled liquid can be described by SBR one should assess the validity of the assumption that fluctuations in the $\beta$ regime are small and can be described by an effective theory. Generically one expects the assumption to be valid provided the dynamical correlation length becomes sufficiently large. Putting these considerations in a precise quantitative form is an open problem and will not be further considered here. In the following instead we will consider models where the validity of an effective theory description is granted by construction. Furthermore we will compute independently the quantitative parameters of SBR and we will show that SBR provides an accurate qualitative and quantitative parameter-free description of the dynamical behavior of these models as obtained from Monte-Carlo numerical simulations.

A classic strategy to build these kind of models is the one of Kac for the Ising model: the original model with nearest-neighbor interactions is generalized in such a way that each spin interacts weakly with all spins at a distance smaller than some parameter $\gamma$. In the limit $\gamma \to \infty$ the model is equivalent to the solvable fully-connected Curie-Weiss mean-field model but for finite, albeit large, values of $\gamma$ the critical exponents of its ferromagnetic transition are non-mean-field. The interesting point is that deviations from MF at large but finite $\gamma$ are described by an effective theory that is easily identified as the $\phi^4$ field theory. The Kac construction in the context of spin-glass models has been explored in [6–8], while alternative strategies have been studied in [9–11] for supercooled liquid models and in
FIG. 1: Left-Bottom: the Diamond Cubic lattice. Right-Top: the three steps of the \( M \)-layer construction illustrated on the regular two-dimensional lattice, from left to right: the original lattice \((M = 1)\), the replicated lattice \((M = 3)\), the rewiring process on a given lattice edge, to be repeated on all edges.

\[12\] for Kinetically-Constrained-Models (KCM). In particular the setting of Ref. \[12\] is closely related to ours, and it is likely that quantitative SBR predictions could be also made in that context. In the following we will consider instead the \( M \)-layer construction, recently introduced in \[13\], that can be easily applied to spin models defined on lattices. The construction is easily illustrated: we start from a lattice of connectivity \( c \), e.g. the square lattice in \( d = 2 \), we replicate it \( M \) times, see Fig. (1) and then we rewire randomly the \( M \) links corresponding to the same link on the original \((M = 1)\) lattice, repeating the procedure for each link of the original lattice. In the limit \( M \to \infty \) the lattice becomes identical to a Bethe lattice with connectivity \( c \) while for \( M \) large but finite the Bethe solution is accurate except close to the Bethe critical point where deviations from mean-field behavior occur. The key point is that these deviations, for sufficiently large but finite values of \( M \), are accurately described by an effective theory. In the next section we will outline the results with reference to the other sections for the details.

II. OUTLINE OF THE RESULTS

We have studied the Fredrickson-Anderson (FA) kinetically constrained model (KCM) \[14, 15\] on the three-dimensional diamond lattice \((c = 4)\) and applied the \( M \)-layer construction to it, see fig. (1). The model is made of independent Ising spins on each site \( H = \sum_i s_i \) that obey a kinetically constrained dynamics: a spin can flip only if it has at least \( m = 2 \) neighbors in the excited (up) state. The \((c = 4, m = 2)\) model can be also implemented on the two-dimensional regular lattice but we choose the diamond lattice to discuss the three-dimensional case.

The Mean-Field approximation to the problem is given by the Bethe lattice solution. The FA model on the Bethe lattice is known to exhibit a dynamical arrest transition at a temperature such that the equilibrium probability \( p \) of a spin to be down is \( p_c = 8/9 \) \[16–21\]. Starting from an equilibrium initial condition the basic observable we consider is the persistence. More precisely the local persistence \( p_i(t) \) is equal to one if the spin was negative at all times smaller than \( t \) and zero otherwise, thus the averaged persistence is the number of negative sites that have never flipped at time \( t \) divided by the total number of spins.

At the critical point \( p_c = 8/9 \) the persistence remains blocked to a plateau value \( p_{plat} \) that can be computed from the solution of bootstrap percolation on the Bethe lattice as given in appendix \[C\]. The persistence \( p(t) \) should not be confused with the probability of negative sites that is fixed to \( p = p_c \). In particular we have:

\[
g(t) \equiv p(t) - p_{plat},
\]

with

\[
g(t) \approx \frac{1}{(t/t_0)\alpha}, \quad p = p_c, \quad t \gg 1,
\]

\[
p_c = \frac{8}{9}, \quad p_{plat} = \frac{21}{32}.
\]
The quantities \( t_0 \) and \( a \) must be estimated numerically from the solution on the Bethe lattice, see appendix [3].

In the \( M \)-layer construction we put \( M \) spins \( s^i_\alpha \), \( \alpha = 1, \ldots, M \) for each site \( i = 1, \ldots, n \) of the original lattice, see fig. [1] and thus the natural local order parameter is the average over layers of the local persistence minus the plateau value

\[
g^\alpha_i(t) \equiv p^\alpha_i(t) - p_{plat} \tag{4}
\]

\[
g(x,t) \equiv \frac{1}{M} \sum_{\alpha=1}^M g^\alpha_i(t), \tag{5}
\]

where \( x \) is the spatial coordinate of site \( i \). For \( M = \infty \) the system displays the Bethe lattice behavior while for large but finite \( M \) we argue that all generic \( K \)-point average obey:

\[
\langle g(x_1, t_1) \cdots g(x_K, t_K) \rangle = [g(x_1, t_1) \cdots g(x_K, t_K)], \quad 1 \ll M < \infty \tag{6}
\]

where in the RHS \( g(x,t) \) is the solution of of the SBR equations [2, 3]:

\[
\sigma + s(x) = -\alpha \nabla^2 g(x,t) - \lambda g^2(x,t) + \frac{d}{dt} \int_0^t g(x,t-s) g(x,s) ds . \tag{7}
\]

The square brackets mean average with respect to the field \( s(x) \) that is a time-independent random fluctuation of the separation parameter, Gaussian and delta-correlated in space:

\[
[s(x)] = 0, \quad [s(x)s(y)] = \Delta \sigma^2 \delta(x-y) . \tag{8}
\]

The equations have to be solved with the small-time condition

\[
\lim_{t \to 0} g(x,t)(t/t_0)^a = 1 . \tag{9}
\]

where \( \lambda \) and \( a \) are related by the MCT relationship \( \lambda = \Gamma^2(1-a)/\Gamma(1-2a) \). Note that we are using the same notation \( g(x,t) \) for the observable on the \( M \)-layer model and for the SBR equations, the difference can be told from the notation representing averages: angled brackets for the \( M \)-layer and squared brackets for SBR. Also note that the \( M \)-layer \( g(x,t) \) is non-zero only on the points of the lattice while the SBR \( g(x,t) \) is defined on the continuum therefore equation (6) is intended to hold only on lattice points. The fact that a lattice theory becomes equivalent to a continuum theory is a consequence that, as we will see in the following, eq. (6) holds on lengthscales considerably larger than the lattice spacing.

Withing SBR, mean-field theory is recovered setting \( \Delta \sigma^2 = 0 \), in this case \( g(x,t) \) is constant in space, the gradient term plays no role and one recovers the critical MCT equation, in particular \( p = p_c \) corresponds to \( \sigma = 0 \) and \( p(t) \) will never reach the plateau value. The \( M \)-layer construction allows to have a finite but small \( \Delta \sigma \) so that the MCT transition is avoided and \( p(t) \) will cross the plateau value at a finite time for all values of \( \sigma \), including \( \sigma = 0 \).

Note that we started from a model with a complex microscopic dynamics and ended up with a (numerically) solvable set of equations depending on few (five) parameters. The numerical values of the SBR parameters depend on the microscopic details of the lattice and can be computed independently of the numerical data on the \( M \)-layer. The three parameters \( \alpha, \Delta \sigma \) and \( \sigma \) can be computed from the properties of the lattice to which we apply the \( M \)-layer construction and from the solution of Bootstrap percolation on the Bethe lattice with connectivity four while the remaining dynamical parameters \( a \) and \( t_0 \) can be estimated from numerical simulations of the FA model on the Bethe lattice. In section [IV] we will show that for \( (c = 4, m = 2) \) we have:

\[
\Delta \sigma^2 = \frac{1}{M \rho} \frac{2187}{8192} (1-\lambda)^2 \tag{10}
\]

\[
\alpha = (1-\lambda)\frac{9}{8} D_{NBW} \tag{11}
\]

\[
\sigma = (1-\lambda)\frac{729}{512} (p-p_c) \tag{12}
\]

The microscopic properties of the original lattice \( (M = 1) \) enter through the density of lattice points \( \rho \) and through the diffusion coefficient on non-backtrackting random walks \( D_{NBW} \). For the diamond lattice we have \( \rho = 1/8 \) and
$D_{NBW} = 1$ as can be obtained from a general formula derived in appendix [14]. As we said already, the remaining two parameters $a$ and $t_0$ have to be estimated from the numerical solution of the FA model on the Bethe lattice and we have (see appendix [13]):

$$a \approx 0.352, \quad t_0 \approx 2.30.$$  \hspace{1cm} (13)

Although we have chosen the three-dimensional diamond lattice the previous relationships are valid for a generic lattice of connectivity four where $p_c = 8/9$ and can be used to repeat the numerical analysis on many other lattices, including notably the regular two-dimensional lattice where $\rho = 1$ and $D_{NBW} = 1/2$.

In the following we shall discuss the arguments leading to eq. (3). In the limit $M \to \infty$, the lattice is locally tree-like, the original three-dimensional structure of the lattice becomes irrelevant and the system is equivalent to the Bethe lattice: at $p = p_c$, dynamics it displays the spurious MCT mean-field dynamical arrest transition, $g(t)$ never becomes negative and approaches zero with a $1/t^\alpha$ power law. For $M$ large but finite the system recovers its finite-dimensional character: it deviates from mean-field theory and the dynamical arrest transition is avoided. Let us see the nature of the deviations from MF theory. One can easily see that on the $M$-layer the two-point correlation of the order parameter at any distance are small ($O(1/M)$) for $M$ sufficiently large at any finite time. They can be expressed as a sum over non-backtracking random walk on the lattice of the correlations computed on the Bethe lattice. In section [IV A] we will show that (neglecting constants that are irrelevant for the present discussion) fluctuations in Fourier space at large times obey:

$$\langle g(k,t)g(k',t) \rangle - \langle g(k,t) \rangle \langle g(k',t) \rangle = \frac{t^{2a}}{M} \delta(k+k') \left( \frac{1}{1+(k\xi)^2} \right)^2, \quad \xi \propto t^{\alpha/2}, \quad t \gg 1$$  \hspace{1cm} (14)

Thus at the MF level the total susceptibility diverges with time as $t^{2a}$ and the correlation length diverges as $t^{\alpha/2}$. Both these behaviors will change when the system starts to deviate from MF: the correlation length will not increase indefinitely and $g(x,t)$ will become negative at a finite time. We note en passant that MF fluctuations do not obey the Ornstein-Zernicke form but rather the square of it. The use of OZ forms to fit numerical data of dynamical fluctuations is widespread in the literature [22–25] but, at present, has no theoretical justification. Be as it may, both MF theory and SBR certainly do not support OZ in the $\beta$ regime, while the $\alpha$ regime predictions are not available at present even in MF theory. In real space the above expression leads to:

$$\langle g(x,t)g(y,t) \rangle - \langle g(x,t) \rangle \langle g(y,t) \rangle = \frac{t^{2a-d/2}}{M} f \left( \frac{x-y}{\xi} \right).$$  \hspace{1cm} (15)

Since the prefactor of the $1/M$ term diverges at large times we can apply a Ginzburg criterion and identify the time where deviations from MF theory must occur with the time where MF fluctuations of $g(x,t)$ on the scale of the correlation length become comparable with its MF average squared, this leads to:

$$\frac{t^{2a-d/2}}{M} \approx \frac{1}{t^{2a}} \to t \approx M^{1/(2-d/2)}$$  \hspace{1cm} (16)

Thus the system starts to deviate from MF at large times $O(M^{1/(2-d/2)})$ so that $g(x,t)$ has already decreased considerably and is very small ($O(M^{-1/(d-2)})$) while the correlation length ($\xi = t^{\alpha/2}$ in MF theory) has increased substantially and is rather large ($O(M^{1/(d-2)})$). The key point is that small order parameter and large (but not necessarily diverging) correlation length define exactly the regime in which an effective theory is expected to be appropriate. Let us recall why: in general the probability of these deviations is described by a complicated action $P(g) \propto e^{A(g)}$, but if the deviations are small we can safely retain only the lowest order terms in the Taylor expansion of $A(g)$ in powers of $g$. Furthermore, if the correlation length is large only the lowest-order space derivatives of the order parameter are important in a coarse-grained description. Thus one concludes that $A(g)$ can be approximated with an effective theory. Typically the precise form of the effective theory can be guessed from symmetry considerations while the microscopic details of the original models determines only the actual value of the coupling constants.

In [2 3] it is argued, starting from an integral representation of dynamics, that the effective theory of MCT is the so-called Glassy-Critical-Theory (GCT) whose form is suggested by the replica-dynamics analogy in the context of the Random-First-Order-Theory. However, identifying the correct effective theory does not at all solve the problem: we claim that deviations are described by an effective theory but we still have to compute them. For example, one thing is to say that deviations from MF in a ferromagnetic model can be described by the $\phi^4$ theory, another thing is to extract the non trivial critical exponents from it. In the case of the GCT the problem is greatly simplified by the fact that the GCT can be shown rigorously to be equivalent to the SBR equations [2]. The great technical advantage
is that the unfeasible computation of the integral of the action is replaced by the average over instances of the random $h(x)$ that can be done numerically.

We should note that the use of the GCT and SBR to describe the FA models on $M$-layer lattices cannot at present be completely justified because we lack an analytic treatment of the dynamics starting from the microscopic. It is not even possible to show analytically that the FA model has a MCT transition on the Bethe lattice. On the other hand there are strong numerical evidences that this is the case [10,21] and further numerical evidences will be presented in appendix [B]. Another important piece of evidence comes from the fact that the glassy phase of FA models (that only exists within MF theory) is closely related to bootstrap percolation. Indeed it has been recently shown [26] that BP on the $M$-layer obeys the static version of the SBR equations [32]. In section IV we will use the mapping to bootstrap percolation to determine the SBR parameters.

We note that the $M$-layer construction, much as the Kac construction, is precisely tailored to make the assumptions of small deviations and large (but not necessarily diverging) correlation length more and more accurate, but they could as well be satisfied in actual supercooled liquid.

In appendix [A] we will show that the solution of SBR corresponding to generic values of $\alpha$, $\Delta\sigma$ and $t_0$ can be expressed in terms of the solution with $\alpha = \Delta\sigma = t_0 = 1$ through appropriate rescalings of the correlators ($g \rightarrow b_\alpha g$), of distances ($x \rightarrow b_x x$) and of times ($t \rightarrow b_t t$). From the corresponding formulas we get that the $M$-dependence of the rescaling factors is the following:

$$\Delta\sigma^2 = O(1/M) \rightarrow b_\phi = O(M^{-\frac{4}{\pi - 2}}), \quad b_x = O(M^{\frac{4}{\pi - 2}}), \quad b_t = O(M^{\frac{4}{\pi - 2} - \frac{4}{\pi}}),$$

and we thus recover the scalings obtained above from the Ginzburg criterion.

In the following we will discuss numerical simulations at $p = p_c$ corresponding to $\sigma = 0$ but we want to stress that SBR holds also for $p \neq p_c$. More precisely all the features of the crossover predicted by SBR as the separation parameter goes from $-\infty$ to $+\infty$ should be observed in a region close to $p_c$ that shrinks with increasing $M$ as

$$|p - p_c| = O(M^{-\frac{4}{\pi - 2}}).$$

These include notably the non-monotonic behavior of the dynamical correlation length and the qualitative change in the nature of fluctuations with the appearance of strong dynamical heterogeneities [5].

We now turn to the comparison of numerical data and SBR predictions. All simulations were performed on diamond cubic lattices with periodic boundary conditions at $p = p_c$. The lattice is generated by repeating in the three directions a basic unit cell of length $L = 4$. In each unit cell there are eight lattice points that can be divided in two groups: blue lattice points have coordinates $(0,0,0)$, $(2,0,2)$, $(0,2,2)$ and $(2,2,0)$, red lattice points have coordinates $(3,3,3)$, $(3,1,1)$, $(1,3,1)$ and $(1,1,3)$. Each red (blue) lattice point is connected with its four blue (red) nearest neighbors, see fig. [1]. All sites in the same group are updated in parallel with a Metropolis rule, further details on the dynamics will be given in appendix [B]. In fig. [2] we plot the time decay of the persistence for a system with $M = 3000$ and for one with $M = 200000$. At initial times the data follows the MF curve corresponding to $M = \infty$ (obtained from numerical simulations on the Bethe lattice (appendix [B]) and deviates from it at large times reaching the plateau value at a
finite time. As expected deviations from MF occur at times increasing with $M$. The dashed line are the prediction of SBR solved numerically with the above values of the five parameters. For $M = 3000$ SBR describe the bulk of the correction and the error is due to the fact that for $M = 3000$ the system departs from the MF curve at times where the small time $1/t^a$ corrections to the asymptotic behavior $1/t^a$ are still significant (see discussion in appendix B). The effect is expected to disappear increasing $M$ and indeed the agreement between numerical simulations and theory is very good for $M = 200000$, we recall that there is no fitting parameter in the SBR predictions.

SBR provides a descriptions of all fluctuations, to illustrate this we consider the persistence density

$$g(t) = \frac{1}{ML^3\rho} \sum_{i,\alpha} g_{\alpha}^i(t)$$

whose fluctuations define the $\chi_4(t)$ function:

$$\chi_4(t) \equiv M L^3 \rho \left( \langle g^2(t) \rangle - \langle g(t) \rangle^2 \right).$$

In the MF regime corresponding to $1 \ll t \ll O(M^{1/(4-d/2)})$ the total susceptibility is the same that one measures on the Bethe lattice (see section IV A) and diverges at large times as:

$$\chi_4(t) \approx \frac{2187}{8192} (1 - \lambda)^2 A_1^2 \left( t/t_0 \right)^{2a}$$

where $A_1$ is the following constant:

$$A_1 \equiv \frac{1}{2(a \pi \csc(a \pi) - \lambda)}.$$

Thus in the MF regime $\chi_4(t)$ diverges as the inverse of $g^2(t)$

$$\chi_4(g) \approx \frac{2187}{8192} (1 - \lambda)^2 A_1^2 \frac{1}{g^2}$$

On the time scales where $g$ deviates from MF and reaches zero $\chi_4(g)$ will deviate from the MF law and remain finite. In fig. 3 we plot parametrically $\chi_4(t)$ vs. $g(t)$, the agreement between numerical data (points) and the solution of SBR (solid line) is again very good. SBR by construction follows the MF asymptote eq. (23) at small times (large values of $g$) while numerical data follows it at intermediate times corresponding to the MF regime $1 \ll t \ll O(M^{1/(4-d/2)})$ but, as expected, deviates from it at microscopic times $t \approx 1$. In order to compare predictions for dynamical fluctuations in space we define

$$\Gamma(x - y, t) \equiv \langle g(x, t)g(y, t) \rangle - \langle g(x, t) \rangle \langle g(y, t) \rangle.$$
In fig. (4) we plot $\Gamma(0,t)$ (the local order parameter squared) and $\Gamma_{av}$ defined as the spatial average of $\Gamma(x,t)$:

$$\Gamma_{av}(t) = \frac{1}{V} \int \Gamma(x,t) dx .$$

Note that $\Gamma_{av}(t)$ is equal to $\chi_4(t)/(ML^3\rho)$. We see that the SBR predictions for both quantities (solid lines) are in good agreement with the numerical data (points) in the regime where deviations from MF are observed.

Note that within SBR $\Gamma(x,t)$ takes its maximum value at $x = 0$ and decreases to zero at infinity, therefore $\Gamma(0,t)$ is always strictly larger than $\Gamma_{av}$. In the thermodynamic limit $\Gamma_{av}$ actually should be proportional to the inverse of the volume, while $\Gamma(0,t)$ remains finite. Note instead that in fig. (4) at large times (small $g(t)$) the two quantities are comparable. This happens because the correlation length is comparable with the system size which is $L = 8$ for the data with $M = 3000$ shown in the previous figures. In this regime the SBR curves depend on the size of the box and to be consistent we had to solve them in a finite cubic box of the same size $L = 8$. This demonstrate that SBR is also able to capture quantitatively finite-size effects when they are expected.

Note that according to the above discussion in the MF regime the correlation length grows with time as $t^{\alpha/2}$ while the time where SBR replaces MF theory depends on $M$. If the correlation length become comparable with the system size before deviations from MF occur, $\Gamma(x,t)$ is a constant for all $x$ also in the SBR regime. In this regime the solution of the SBR equations become identical to the zero-dimensional SBR equation corresponding to a $g(x,t)$ that has no spatial fluctuations.

This situation will always occur if we increase $M$ keeping the size $L$ of the original ($M = 1$) lattice fixed. This can be also seen noticing that according to eqs. (17) if we go from $M$ to a large value $M'$ we should also increase the system size $L$ of the original lattice by a factor $(M'/M)^{1/(8-d)}$, if we do not do that we are systematically reducing the effective system size so that spatial fluctuations are negligible due to the gradient term. The data at $M = 20000$ in fig. (2) correspond to $L = 4$ and are indeed in this regime, correspondingly the SBR prediction displayed in the figure was computed with the zero-dimensional equations. In the section III we will discuss numerical data at $L = 4$ for values of $M$ corresponding to the zero-dimensional regime. The data for $M = 3000$ and $L = 8$ are instead strictly three-dimensional because, although we are not in the thermodynamic limit, we had to consider the full three-dimensional SBR equation that depends on the the value of $D_{NBW}$ on the lattice and on the system size.

III. FINITE-SIZE CORRECTIONS TO MEAN-FIELD MODELS (SBR IN ZERO DIMENSION)

We will consider simulations for $L = 4$ and values of $M$ that are large enough that, according to the discussion of the previous section, the SBR equations to be used are the zero dimensional ones.

More precisely considering the shifted total persistence

$$g(t) = \frac{1}{ML^3\rho} \sum_{i,\alpha} g_i^\alpha(t)$$

FIG. 4: Volume averaged dynamical fluctuation (bottom) and zero-distance dynamical fluctuations (top) vs. average order parameter for $M = 3000$ for the diamond cubic lattice. Numerical data (points) are well described by SBR predictions (solid lines). SBR reproduces accurately the finite-size effects displayed by the data, see text.
FIG. 5: Shifted persistence for $L = 4$ and $M = 5 \times 10^4$, $M = 10^5$ and $M = 2 \times 10^5$, the data follows the mean-field curve (corresponding to $M = \infty$) at small times and deviate from it at times that increase with $M$, the dotted lines represent the corresponding SBR predictions

we have:

$$\langle g^N(t) \rangle = [g^N(t)], \quad 1 \ll M < \infty$$

where in the R.H.S. $g(t)$ is the solution of of the zero-dimensional SBR equations [245]:

$$\sigma + s = -\lambda g^2(t) + \frac{d}{dt} \int_0^t g(t-s)g(s)ds.$$ (28)

The square brackets mean average with respect to the field $s$ that is a time-independent Gaussian random fluctuation of the separation parameter:

$$[s] = 0, \quad [s^2] = \Delta \sigma^2.$$ (29)

The equations have to be solved with the small-time condition

$$\lim_{t \to 0} g(t)(t/t_0)^a = 1.$$ (30)

Note that the gradient term is not present and indeed the actual geometrical structure of the lattice is irrelevant, the only relevant control parameter for SBR is the total number $N$ of sites of the lattice that controls $\Delta \sigma^2$ through

$$\Delta \sigma^2 = \frac{1}{N} \frac{2187}{8192} (1 - \lambda)^2.$$(31)

The above expression follows from expression [10]). The above expression does not depend on how the lattice is actually generated, in particular it is also correct for a random-regular-graph of size $N$ [19]. In the present setting $N = M L^2 \rho = 8 M$.

In fig. (5) we show the relaxation for $L = 4$ and $M = 5 \times 10^4$, $M = 10^5$ and $M = 2 \times 10^5$ averaged over respectively 10840, 3840 and 3840 samples. As expected the data follows the mean-field curve (corresponding to $M = \infty$) at small times and deviate from it at times that increase with $M$. The dotted lines represent the predictions of SBR corresponding to the different values of $M$. In fig. (6) we show the same data rescaled horizontally and vertically with the appropriate powers of $M$ that lead to collapse of the data on a single SBR curve. They are obtained from (17) with $d = 0$ (this is the technical reason why this is called the zero-dimensional regime). The rescaling factors are $M^{1/4}$ for the shifted persistence and $M^{-1/(4a)}$ for time.

The dotted lines in the fig. (6) represents respectively the mean-field asymptotic expression $(t/t_0)^{-a}$ (top) and the SBR prediction. We see that the curves tend indeed to collapse onto the SBR prediction. Note that the agreement is not yet perfect and this can be tracked to the fact that at the values of $M$ considered the curves start to deviate from the mean-field curve at times where the $M = \infty$ mean-field curve itself has still relevant small-time corrections of order $(t/t_0)^{-2a}$ to the leading asymptotic behavior $(t/t_0)^{-a}$, as discussed in appendix [3]. As a consequence in
FIG. 6: Rescaled shifted persistence for $L = 4$ and $M = 5 \times 10^4$, $M = 10^5$ and $M = 2 \times 10^5$. The dotted lines are the mean-field asymptotic expression $(t/t_0)^{-a}$ (top) and the SBR prediction (bottom). The data for different $M$ display a good collapse on the SBR curve at large rescaled times.

FIG. 7: Parametric plot of the fluctuations vs. shifted persistence for $L = 4$ and $M = 5 \times 10^4$, $M = 10^5$ and $M = 2 \times 10^5$ (bottom to top). The data approaches the MF asymptote (dashed) at large times and deviates from it at times that increase with $M$, corresponding to smaller values of the persistence.

fig. 6 the data for small rescaled times $tM^{1/(4a)}$ of order $5 - 20$ display an approximate 20% deviation from the asymptotic expression $(t/t_0)^{-a}$ that should describe the data for small rescaled times at large values of $M$. The small rescaled-time agreement improves considering larger values of $M$, as confirmed by simulations at $M = 1.610^6$ run for times corresponding to rescaled times $tM^{1/(4a)} \approx 8$. On the other hand small time corrections gets smaller at larger rescaled times $\approx 50$ and correspondingly the data agree much better with the SBR curve. We recall that in order to fully appreciate the agreement between theory and data it is important to remember that the SBR curves in fig. 5 and 6 were computed independently of the data. We note that the rescaling factor $M^{1/4}$ that leads to the collapse of the curves was obtained for the first time in [27] from a static computation, the resulting scaling function, apart from being independent of time, is ill-defined and one needs a full-fledged dynamical treatment to compute the time-dependent scaling curve.

We now turn to the analysis of fluctuations defined as before:

$$\chi_4(t) \equiv N (\langle g^2(t) \rangle - \langle g(t) \rangle^2)$$

(32)

where again $N = 8M$. In the MF regime corresponding to $1 \ll t \ll O(M^{1/4})$ the total susceptibility is the same that one measures on the Bethe lattice (see section IV A) and diverges at large times as:

$$\chi_4(t) \approx \frac{2187}{8192} (1 - \lambda)^2 A_1^2 (t/t_0)^{2a}$$

(33)
while deviations are observed for times of order $M^{1/(4a)}$. In fig. (7) we plot parametrically the susceptibility vs. the shifted persistence for different values of $M$. As expected the data approach the MF asymptote at large times (small persistence) and then deviates from it at times that increase with increasing $M$.

In fig. (8) we plot parametrically the rescaled susceptibility vs. the rescaled persistence for different values of $M$. The data display an almost perfect collapse on the corresponding universal curve computed through SBR. The SBR curve at small times (large values of the persistence) follows the large times MF behavior that corresponds to the straight line with a slope given by (23). Note that although time has been removed one still needs a dynamical SBR equations to obtain the scaling function.

**IV. COMPUTING THE PARAMETERS OF SBR**

In this section we will compute the parameters $\alpha$, $\Delta \sigma$ and $\sigma$ of SBR for FA models on $M$-layer lattices. The computation of three parameters can be done by comparing any three quantities of the theory. We will extract them comparing the equal-time fluctuations of the density $g(x,t)$ in the Mean-Field regime.

According to the analysis of [13] on the $M$-layer model the mean-field expression of the correlation of the order parameter a two points of the lattice is given by the correlation on all non-backtracking paths connecting the two sites on the original lattice divided by $M$:

$$\langle g(x,t)g(y,t) \rangle - \langle g(x,t) \rangle \langle g(y,t) \rangle = \frac{1}{M} \sum_{L=0}^{\infty} N_L(x,y)c_{L}^{\text{Bethe}}(t)$$

(34)

where $c_{L}^{\text{Bethe}}(t)$ is the correlation of the shifted persistence between two sites $i$ and $j$ at distance $L$ on the Bethe lattice:

$$c_{L}^{\text{Bethe}}(t) = \langle g_i(t)g_j(t) \rangle - \langle g_i(t) \rangle \langle g_j(t) \rangle$$

(35)

and $N_L(x,y)$ is the number of non-backtracking walks between point $x$ and point $y$ on the original lattice (corresponding to $M = 1$).

As we have saw before, for $M$ finite but large, in the regime where SBR provides a quantitatively accurate description, the length-scale of the fluctuations of $g(x,t)$ is large and thus we are interested in the regime where $x - y$ (and thus $L$) in $N_L(x,y)$ is also large. For large $L$, $N_L(x,y)$ tends to a Gaussian with a $O(L)$ variance:

$$N_L(x,y) \approx \frac{c(c-1)^{L-1}}{\rho} G(x - y).$$

(36)

The above relationship is valid only if $x$ and $y$ corresponds to coordinates of points of the lattice and is zero otherwise. This explains the prefactor, indeed since as we will see the variance of the Gaussian is much large than the lattice.
spacing we can replace the sum over lattice points as an integral on the continuum \( \sum_j \rightarrow \int \rho d^d x \). The integral on the other hand must be equal to the total number of paths of length \( L \) originating from a point, \( i.e. \ c(c-1)^L-1 \). In terms of the unitary Fourier transform we can then write

\[
N_L(k,k') \approx \delta(k+k') \frac{c}{\rho(c-1)} (c-1)^L \exp[-L D_{NBW} k^2]
\]

(37)

Where \( D_{NBW} \) is the diffusion coefficient of non-backtracking random walks \( \{x_1, \ldots, x_t\} \) on the original lattice with \( M = 1 \):

\[
D_{NBW} \equiv \lim_{t \rightarrow \infty} \frac{\langle ||x_t||^2 \rangle}{2 \rho t}
\]

(38)

In appendix [D] we will provide a simple expression for \( N_{DBW} \) in terms of the dimension of the lattice and of its connectivity valid for a huge class of lattices.

If we compute quantities at the level of the MF approximation we will observe the spurious dynamical arrest transition. In particular the FA model on the Bethe lattice below the critical temperature (corresponding to \( p < p_c = 8/9 \)) displays a glassy phase. In this regime it is convenient to study the persistence and its fluctuations in the infinite-time limit. Besides in this limit the critical properties of the blocked (negative) sites are exactly the same of the \( k \)-core of bootstrap percolation. In particular we have

\[
\lim_{t \rightarrow \infty} \langle g(x,t) \rangle = P_{site}(p) - P_{site}(p_c) \approx \frac{27}{16\sqrt{2}} \delta p^{1/2}
\]

(39)

and

\[
\lim_{t \rightarrow \infty} c_L^{\text{Bethe}}(t) = c_L^{\text{Bethe}}(p) \approx c_1 L \mu_L(p)
\]

(40)

where \( P_{site}(p) \) is the \( k \)-core density, \( \delta p \equiv p - p_c \) and \( c_L^{\text{Bethe}}(p) \) is the two-point correlation of the \( k \)-core. The above formula is derived in in the supplemental material of Ref. [26], (see also appendix [C]) where it is also shown that \( \mu(p) \) tends to the critical value \( (c-1)^{-1} \) at large times with a correction of order \( \sqrt{\delta p} \)

\[
\mu(p) \approx \frac{1}{c-1} (1 - c_2 \delta p^{1/2})
\]

(41)

and the numerical constants \( c_1 \) and \( c_2 \) read for \( c = 4 \) and \( m = 2 \) [26]:

\[
c_1 = \frac{81}{512}, \quad c_2 = \frac{3}{\sqrt{2}}
\]

(42)

In appendix [C] we report the general formulas from which \( c_1 \) and \( c_2 \) were obtained. Putting eqs. (34), (37) and (40) together and performing the summation over \( L \) we can write for the unitary Fourier transform of the fluctuations:

\[
\lim_{t \rightarrow \infty} \langle (g(k,t)g(k',t)) - \langle g(k,t) \rangle \langle g(k',t) \rangle \rangle = \frac{1}{M} \delta(k+k') \frac{c_1 c}{\rho (c-1)} \left( \frac{1}{c_2 \delta p^{1/2} + D_{NBW} k^2} \right)^2
\]

(43)

note the difference from a simple Ornstein-Zernicke form due to the square that appears because of the \( O(L) \) prefactor in (40).

The above expression has to be compared with the long-time limit of the mean-field approximation to SBR in the glassy phase \( \sigma > 0 \). The MF approximation to SBR corresponds to assume \( \Delta \sigma^2 \) is negligible and leads to:

\[
\lim_{t \rightarrow \infty} [g(x,t)] = \left( \frac{\sigma}{1 - \lambda} \right)^{1/2}
\]

(44)

fluctuations can be computed treating the fields \( h(x) \) as small compared to \( \sigma \) and lead to:

\[
\lim_{t \rightarrow \infty} [g(k,t)g(k',t)] - [g(k,t)][g(k',t)] = \Delta \sigma^2 \delta(k+k') \left( \frac{1}{2 \sqrt{(1 - \lambda) \sigma + a k^2}} \right)^2
\]

(45)

Note that the above MF approximations are completely wrong because in reality the solution of SBR has no static limit. By equating expressions (44) and (39) we obtain, as quoted before,

\[
\sigma = (1 - \lambda) \frac{729}{512} (p - p_c)
\]

(46)
By equating expressions (45) and (43) we obtain:

$$\Delta \sigma^2 = \frac{1}{M \rho} \frac{2187}{8192} (1 - \lambda)^2$$ (47)

$$\alpha = (1 - \lambda) \frac{9}{8} D_{NBW}$$ (48)

A. Dynamical Mean-Field Fluctuations

In the previous section we have computed the mean-field expression (34) in the glassy phase in the infinite time limit. In the following we will obtain its expression at large times at the critical point ($p = p_c = 8/9$). In this case one expects (and may confirm numerically) that:

$$c_Bethe(t) \approx c_1 L \mu_L(t)$$ (49)

where $\mu(t)$ tends to the critical value $(c - 1)^{-1} \Delta \sigma^2$ at large times with a correction of order $t^{-a}$

$$\mu(t) \approx \frac{1}{c - 1} (1 - \tilde{c}_2 t^{-a})$$ (50)

The constants $c_1$ is the same obtained above in the glassy phase, while $\tilde{c}_2$ is different from $c_2$ and at present can only be extracted from the numerics. Repeating the same steps above we then obtain:

$$\langle g(k, t)g(k', t) \rangle - \langle g(k, t) \rangle \langle g(k', t) \rangle = \frac{1}{M} \delta(k + k') - \frac{c_1 c}{\rho (c - 1)} \left( \frac{1}{\tilde{c}_2 t^{-a} + D_{NBW} k^2} \right)^2$$ (51)

this expression has to be compared with the mean-field small-time approximation of the SBR equations. This is obtained solving the equations perturbatively around the small-time limit $g(x, t) \approx (t/t_0)^{-a}$. In Fourier space one computes the correction to the $(t/t_0)^{-a}$ mean-field behavior due to the $h(x)$, then after averaging one easily obtains:

$$\langle g(k, t)g(k', t) \rangle - \langle g(k, t) \rangle \langle g(k', t) \rangle = \Delta \sigma^2 \delta(k + k') \left( \frac{1}{A_1^{-1} (t/t_0)^{-a} + \lambda k^2} \right)^2$$ (52)

where we have used Gotze's notation (eq. 6.63a in [1])

$$A_1 \equiv \frac{1}{2(a \pi \csc(a \pi) - \lambda)}$$ (53)

Note the above expression will be modified at larger times and should be replaced with the full SBR average. Equating (51) and (52) provides an alternative way to determine the SBR parameters $\Delta \sigma$ and $\alpha$ from the dynamics of FA models on the Bethe lattice (through the constants $c_1$ and $\tilde{c}_2$). Setting $\Delta \sigma^2 = 1/M$, $\alpha = t_0 = 1$ we obtain expression (14) used to derive the finite-time Ginzburg criterion discussed in section II.

V. CONCLUSIONS

We have considered supercooled liquid models where deviations from mean-field MCT predictions in finite dimension are expected to be described by an effective theory and we have demonstrated that the corresponding theory is SBR.

The parameters of SBR could be independently computed, in part analytically (through the connection to bootstrap percolation) and in part numerically, from the Bethe lattice solution of the problem. We thus provide the first instance of a model that can be described both qualitatively and quantitatively beyond mean-field theory without any fitting parameter.

The present analysis could not only be repeated for FA models on lattices in other dimensions, but it can be applied to any model that exhibits a MCT-like transition on the Bethe lattice. First we should identify a lattice in finite dimension that has the same local structure so that we could apply the $M$-layer construction to it. The case of spin-glass models is particularly interesting because in that case the replica (and cavity) method allows to compute in a static context also the parameter exponent $\lambda$ [28, 30] and thus all the four mesoscopic parameters $\Delta \sigma$, $\alpha$, $\sigma$ and $\lambda$ of SBR could be computed analytically.
FIG. 9: Shifted persistence at $L = 4$ and $M = 1.6 \times 10^6$, the dotted lines represent the asymptotic expression $(t/t_0)^{-a}$ and the asymptotic expression plus the negative sub-leading correction $c_1 (t/t_0)^{-2a}$

Appendix A: SBR in general form

The equations of SBR are:

$$\sigma + s(x) = -\alpha \nabla^2 g(x,t) - \lambda g^2(x,t) + \frac{d}{dt} \int_0^t g(x,t-s)g(x,s)ds$$

(A1)

where the field $s(x)$ is a time-independent random fluctuation of the separation parameter, Gaussian and delta-correlated in space:

$$[s(x)] = 0, \quad [s(x)s(y)] = \Delta \sigma^2 \delta(x-y)$$

(A2)

and they have to be solved with the condition

$$\lim_{t \to 0} g(x,t)(t/t_0)^a = 1.$$  

(A3)

Thus the equations depend on five coupling constants: $\lambda$, $t_0$, $\alpha$, $\sigma$ and $\Delta \sigma$. However one can fix $c_0 = \alpha = \Delta \sigma = 1$ and then the general solution can be obtained by rescalings. This means that in practice the SBR equations at fixed $\lambda$ need only to be solved for varying values of $\sigma$. More precisely one can easily verify that for a generic $K$-point function we have

$$[g(x_1,t_1) \ldots g(x_K,t_K)]_{\alpha,\Delta \sigma, t_0, \sigma, \lambda} = b_{\phi}^K [g(x_1/b_x,t_1/b_t) \ldots g(x_K/b_x,t_K/b_t)]_{1,1,1,\sigma/b_{\sigma},\lambda}$$

(A4)

where the notation $[\ldots]_{\alpha,\Delta \sigma, t_0, \sigma, \lambda}$ means that the SBR equations above are to be solved with the corresponding values of the five parameters. The rescaling parameters read:

$$b_{\phi} = \Delta \sigma^{1/2} \alpha^{-\frac{d+2}{2d}}$$

(A5)

$$b_x = (\alpha/b_{\phi})^{1/2}$$

(A6)

$$b_{\sigma} = b_{\phi}^2$$

(A7)

$$b_t = t_0 b_{\phi}^{-1/a}.$$  

(A8)

Appendix B: Mean-Field Behavior

In figure (9) we plot $g(t) = p(t) - p_{\text{plat}}$ for $L = 4$ and $M = 1.6 \times 10^6$. In the range of times shown these data have converged on the Bethe lattice solution corresponding to $M \to \infty$. In fig. (10) we plot parametrically the (discrete) logarithmic derivative of $g(t)$ vs. $g(t)$. For a $g(t)$ decaying at large times as $1/t^a$ the logarithmic derivative should converge to $-a$ in the limit of $g(t) \to 0$. The data display linear behavior in the small $g$ region corresponding to $a$
correction of order $1/t^{2a}$. This is precisely the small-time correction that one would expect within MCT: the leading term is given by a quadratic equation while various subleading cubic terms induce a $1/t^{2a}$ correction. The following behavior

$$g(t) = \frac{1}{(t/t_0)^a} + \frac{\delta_1}{(t/t_0)^{2a}} + \ldots$$  \hspace{1cm} (B1)

leads to

$$\frac{\Delta \ln g}{\Delta \ln t} = -a - a \delta_1 + \ldots$$  \hspace{1cm} (B2)

and thus from a linear fit we can extract $a$ and $\delta_1$. The constant $t_0$ can then be estimated fitting the numerical data with the asymptotic form (B1), in fig. (9) we show both the $1/(t/t_0)^a$ term and the corrected expression. We note that in the range of times accessed numerically the $1/t^{2a}$ correction is significant and a linear fit of the data in fig. (9) would give an incorrect smaller exponent $a$. Instead the analysis of the second derivative is much safer and inconsistent with the value $28 - .3$ reported in previous studies.

We note that within the replica treatment the parameter exponent (and thus the exponent $a$) is available from the statics ([28, 29]) and only the coefficient $t_0$ must be extracted numerically. Unfortunately a static replica treatment of the problem is not available at present for these problems.

1. Comparison between different dynamics

The numerical simulations were performed in a Chessboard/Metropolis setting: we have divided the lattice in red and blue sites in such a way that blue sites are connected only to red sites and vice versa. Then we perform a sequential update of all the red spins followed by a sequential update of all the blue spins. The single update is made with a Metropolis move: a negative mobile spin is flipped with probability $e^{-\beta}$ and a positive mobile spin is flipped with probability one. The standard dynamics of the FA model is instead a Random-site/Metropolis one in which the site to be updated is chosen at random. The choice of the Chessboard setting is more convenient because each Monte-Carlo step (MCS) requires less CPU time, besides in MCS unit the relaxation is faster as can be seen in fig. (11) where we display the relaxation for a system with $L = 4$ and $M = 4 \times 10^5$ with different dynamics, including Chessboard with heat-bath (Glauber) update. The key point is that at large enough times the different curves differ only by a constant shift in time: this can be seen more clearly considering the parametric plot of the logarithmic derivative of $g$ vs. $g$. In fig. (12) we see that at small values of $g$ the parametric curves collapse onto a single curve independently of the dynamics. Note that the collapsing curve deviates from the linear fit corresponding to the asymptotic behavior implying that finite-size critical deviations are also independent on the dynamics. This is consistent with the fact that as we saw already these deviations are described by SBR in which the different microscopic dynamics enter only through the parameter $t_0$. We also mention that the Chessboard dynamic satisfies detailed balance but is not invertible, furthermore irreducibly of the Markov chain is not granted in general in sequential update with Metropolis moves, but these features, as usual, appear to be harmless in the interesting region of long-time critical behavior.
FIG. 11: Relaxation vs. time for a system with \( L = 4 \) and \( M = 4 \times 10^5 \) with different dynamics, from top to bottom: Random-site/Metropolis, Chessboard/Heat-Bath, Chessboard/Metropolis.

FIG. 12: Parametric the (discrete) logarithmic derivative of \( g(t) \) vs. \( g(t) \) for \( L = 4 \) and \( M = 4 \times 10^5 \) from top to bottom: Random-site/Metropolis, Chessboard/Heat-Bath, Chessboard/Metropolis. The curves from different dynamics are on top of one another at small values of \( g \). The dotted line is the linear fit describing the data at large \( M \).

Appendix C: Bootstrap Percolation on the Bethe lattice

The dynamics transition of FA models on the Bethe lattice of connectivity \( c \) is intimately related with \( k \)-core, or bootstrap, percolation [16]. Let us recall the definition of BP: first the sites of a given lattice are populated with probability \( p \), then each site with less than \( k \) neighbors is removed and the process is repeated until each site has at least \( k \) neighbors. When the process is completed the remaining occupied sites, if any, form the so-called the \( k \)-core.

One can argue that the cluster of blocked negative sites on the FA model is exactly the \( k \)-core with \( k \) given by

\[
k = c - m + 1
\]

therefore \( k = 3 \) for the \((c = 4, m = 2)\) FA models considered here. The relationship between temperature in the FA model and \( p \) in BP is easily found:

\[
p = \frac{e^\beta}{1 + e^\beta}
\]

With the definition \( P_{s,t} \equiv \sum_{i=s}^t \binom{t}{i} P^i (1 - P)^{t-i} \) the Bethe solution of BP is determined by the following equations:

\[
P = p P_{k-1,c-1}, \quad P_{site} = p P_{k,c}.
\]
Where $P_{\text{site}}$ is the density of the $k$-core and $p$. For all $k > 2$ a solution with non-zero $P$ is found at large values of $p$; the solution disappears at a critical value $p = p_c$ with a square-root singularity thus exhibiting the celebrated mixed first-order/second-order character.

As shown in the Supplemental Material for [26] the connected probability that two points at large distances $L$ on the Bethe lattice are both on the $k$-core is given:

$$c_L(p) \approx \frac{p^2 \Delta}{\mu} L^{\mu^2} + O(\lambda^L)$$  \hspace{1cm} (C4)

where

$$\mu = p \left( \frac{c-2}{k-2} \right) P^{k-2}(1-P)^{c-k}$$  \hspace{1cm} (C5)

Note that in [26] the parameter $\mu$ is called $\lambda$ but we changed notation to avoid confusion with the MCT parameter exponent.

$$\Delta \equiv \frac{1-\mu}{\mu} P(1-P)$$  \hspace{1cm} (C6)

$$p_1 \equiv p \left( \frac{c-1}{k-1} \right) P^{k-1}(1-P)^{c-k}$$  \hspace{1cm} (C7)

In the case $(c = 4, k = 3)$ we have

$$P = \frac{3}{4}, \mu = \frac{1}{3}, P_{\text{site}} = \frac{21}{32} \text{ for } p = p_c = \frac{8}{9}$$  \hspace{1cm} (C8)

From which the expressions for $c_1$ and $c_2$ given in section (IV) can be obtained.

**Appendix D: Diffusion Coefficient of Non-Backtracking random walks on generic lattices**

We consider lattices with connectivity $c$ such that we can classify sites in two classes, say red and blue, such that a blue site is connected to the $c$ red sites located in the $c$ directions $\mathbf{v}_\mu$. Similarly a red site is connected to the $c$ blue sites located in the $c$ directions $-\mathbf{v}_\mu$. We will not require the condition $\{\mathbf{v}_\mu\} = -\{\mathbf{v}_\mu\}$, meaning that the set of directions need not to be invariant under inversion of the coordinate axes. Thus we extend previous results [13, 31] to include e.g. the honeycomb lattice in $d = 2$ and the diamond lattice in $d = 3$. We assume instead that

$$\sum_\mu \mathbf{v}_\mu = 0, \quad ||\mathbf{v}_\mu|| = v.$$  \hspace{1cm} (D1)

A non-backtracking random walk can be described as a sequence of steps at consecutive times $s$ in the directions $\mu(s)$:

$$x_t = \sum_{s=0}^{t-1} (-1)^s \mathbf{v}_{\mu(s)}$$  \hspace{1cm} (D2)

where the minus sign comes from the fact that the set of possible directions changes as the walker moves from a blue and to a red site. The average can then be written in terms of the joint probability $P_{\mu,\mu'}^{(s,s')}$

$$\langle ||x_t||^2 \rangle = \sum_{s,s'=0}^{t-1} (-1)^{s+s'} \mathbf{v}_\mu \cdot \mathbf{v}_{\mu'} P_{\mu,\mu'}^{(s,s')}$$  \hspace{1cm} (D3)

note that we use Einstein’s convention of implicit summation over repeated indexes $\mu$. Since the directions are uniformly distributed at zero time we have exactly

$$P_{\mu,\mu'}^{(s,s')} = P_{\mu,\mu'}^{(0,s-s')}$$  \hspace{1cm} (D4)
and we can write for large \( t \)

\[
\langle \| x_t \| \rangle^2 \approx t \, v^2 \left( 2 \sum_{s=1}^{\infty} (-1)^s \hat{\mathbf{v}}_\mu \cdot \hat{\mathbf{v}}_{\mu'} P_{\mu,\mu'}^{(0,s)} + 1 \right)
\]  \( \text{(D5)} \)

The probability can be computed recursively

\[
P_{\mu,\mu'}^{(0,s+1)} = T_{\mu,\mu'} P_{\mu,\mu'}^{(0,s)}
\]  \( \text{(D6)} \)

where \( T_{\mu,\mu'} \) is the \( c \times c \) matrix with zero diagonal elements and off-diagonal elements equal to \( 1/(c-1) \). We can thus write

\[
T = \mathcal{P}_s - \frac{1}{c-1} \mathcal{Q}_s
\]  \( \text{(D7)} \)

where \( \mathcal{P}_s \) is the projector on the vector with all equal components and \( \mathcal{Q}_s = I - \mathcal{P}_s \) is the orthogonal projector. It follows that

\[
P_{(0,0)} = \frac{1}{c} I, \quad P_{(0,s)} = \frac{1}{c} T^s = \frac{1}{c} \mathcal{P}_s + \frac{1}{c} \left( \frac{-1}{c-1} \right)^s \mathcal{Q}_s
\]  \( \text{(D8)} \)

since \( \mathcal{P}_s \) is a matrix with all elements equal it gives zero contribution when summed over the directions because of the condition \( \sum_\mu \hat{\mathbf{v}}_\mu = 0 \) and we have

\[
\hat{\mathbf{v}}_\mu \cdot \hat{\mathbf{v}}_{\mu'} P_{\mu,\mu'}^{(0,s)} = \left( \frac{-1}{c-1} \right)^s v^2
\]  \( \text{(D9)} \)

replacing the above expression in the large time expression \( \text{[D5]} \) we obtain:

\[
\langle \| x_t \| \rangle^2 \approx t \, \frac{v^2}{c-2}
\]  \( \text{(D10)} \)

and therefore (taking into account that each vector has \( d \) components)

\[
D_{NBW} \equiv \lim_{t \to \infty} \frac{\langle \| x_t \| \rangle^2}{2 \, dt} = \frac{c \, v^2}{2 \, d \, (c-2)}.
\]  \( \text{(D11)} \)

On regular lattice \( c = 2d \) and we recover the result \( D_{NBW} = 1/(2d-2) \). On the diamond cubic lattice studied in the paper we have \( c = 4 \) and \( v^2 = 3 \) leading to \( D_{NBW} = 1 \). The honeycomb lattice can be realized repeating a \( 3 \times \sqrt{3} \) unit cell with two blue points at coordinates \((0,0),(\sqrt{3}/2,3/2)\) and two red points with coordinates \((0,1),(\sqrt{3}/2,5/2)\). This leads to \( \rho = 43/52 \) and \( D_{NBW} = 3/4 \).

The static SBR equations are ill-defined and this implies that the hybrid BP transition observed in mean-field theory does not survive in physical dimension.